Vol 11 No 1, January 2002
1009-1963/2002/11(01)/0017-04

Chinese Physics

© 2002 Chin. Phys. Soc.
and IOP Publishing Ltd

Scaling of torus-doubling terminal points in a quasi-periodically forced map*

Fu Wu-Jiu(符五久), He Dai-Hai(何岱海)†, Shi Peng-Liang(史朋亮),
Kang Wei(康 炜), and Hu Gang(胡 岗)
Department of Physics, Beijing Normal University, Beijing 100875, *China*

(Received 3 June 2001; revised manuscript received 24 July 2001)

The torus-doubling bifurcations of a quasi-periodically forced two-dimensional map are investigated numerically. The scaling law on the terminal points of the torus-doubling bifurcation sequences is obtained by a simple method, based on hyper-stable period point and phase sensitivity exponent analyses.

Keywords: strange non-chaotic attractor, phase sensitive exponent
PACC: 0545

1. Introduction

The concept of the strange non-chaotic attractor (SNA) was first proposed by Grebogi *et al.*,[1] and since then the related topic has been extensively studied numerically[2−9] and experimentally.[10,11] An attractor is chaotic if its typical orbits have a positive Lyapunov exponent. We will use the definition for a strange attractor as given in Ref.[1]: "A strange attractor is an attractor which is not a finite set of points and is not piecewise differentiable. We say that it is piecewise differentiable if it is either a piecewise differentiable curve or surface, or a volume bounded by a piecewise differentiable close surface." An SNA is, therefore, an attractor which is geometrically strange, but for which the typical orbits have non-positive Lyapunov exponents. Recently, the terminal points of torus-doubling bifurcation curves, the so-called torus-doubling terminal (TDT) points, have been analysed based on the concept of scaling.[12] This critical point is of crucial importance for understanding the entire picture of the dynamics because the regimes of quasi-periodic motion, SNA and chaos meet at this point. The analysis of the TDT critical point links the problems of the quasi-periodic transition to chaos and SNA to the concept of scaling. The phase sensitive exponent (PSE)[3] characterizing the sensitivity of the SNA to the phase of the external force has been proposed to identify the SNA objects. In this paper, we present a new approach, based on the PSE, to obtain the scaling relation of the driven frequency and amplitude at the TDT points in a two-dimensional quasi-periodically driven map. At the same time, the low-frequency character of the map is revealed in double precision and high precision for the PSE.

2. PSE method and scaling results

We consider the following driven map

$$\begin{aligned} x_{n+1} &= a x_n(1 - x_n) + b\cos(\theta_n), \\ \theta_{n+1} &= \theta_n + 2\pi\frac{\sqrt{5}-1}{2}\omega, \end{aligned} \tag{1}$$

where b is the driving amplitude and $\dfrac{\sqrt{5}-1}{2}\omega$ is the driving frequency. For zero amplitude b, Eq.(1) becomes the usual logistic map and exhibits a cascade of period-doubling bifurcations. For finite but small b, these bifurcations are transformed into torus-doubling bifurcations, at which an attractor consisting of 2^n torus branches (called a 2^n torus) splits into an attractor consisting of 2^{n+1} torus branches. The smaller the amplitude b, the larger the number of the torus doublings seen; for any constant amplitude this number is finite. For the system with a certain driving frequency ω, it is not difficult to determine the hyper-stable point of a certain torus both analytically and

*Project supported by the National Natural Science Foundation of China (Grant No. 19835020), the National Basic Research Foundation for "Nonlinear Science" of China, and the Doctorate Foundation of the Chinese Ministry of Education.
†E-mail: hedaihai@china.com

