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Entropy and topology of the Kerr-de Sitter black hole*

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By using the path integral method of Gibbons and Hawking, the entropy of the Kerr-de Sitter black hole is investigated under the microcanonical ensemble. We find that the entropy is one eighth the sum of the products of the Euler number of its cosmological horizon and event horizon with their respective areas. It is shown that the origin of the entropy of the black hole is related to the topology of its instanton.

Keywords: entropy of black hole, constrained gravitational instanton, Euler number
PACC: 9880H, 0470D, 9760L

1. Introduction

It is well known that black holes are endowed with an intrinsic entropy proportional to the horizon area[1,2] which is interpreted as the measure of our lack of information beyond the horizon. The discovery of the entropy of the black hole is one of the most profound achievements in theoretical physics, and the origin of the entropy has attracted much attention in recent years.[3−7] In Refs.[8] and [9], Hawking *et al.* found that the Bekenstein-Hawking entropy, $S = A/4$, is not valid for some extremal black holes because these extremal black holes with nonzero areas of the event horizons possess zero entropy. They[8] observed that the source of such a different behaviour between extremal and nonextremal black holes comes from the difference in the topological structures. Hawking *et al.*[8,9] have suggested that the entropy is the result of the nontrivial topology of its Euclidean spacetime in Euclidean quantum gravity theory. This idea is supported by Gibbons and Kallosh[10] and Liberati[11] since they found that there is a concrete relation of the entropy and topology of a black hole, i.e. $S = \chi A/8$, where χ is the Euler number of the black hole. Recently, Wu[12] has studied entropies of static black holes embedded in a cosmological background. He concluded that the entropy of the black holes is one quarter of the sum of the products of the Euler number of its horizons with their respective areas. Whether the conclusion remains valid for stationary black holes is an open question. In this paper, we try to extend Wu's study to the Kerr-de Sitter black hole.

This paper is organized as follow. In Section II, we work out the entropy of the Kerr-de Sitter black hole by using the Euclidean path integral approach. In Section III, we calculate its Euler number by applying the Gauss-Bonnet action. Then we draw a conclusion and give a discussion.

2. Entropy of the Kerr-de Sitter black hole

The Kerr-de Sitter black hole can be considered as a rotating black hole embedded in a cosmological background. Its study is of great interest in understanding the properties of astrophysical objects and in checking any conjecture about the thermodynamic properties of the black hole. In gravitational thermodynamics,[13,14] as in other black holes, we can still define the Kerr-de Sitter black hole temperature as $\kappa/2\pi$, where κ is the surface gravity of the black hole. For the Kerr-de Sitter black hole, there are four horizons. We only take into account the outermost event horizon and the cosmological outermost horizon because these two horizons are stable. Then there are two different temperatures because of the difference of κ on each horizon. This implies that there is no thermal equilibrium state with a uniform temperature. However, we cannot apply the canonical or grand canonical ensemble to settle it. Fortunately, we can use the microcanonical ensemble. The temperature of each microscopic state may be not identical in the microcanonical ensemble, but all the conserved quantities must be given.[12] Also, the probability of

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each microscopic state which is constrained by the conserved quantity is equal, so the entropy of microcanonical ensemble is the logarithm of the number of these states.

In gravitational theory, the partition function for microcanonical ensemble is defined as

$$Z = \int \mathrm{d}[g_{\mu\nu}]\mathrm{d}[\phi]\exp[-I], \tag{1}$$

where the path integral is over all spacetime metrics $g_{\mu\nu}$ and the matter configurations ϕ which are constrained by the microcanonical ensemble. I is the Euclidean action which is given by the following integral

$$\begin{aligned} I =& -\frac{1}{16\pi}\int_{M^n}(R-2\Lambda+L_m)\mathrm{d}^4x \\ &+\frac{1}{8\pi}\oint_{\partial M^n}(K-K_0)\sqrt{\gamma}\mathrm{d}^3x, \end{aligned} \tag{2}$$

