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Integrating factors and conservation theorem for holonomic nonconservative dynamical systems in generalized classical mechanics*

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In this paper, we present a general approach to the construction of conservation laws for generalized classical dynamical systems. Firstly, we give the definition of integrating factors and, secondly, we study in detail the necessary conditions for the existence of conserved quantities. Then we establish the conservation theorem and its inverse for the Hamilton's canonical equations of motion of holonomic nonconservative dynamical systems in generalized classical mechanics. Finally, we give an example to illustrate the application of the results.

Keywords: generalized classical mechanics, holonomic nonconservative system, Hamilton's canonical equation, integrating factor, conservation theorem

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1. Introduction

The generalized classical mechanical theory initiated by Ostrogradsky and Jacobi in 1859 has been developed greatly in the last fifty years, and many important results have been obtained. Especially, concerning physics, the Lagrangian equations with high-order derivatives in mechanics and field theories have been studied; for example, Podolsky has introduced an electromagnetic theory with second-order derivatives. Concerning mathematics, the description of the modern geometrical method has obtained good results.[1]

The conservation law of a dynamical system plays an important role in the study of mechanics. The existence of a certain conserved quantity will help us understand the partial physical state and the quality of the system studied when it is hard to solve the differential equations of motion of the system. Furthermore, it has become one of the most important directions for research. At present, there are two main methods to seek the conserved quantity. One is based on the theory of invariance of Hamilton's action under infinitesimal transformations, namely, the Noether theory.[2] The other was given by Lutzky[3] in 1979, who introduced Lie's method of the expansive group of invariance for ordinary differential equations under infinitesimal transformations to mechanical systems and presented the Lie symmetries of differential equations of motion. In recent years, this kind of method has been extended to generalized classical mechanics, and significant achievements have been obtained in the study of this method.[4−11]

In this paper, we extend the theory of integrating factors of differential equations[12] to generalized classical mechanics to seek the conserved quantity for Hamilton's canonical equation of motion. Compared with the two famous methods mentioned above, this method has the benefits of fewer conditions, easy calculation and wide practical applications. Also, it will have a great future in the integration theory of generalized classical mechanics. Firstly, we give the definition of integrating factors of differential equations of motion. Secondly, we study the necessary conditions of the existence of conserved quantities. Then we establish the conservation theorem and its inverse theorem of the Hamilton's canonical equations of motion of generalized holonomic nonconservative dynamical systems. Finally, an example is given to illustrate the application of the results.

2. Generalized Hamilton's canonical equations of motion

If the configuration of generalized mechanical sys-

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tems is determined by n generalized coordinates q_s $(s=1,2,\cdots,n)$, then the Lagrangian function is

$$L=L(t,q_s,\dot{q}_s,\ddot{q}_s,\cdots,\overset{(\omega)}{q_s})$$
$$(s=1,2,\cdots,n). \tag{1}$$

In a lower-dimensional space, the generalized nonpotential force is $Q_\alpha=Q_\alpha(t,q_s,\dot{q}_s)$, while in a higher-dimensional space, the generalized nonpotential force is written as

$$Q_\alpha=Q_\alpha(t,q_s,\dot{q}_s,\dot{q}_s,\ddot{q}_s,\cdots,\overset{(\omega)}{q_s})$$
$$(s=1,2,\cdots,n). \tag{2}$$

The Lagrangian equation of the generalized nonconservative system is

$$\sum_{j=0}^{\omega}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{\partial L}{\partial\overset{(j)}{q_s}}\right)+Q_s=0$$
$$(s=1,2,\cdots,n). \tag{3}$$

The generalized momentum is

$$p_{s/m}=(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{\partial L}{\partial\overset{(j+m)}{q_s}}\right)$$
$$(j=0,1,2,\cdots,\omega-m;\quad m=1,2,\cdots,\omega;$$
$$s=1,2,\cdots,n). \tag{4}$$

We introduce the Hamilton's function

$$H(t,\overset{(m-1)}{q_s},p_{s/m})=p_{s/m}\overset{(m)}{q_s}-L(t,q_s,\dot{q}_s,\ddot{q}_s,\cdots,\overset{(\omega)}{q_s})$$
$$(s=1,2,\cdots,n;\quad m=1,2,\cdots,\omega). \tag{5}$$

Hence, the Hamilton's canonical equations of motion of the generalized mechanical systems are

$$\overset{(m)}{q_s}=\frac{\partial H}{\partial p_{s/m}}, \tag{6}$$

$$p_{s/m}=-\frac{\partial H}{\partial\overset{(m-1)}{q_s}}+\tilde{Q}_s$$
$$(s=1,2,\cdots,n;\quad m=1,2,\cdots,\omega). \tag{7}$$

Here, the symbol $\sim$ represents the expression obtained by substituting $t,\overset{(m-1)}{q_s},p_{s/m}$ for $\overset{(m)}{q_s}$.