numerically when the amplitude b is zero. When the amplitude b is small enough, the torus at the hyper-stable point can be maintained. As b increases, the given torus will be kept until it is destroyed due to the merging of the torus, leading to the SNA, further to chaos. The point corresponding to the end of torus doubling with finite n is called the terminal point of torus doubling. The transition interval from the merging of different torus branches to the appearance of the SNA or chaos is very short in the whole process. Consequently, the starting point of the SNA or chaos can be taken as the terminal point of the initial torus. We can plot the curve of the PSE versus the amplitude b, from which various terminal points can be determined. The scaling relation of torus numbers and the corresponding terminal points can be obtained accordingly. Figure 1 shows, for $\omega = 0.1$, 0.07, 0.05 and 0.03, the PSE curves of $2^{10}, 2^9, \ldots, 2^2$ torus versus amplitude b. The sudden jump of each PSE curve is obvious. The value of amplitude b at the sudden jump is the extreme value which the corresponding torus can suffer. Two methods have been introduced in Ref.[3] for computing the PSE. The first method is to compute in the tangent space, namely,

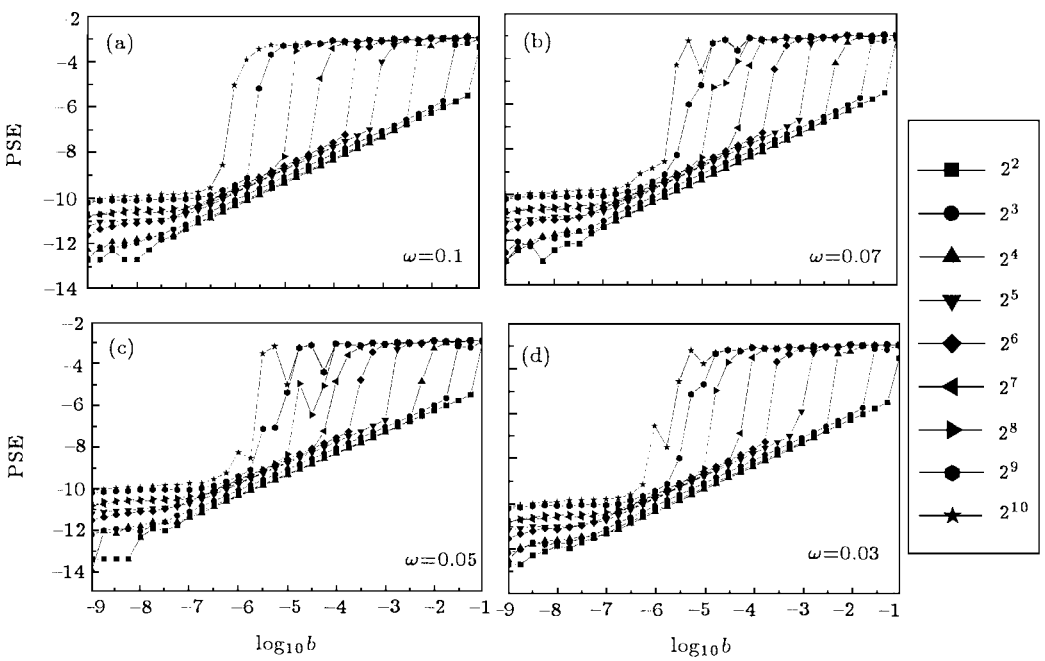


Fig.1. The PSE for (a) $\omega = 0.1$, (b) $\omega = 0.07$, (c) $\omega = 0.05$ and (d) $\omega = 0.03$ of $2^{10}, 2^9, ..., 2^2$ torus versus amplitude b.

$$\lambda_p = \left\langle \frac{1}{N} \log_2 \left| \frac{\partial x_N}{\partial \theta} \right| \right\rangle, \tag{2}$$

where

$$\frac{\partial x_N}{\partial \theta} = \sum_{k=1}^{N} -b \sin(\theta_n) \prod_{i=0}^{N-k-1} a(1 - 2x_{k+i}) + \prod_{i=0}^{N-1} a(1 - 2x_0) \frac{\partial x_0}{\partial \theta}, \tag{3}$$