where M^n is the spacetime manifold with metric $g_{\mu\nu}$, R is the scalar curvature of M^n, K is the trace of extrinsic curvature of the boundary $\partial M^n(r{=}\text{const.})$, K_0 is obtained by substituting into K the flat spacetime metric, γ is the determinant of the induced metric on the boundary, Λ is the cosmological constant, and L_m is the Lagrangian of the matter content. In Ref.[15], Wu has proven that the main contribution to the partition function for the microcanonical ensemble comes from the constrained instanton. Then, by using the same method as in Ref.[12], we can construct a compact constrained instanton in the Kerr-de Sitter manifold by applying a section between the cosmological outermost horizon (r_l) and the outermost event horizon (r_k). We use M_l^n to denote the small neighbourhood of horizon r_l with a boundary ($r{=}$const) and use $M^{n\prime}$ to denote M^n minus M_l^n and M_k^n. The total Euclidean action of M^n is the sum of three submanifolds

$$I = I_l + I_k + I_{M'}, \tag{3}$$

where I_l, I_k and $I_{M'}$ are the action for M_l^n, M_k^n and $M^{n\prime}$, respectively. Because the Euclidean manifold of $M^{n\prime}$ is regular for any temperature, the action of $M^{n\prime}$ can be written in the canonical form. For the vacuum gravity field, the action of $M^{n\prime}$ is given by[9,16]

$$I_{M'} = \int_{M^{n\prime}}(\pi^{ij}h_{ij} - NH - N_iH^i)\mathrm{d}x\mathrm{d}\tau, \tag{4}$$

where H is the Hamiltonian constraint and H^i is the momentum constraint, N is the lapse function and N_i is the shift vector, h_{ij} and π^{ij} are three-metric and the conjugate momenta, respectively.

For the Kerr-de Sitter black hole without matter configurations, the constraint vanishes. We can prove that the integral $I_{M'}$ over $M^{n\prime}$ is zero. Then both the actions I_l and I_k can be written as

$$\begin{aligned} I_i =& -\frac{1}{16\pi}\int_{M_i^n}\sqrt{g}(R-2\Lambda)\mathrm{d}^4x \\ &+\frac{1}{8\pi}\int_{\partial M_i^n}\sqrt{\gamma}(K-K_0)\mathrm{d}^3x \quad (i=l,k). \end{aligned} \tag{5}$$

The Euclidean metric of the Kerr-de Sitter black hole can be expressed as

$$\begin{aligned} \mathrm{d}s^2 =& \frac{1}{\rho^2\Sigma^2}(\Delta_r - \Delta_\theta a^2\sin^2\theta)\mathrm{d}\tau^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ &- \frac{\sin^2\theta}{\rho^2\Sigma^2}[\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta]\mathrm{d}^2\varphi \\ &+ \frac{2a}{\rho^2\Sigma^2}[\Delta_\theta(r^2+a^2) - \Delta_r]\sin^2\theta\mathrm{d}\tau\mathrm{d}\varphi, \end{aligned} \tag{6}$$

with

$$\begin{aligned} \rho^2 =& r^2 + a^2\cos^2\theta, \\ \Delta_r =& (r^2+a^2)\left(1-\frac{\lambda r^2}{3}\right) - 2Mr, \\ \Delta_\theta =& 1 + \frac{\lambda}{3}a^2\cos^2\theta, \\ \Sigma =& 1 + \frac{\lambda a^2}{3}. \end{aligned}$$

In order to calculate action I, we define the trace of extrinsic curvature K of the hypersurface

$$K \equiv \gamma^{\mu\nu}\nabla_\mu n_\nu, \tag{7}$$

where

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu. \tag{8}$$

Here n^μ is the unit normal vector on the spacelike hypersurface, and $\gamma_{\mu\nu}$ is the induced metric on it.