3. Integrating factors and conservation theorems

3.1. Integrating factors

If the invariant

$$\left(\dot{p}_{s/m}+\frac{\partial H}{\partial\overset{(m-1)}{q_s}}-\tilde{Q}_s\right)G_s$$

reduces identically to

$$\left(\dot{p}_{s/m}+\frac{\partial H}{\partial\overset{(m-1)}{q_s}}-\tilde{Q}_s\right)G_s$$
$$\equiv\frac{\mathrm{d}}{\mathrm{d}t}(p_{s/m}G_s-HR-\Lambda)$$
$$+\lambda_s\left(\dot{p}_{s/m}+\frac{\partial H}{\partial\overset{(m-1)}{q_s}}-\tilde{Q}_s\right), \tag{8}$$

where R,Λ and λ_s are functions of $t,\overset{(m-1)}{q_s}$ and $p_{s/m}$, then functions $G_s=G_s(t,\overset{(m-1)}{q_\beta},p_{\beta/m})$ are called integrating factors of the Hamilton's canonical equations of motion (7).

Combining Eqs.(7) and (8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_{s/m}G_s-HR-\Lambda)=\lambda_s\left(\dot{p}_{s/m}+\frac{\partial H}{\partial\overset{(m-1)}{q_s}}-\tilde{Q}_s\right), \tag{9}$$

and the following theorem can be obtained.

3.2. Conservation theorems

Theorem 1. If functions G_s are integrating factors of Eq.(7), then there exists a conserved quantity (first integration)

$$D=(p_{s/m}G_s-HR-\Lambda). \tag{10}$$

For a given dynamical system, if functions G_s are integrating factors of Eq.(7), the necessary condition (8) must be satisfied for every set of functions G_s,R,Λ and λ_s, which can be written as

$$-R\frac{\partial H}{\partial t}+\tilde{Q}_s\left(G_s-R\frac{\partial H}{\partial p_{s/m}}\right)$$
$$+p_{s/m}\dot{G}-H\dot{R}-\dot{\Lambda}+I=0. \tag{11}$$

Here, we have used Eqs.(6) and (7) to calculate, $\dot{H}$ and

$$I=\lambda_s\left((\dot{p}_{s/m}+\frac{\partial H}{\partial\overset{(m-1)}{q_s}}-\tilde{Q}_s\right). \tag{12}$$

If the set of functions G_s,R,Λ and λ_s satisfy the necessary condition (11) and the right-hand side of Eq.(10) is reduced to a constant, then we have a singular set of functions G_s,R,Λ and λ_s.

According to these considerations and Theorem 1, the following theorem can be formulated, which is convenient for finding first integral Eqs.(6) and (7).

Theorem 2. For every nonsingular set of functions G_s, R, Λ and λ_s, which satisfies the necessary condition (11), there exists a conserved quantity (first integration) (10) of the given holonomic nonconservative dynamical system.

By integrating Eq.(11), we can obtain the nonsingular set of functions G_s, R, Λ and λ_s which gives a conserved quantity. In any case, if we have any solution of Eq.(11), which does not contain any integration constant, we call the solution a functional solution of Eq.(11). When a nonsingular functional solution of Eq.(11) is substituted into Eq.(10), it produces a usual first integral of the dynamical system. Here, the only integration constant is the value of D. Vujanovic[13] has shown that, by using a complete solution of Eq.(11) which contains a sufficient number of integration constants, we can finally form a finite solution of the equations of motion (6) and (7).

Obviously, in Eq.(11) $\dfrac{\mathrm{d}(\ )}{\mathrm{d}t}$ must be interpreted as the following operator

$$\frac{\mathrm{d}(\ )}{\mathrm{d}t}=\frac{\partial(\ )}{\partial t}+\frac{\partial H}{\partial p_{s/m}}\frac{\partial(\ )}{\partial\overset{(m-1)}{q_s}}+\dot{p}_{s/m}\frac{\partial(\ )}{\partial p_{s/m}}. \quad (13)$$

4. Killing equations

The key is to find the functions $R=R(t,\overset{(m-1)}{q_s},p_{s/m})$, $\Lambda=\Lambda(t,\overset{(m-1)}{q_s},p_{s/m})$ and $G_s=G_s(t,\overset{(m-1)}{q_\beta},p_{\beta/m})$ which are used together with the theory of integrating factors to seek the conserved quantity of the generalized classical dynamical systems. Usually, we expand the necessary condition (11) and split it into first-order partial differential equations for R, Λ and G_s. This kind of partial differential equation is called the Killing equation. We can obtain the above functions through solving the Killing equation.