and $< \cdots >$ denotes an average on the given trajectory. The second method is to calculate the PSE from the time series. In our numerical work, the results of the two methods are the same. Our main idea is to use the hyper-stable point of the torus to calculate each PSE curve for the corresponding torus. For a 2^n torus, there is an extreme value of amplitude b below which the 2^n torus can be observed. In our numerical experiment, the 2^n torus at the hyper-stable point with zero amplitude b is the most stable. From Fig.1, it can be seen that the PSE curves of different tori jump to a flat plateau one after another. Once reaching the plateau, the system behaviour is changed into the SNA or chaos. The sudden jump point is the TDT point. In Fig.2, the TDT points are plotted for $\omega = 0.1$, 0.07, 0.05 and 0.03, respectively. The power-law scaling relation between $\log_{10} b$ and $\log_2 n$ (where n is the number of branches of the given torus) is obvious. Figure 3 shows the slopes of the $\log_2 n$–$\log_{10} b$ curves versus driving frequency ω. It can be seen that the scaling relation is not sensitive to the change of the frequency ω. All the slopes are located in $[-0.8, -0.6]$.

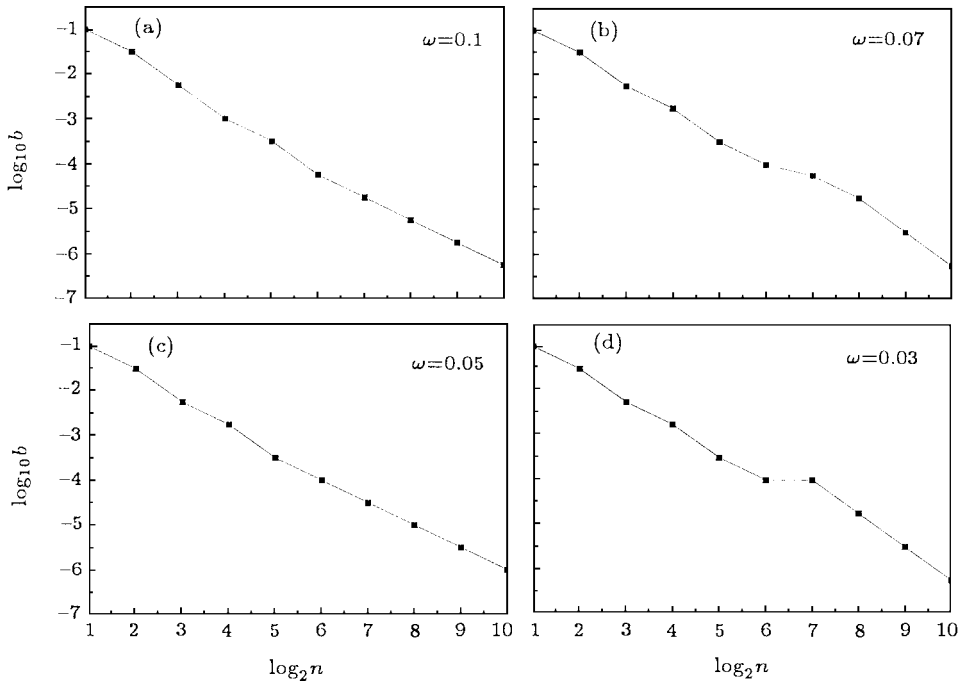


Fig.2. The TDT points in scaling line, i.e. the number of torus n versus amplitude b, for (a) $\omega = 0.1$, (b) $\omega = 0.07$, (c) $\omega = 0.05$ and (d) $\omega = 0.03$.

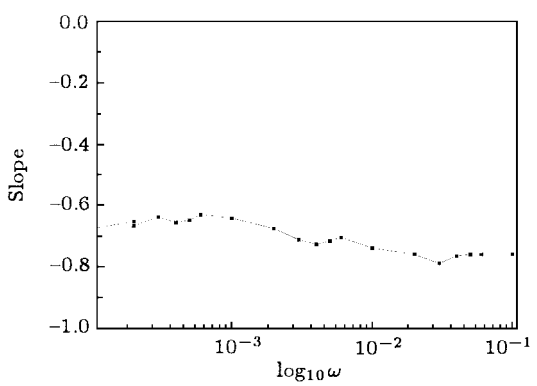


Fig.3. The slope of the scaling relation versus driving frequency ω.