For the stationary metric, setting $n_\mu = (0, \sqrt{g_{rr}}, 0, 0,)$ as in Ref.[17], we find that

$$\begin{aligned} K =& \frac{1}{2}g^{rr}n_r[g^{\tau\tau}(g_{\tau\tau,r}) \\ &+ g^{\theta\theta}(g_{\theta\theta,r}) + g^{\varphi\varphi}(g_{\varphi\varphi,r}) + 2g^{\tau\varphi}(g_{\tau\varphi,r})]. \end{aligned} \tag{9}$$

Substituting Eq.(6) into Eq.(9), we obtain

$$K = \frac{r\sqrt{\Delta_r}}{\rho^3} + \frac{\Delta_{r,r}}{2\rho\sqrt{\Delta_r}}. \tag{10}$$

Inserting Eq.(10) into Eq.(5), we then find that the entropy is

$$S_i = \frac{\beta_i}{4\Sigma^2}\Delta_{r,r}(r_i), \tag{11}$$

where β_i is the inverse temperature on the horizon r_i.

3. Euler number of the Kerr-de Sitter black hole

In a closed Riemannian manifold M^n without boundary, the Gauss-Bonnet action is[18]

$$S_{\rm GB}=\int_{M^n}\Omega, \tag{12}$$

with

$$\Omega=\frac{(-1)^{n/2}}{2^n\pi^2(n/2)!}\varepsilon_{a_1\cdots a_{n/2}} \times R^{a_1a_2}\wedge R^{a_2a_3}\wedge\cdots\wedge R^{a_{n-1}a_n}, \tag{13}$$

where the curvature two-form $R^a{}_b$ is defined by the spin connection one-form $\omega^a{}_b$ as

$$R^a{}_b=\mathrm{d}\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b. \tag{14}$$

Ω can be expressed as the exterior derivative of a differential $(n-1)$-form in M^{2n-1} which can be formed by the unit tangent vectors of M^n

$$\Omega=-\mathrm{d}\Pi. \tag{15}$$

We can construct an n-dimensional submanifold V^n in M^{2n-1}, which is made of the image in M^{2n-1} of a continuous unit tangent vector field defined over M^n. Chern[18] has proven that the original integral of Ω over M^n can be performed over submanifold V^n. By using Stokes' theorem, we can obtain[19]

$$S_{\rm GB}^{\rm volume}=\int_{M^n}\Omega=\int_{V^n}\Omega=\int_{\partial V^n}\Pi. \tag{16}$$

The integral of Π over the boundary of V^n is equal to the Euler number χ in the manifold M^n. For a manifold with a boundary, this formula can be generalized as [19]

$$\begin{aligned}S_{\rm GB}=&S_{\rm GB}^{\rm volume}+S_{\rm GB}^{\rm boundary}\\ =&\int_{\partial V^n}\Pi-\int_{\partial M^n}\Pi.\end{aligned} \tag{17}$$

The boundaries of the manifolds V^n and M^n are represented by $\partial V^n=(r_h,r_{\rm out})$ and $\partial M^n=(r_{\rm in},r_{\rm out})$, respectively. The inner boundary of M^n usually vanishes since the horizon, after removal of the conical singularity, becomes a regular surface. Thus, Eq.(17) can be expressed as

$$S_{\rm GB}=\int_{r_h}\Pi. \tag{18}$$

For the Kerr-de Sitter submanifolds M_l^n and M_k^n, both the Gauss-Bonnet action $S_{{\rm GB}_l}$ on the cosmological horizon and $S_{{\rm GB}_k}$ on the event horizon can be written as

$$S_{{\rm GB}_i}=\int_{r_i}\Pi. \tag{19}$$

We now introduce the vierbein forms

$$\begin{aligned}e^0=&\frac{\sqrt{\Delta_r}}{\rho\Sigma}(\mathrm{d}t-a\sin^2\theta\mathrm{d}\varphi),\\ e^1=&\frac{\rho}{\sqrt{\Delta_r}}\mathrm{d}r,\\ e^2=&\frac{\rho}{\sqrt{\Delta_\theta}}\mathrm{d}\theta,\\ e^3=&\frac{\sqrt{\Delta_\theta}}{\rho\Sigma}[(r^2+a^2)\mathrm{d}\varphi-a\mathrm{d}t]\sin\theta.\end{aligned} \tag{20}$$