4.1. We can expand Eq.(11) using Eqs. (12) and (13) and separate it into terms with and without $\dot{p}_\beta$, and set them equal to zero individually. Then we have

$$\begin{aligned}&-R\frac{\partial H}{\partial t}+\tilde{Q}_s\left(G_s-R\frac{\partial H}{\partial p_{s/m}}\right)\\&+p_{s/m}\left(\frac{\partial G_s}{\partial t}+\frac{\partial G_s}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}\right)-G_s\frac{\partial H}{\partial\overset{(m-1)}{q_s}}\\&-H\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}\right)-\frac{\partial\Lambda}{\partial t}\\&-\frac{\partial\Lambda}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}+\lambda_\beta\left(\frac{\partial H}{\partial\overset{(m-1)}{q_\beta}}-\tilde{Q}_\beta\right)=0, \quad (14)\end{aligned}$$

$$p_{s/m}\frac{\partial G_s}{\partial p_{\beta/m}}-H\frac{\partial R}{\partial p_{\beta/m}}-\frac{\partial\Lambda}{\partial p_{\beta/m}}+\lambda_\beta=0. \quad (15)$$

If the dynamical systems admits a nonsingular solution with respect to $R(t,\overset{(m-1)}{q_s},p_{s/m})$, $\Lambda(t,\overset{(m-1)}{q_s},p_{s/m})$, $G_s(t,\overset{(m-1)}{q_\beta},p_{\beta/m})$, and $\lambda_s(t,\overset{(m-1)}{q_\beta},p_{\beta/m})$, then a first integral (10) exists automatically. Here, we have a system of $(n+1)$ linear partial differential equations with $(2n+2)$ unknown functions in most cases. Thus, there is greater freedom for their solutions. The most logical suggestion is to prescribe $(n+1)$ functions in advance and, after that, to solve the equations with respect to the remaining $(n+1)$ unknowns. Usually, the prescribed functions are Λ and λ_s.

4.2. We can expand the necessary condition (11) using Eqs.(12), (13) and (17), then we have

$$\begin{aligned}&-R\frac{\partial H}{\partial t}+\tilde{Q}_s\left(G_s-R\frac{\partial H}{\partial p_{s/m}}\right)\\&+p_{s/m}\left(\frac{\partial G_s}{\partial t}+\frac{\partial G_s}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}\right)-G_s\frac{\partial H}{\partial\overset{(m-1)}{q_s}}\\&-H\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}\right)-\frac{\partial\Lambda}{\partial t}\\&-\frac{\partial\Lambda}{\partial\overset{(m-1)}{q_\beta}}\frac{\partial H}{\partial p_{\beta/m}}+\left(\tilde{Q}_\beta-\frac{\partial H}{\partial\overset{(m-1)}{q_\beta}}\right)\\&\times\left(p_{s/m}\frac{\partial G_s}{\partial p_{\beta/m}}-H\frac{\partial R}{\partial p_{\beta/m}}-\frac{\partial\Lambda}{\partial p_{\beta/m}}\right)=0. \quad (16)\end{aligned}$$

This is a linear partial differential equation. In the general case, each of the $(n+2)$ functions G_s, R and Λ can be regarded as unknown. According to Theorem 2, any nonsingular solution (16) with respect to G_s, R and Λ yields a first integral (10).

5. Inverse theorems to theorems 1 and 2

The connection among the first integral of the dynamical systems, which is based on the theory of integration, the corresponding integrating factors G_s and the functions R and Λ must be consistent with the necessary condition (11). It is shown that the necessary condition (11) is equivalent to Eqs.(14), (15) or (16). Therefore, the set of $(n+2)$ functions G_s, R and Λ must satisfy Eqs.(10), (14) and (15) or (10) and (16).

Calculating $\dfrac{\partial\Lambda}{\partial p_{\beta/m}}$ from Eq.(10), and forcing this result to satisfy Eq.(5), we have

$$G_s=\frac{\partial D}{\partial p_{s/m}}+R\frac{\partial H}{\partial p_{s/m}}. \quad (17)$$

Now, combining Eqs.(10) and (17), we obtain

$$\Lambda = p_{s/m}\frac{\partial D}{\partial p_{s/m}} - D + R\left(p_{s/m}\frac{\partial H}{\partial p_{s/m}} - H\right). \quad (18)$$

Next, setting the first integral $D = D(t, {}^{(m-1)}_{q_s}, p_{s/m})$ equal to the conserved quantity, we have

$$p_{s/m}G_s - HR - \Lambda = D(t, {}^{(m-1)}_{q_s}, p_{s/m}). \quad (19)$$

From Eqs.(17)–(19), G_s, R and Λ can be obtained.