3. The PSE behaviour in a low driving frequency regime

A model similar to Eq.(1) has been studied by Shuai.[13] It was concluded that different pseudo-periodic orbits can be obtained for different computation precisions. We find that it is essentially the low driving frequency which causes such differences (in Ref.[9] the authors used very low frequency). With small frequency ω, Eq.(1) can be regarded as driven adiabatically; with $a = 3.4$ and $b = 0.1$, it is in a range of two torus states and some segments can manifest local positive Lyapunov exponents. In Fig.4(a), we present the PSE versus frequency ω in double precision (i.e. 10^{-16} precision) in comparison with that in high computing precision 10^{-50} (Fig.4(b)). The behaviour of the system dynamics can be divided into three regimes: a regime where the torus is sensitively dependent on computing precision ($0 < \omega \lesssim 0.03$); the second regime where the torus is not dependent on computing precision ($0.03 \lesssim \omega \lesssim 0.08$); and the third regime of the SNA and chaos ($\omega \gtrsim 0.08$). In the first regime, the PSE calculated in double precision is larger than that in high precision. When the driving frequency $\omega < 0.03$ with $a = 3.4$ and $b = 0.1$, the behaviour of the system is seriously affected by the computing precision. The reason for this peculiar behaviour is discussed in Ref.[14].

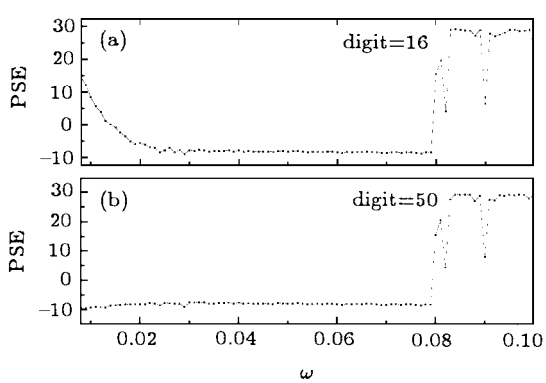


Fig.4. The PSE versus driving frequency ω with $a = 3.4$ and $b = 0.1$ (a) in double precision, and (b) in high computing precision 10^{-50}.

4. Conclusion

In this paper the scaling law of torus-doubling processes in a quasi-periodically driven system is obtained conveniently by using the PSE method. The behaviour of the system versus the changing of the frequency can be divided into regimes of tori sensitively dependent on computing precision, tori not dependent on computing precision, and the SNA and chaos. This classification is novel.

References

[1] Grebogi C, Ott E, Pelikan S and Yorke J A 1984 *Physica* D **13** 261
[2] Ding M, Grebogi C and Ott E 1989 *Phys. Rev.* A **39** 2593
[3] Pikovsky A and Feudel U 1995 *Chaos* **5** 253
[4] Lai Y C 1996 *Phys. Rev.* E **53** 57
[5] Nishikawa T and Kaneko K 1996 *Phys. Rev.* E **54** 6114
[6] Luo X C 2000 *Chin. Phys.* **9** 333
[7] Zhu S Q and Lu X 2000 *Chin. Phys.* **9** 337
[8] Zheng Z G, Hu G, Zhou C S and Hu B B 2000 *Acta Phys. Sin.* **49** 2320 (in Chinese)
[9] He D H, Xu J X and Chen Y H 1999 *Acta Phys. Sin.* **48** 1611 (in Chinese)
[10] Ditto W L *et al* 1990 *Phys. Rev. Lett.* **65** 533
[11] Zhou T, Moss F and Bulsara A 1992 *Phys. Rev.* A **45** 5394
[12] Kuznetsov S, Feudel U and Pikovsky A 1998 *Phys. Rev.* E **57** 1585
[13] Shuai J W and Wong K W 1998 *Phys. Rev.* E **57** 5332
[14] Shi P L, He D H, Kang W, Fu W J and Hu G 2001 *Phys. Rev.* E **63** 046310