From Eq.(20), one can obtain the spin connection one-forms $\omega^\mu{}_\nu$ as

$$\begin{aligned}\omega^0{}_1=&\frac{\partial}{\partial r}\left(\frac{\sqrt{\Delta_r}}{\rho}\right)e^0+\frac{ra\sqrt{\Delta_\theta}}{\rho^3}\sin\theta e^3,\\ \omega^0{}_2=&-\frac{a^2\sqrt{\Delta_\theta}}{\rho^3}\sin\theta\cos\theta e^0-\frac{a\sqrt{\Delta_r}}{\rho^3}\cos\theta e^3,\\ \omega^0{}_3=&\frac{ra\sqrt{\Delta_\theta}}{\rho^3}\sin\theta e^1+\frac{a\sqrt{\Delta_r}}{\rho^3}\cos\theta e^2,\\ \omega^1{}_2=&-\frac{a^2\sqrt{\Delta_\theta}}{\rho^3}\sin\theta\cos\theta e^1-\frac{r\sqrt{\Delta_r}}{\rho^3}e^2,\\ \omega^1{}_3=&-\frac{\sqrt{\Delta_\theta}ar}{\rho^3}\sin\theta e^0-\frac{r\sqrt{\Delta_r}}{\rho^3}e^3,\\ \omega^2{}_3=&\frac{\sqrt{\Delta_r}a}{\rho^3}\cos\theta e^0-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sqrt{\Delta_\theta}\sin\theta}{\rho}\right)e^3.\end{aligned} \tag{21}$$

By virtue of Eqs.(19) and (21), we obtain the Gauss-Bonnet action

$$\begin{aligned}S_{{\rm GB}_i}=&-\frac{1}{4\pi^2}\int(\omega^{01}\wedge R^{23}+\omega^{02}\wedge\omega^{21}\wedge\omega^{23}\\ &+\omega^{03}\wedge\omega^{31}\wedge\omega^{23}+\omega^{02}\wedge\mathrm{d}\omega^{31})_{r_i}.\end{aligned} \tag{22}$$

Substituting Eqs.(21) and (14) into Eq.(22), we find

$$\begin{aligned}\chi_i=&-\frac{1}{4\pi^2\Sigma^2}\int_0^{\beta_i}\mathrm{d}\tau\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\frac{\Delta_{r,r}(r_i)}{2}\frac{\partial}{\partial\theta}\\ &\times\left[\frac{\Delta_\theta\cos\theta(r^2+a^2)-\frac{\lambda}{3}a^2\cos\theta\sin^2\theta\rho^2}{\rho^4}\right]\mathrm{d}\theta\\ =&\frac{\beta_i}{\pi\Sigma^2}\frac{\left(1+\frac{\lambda a^2}{3}\right)}{(r_i^2+a^2)}\frac{\Delta_{r,r}(r_i)}{2}.\end{aligned} \tag{23}$$

By employing Eq.(23) we know that the entropy (11) can be expressed as

$$S_i=\frac{A_i\chi_i}{8}. \tag{24}$$

Thus, the total entropy of the Kerr-de Sitter black hole is given by

$$S = \frac{1}{8}(A_l\chi_l + A_k\chi_k). \tag{25}$$

4. Summary and discussion

By using the Gibbons-Hawking path integral method, we find that the entropy of the Kerr-de Sitter black hole is one eighth of the sum of products of the Euler number of its horizons and their corresponding areas. It is shown that the origin of the black hole entropy is related to the topology of its instanton. The conclusion can easily be generalized to the Kerr-Newman-de Sitter black hole. It is interesting to note that the charge e of the Kerr-ewman black hole has no effect on the calculation of χ and K. The action I_A which is the result of electromagnetic field do not affect the entropy of a black hole. So the entropy (25) remains true in the Kerr-Newman-de Sitter black hole.

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