In this way, we have the following theorem.

Theorem 3. If every constant of motion $D(t, {}^{(m-1)}_{q_s}, p_{s/m})$ for the dynamical systems can be described by the Hamiltonian H and nonconservative generalized forces $\tilde{Q}_s$, then the integrating factors G_s and functions R and Λ corresponding to D are determined by Eqs.(17)–(19).

6. Example

Suppose that the Lagrange function of a mechanical system is

$$L = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\ddot{y}^2, \quad (20)$$

then the nonpotential generalized force is

$$Q = -\ddot{y}. \quad (21)$$

We try to find the conserved quantities of the system.

Firstly, we study the direct problem to obtain the conserved quantities of the system.

The first step is to establish the Hamilton function. From Eq.(4), the generalized moments are

$$p_1 = \frac{\partial L}{\partial \dot{y}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{y}} = \dot{y} - \dddot{y},$$

$$p_2 = \frac{\partial L}{\partial \ddot{y}} = \ddot{y}. \quad (22)$$

Letting $q_1 = y, q_2 = \dot{y}, \tilde{Q} = -p_2$, then the Hamilton function is

$$\begin{aligned} H =& p_1\dot{q}_1 + p_2\dot{q}_2 - L \\ =& p_1 q_2 + \frac{1}{2}p_2^2 - \frac{1}{2}q_2^2. \end{aligned} \quad (23)$$

The second step is to establish the Killing equation and find its solution.

From Eq.(16), we have

$$\begin{aligned} &\tilde{Q}_1\left(G_1 - R\frac{\partial H}{\partial p_1}\right) + p_1\left(\frac{\partial G_1}{\partial t} + \frac{\partial G_1}{\partial q_1}q_2 + \frac{\partial G_1}{\partial q_2}p_2\right) \\ &+ p_2\left(\frac{\partial G_2}{\partial t} + \frac{\partial G_2}{\partial q_1}q_2 + \frac{\partial G_2}{\partial q_2}p_2\right) \\ &- G_2(p_1 - q_2) - \left(p_1 q_2 + \frac{1}{2}p_2^2 - \frac{1}{2}q_2^2\right) \\ &\times \left(\frac{\partial R}{\partial t} + \frac{\partial R}{\partial q_1}q_2 + \frac{\partial R}{\partial q_2}p_2\right) - \frac{\partial \Lambda}{\partial t} - \frac{\partial \Lambda}{\partial q_1}q_2 - \frac{\partial \Lambda}{\partial q_2}p_2 \\ &+ \tilde{Q}_1\left(p_1\frac{\partial G_1}{\partial p_1} + p_2\frac{\partial G_2}{\partial p_1} - H\frac{\partial R}{\partial p_1} - \frac{\partial \Lambda}{\partial p_1}\right) \\ &- \frac{\partial H}{\partial q_2}\left(p_1\frac{\partial G_1}{\partial p_2} + p_2\frac{\partial G_2}{\partial p_2} - H\frac{\partial R}{\partial p_2} - \frac{\partial \Lambda}{\partial p_2}\right) = 0. \end{aligned} \quad (24)$$

The solutions of Eq. (24) are as follows

$$R = 0, \quad G_1 = 1, \quad G_2 = 0, \quad \Lambda = -\dot{y}. \quad (25)$$

The third step is to seek the conserved quantity of the system.

For the integrating factors G_1, G_2 and functions R, Λ all satisfy the necessary condition (11); according to Theorem 1, the conserved quantity (10) can be given as

$$D = 2\dot{y} - \dddot{y} = \text{const}. \quad (26)$$

Next, we study the inverse problem. The integrating factors G_s and functions R and Λ can be obtained from the known first integral.

Suppose the system studied has a first integral

$$D = 2\dot{y} - \dddot{y} = \dot{y} + p_1 = \text{const}. \quad (27)$$

From Eqs.(17) and (18), we have

$$G_1 = 1 + Rq_2, \quad G_2 = Rp_2, \quad (28)$$

$$R = \frac{\Lambda + \dot{y}}{\frac{1}{2}\dot{y}^2 + \frac{1}{2}\ddot{y}^2}. \quad (29)$$

When $\Lambda = -\dot{y}$, we have

$$G_1 = 1, \quad G_2 = 0, \quad R = 0. \quad (30)$$

These are consistent with the expression (25).

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