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Lie symmetries and conserved quantities of non-holonomic mechanical systems with unilateral Vacco constraints

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In this paper, we study the Lie symmetries and the conserved quantities of non-holonomic mechanical systems with unilateral Vacco constraints. We give the conditions and the form of conserved quantities due to the Lie symmetries of the systems, and present the inverse problem of the above problem, i.e. finding the corresponding Lie symmetry transformation according to a given integral of the system. Finally, we give an example to illustrate the application of the results.

Keywords: non-holonomic mechanical systems with unilateral Vacco constraints, Lie symmetry, conserved quantity

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1. Introduction

In the past twenty years, great progress has been made in the study of Lie symmetries and conserved quantities of constrained mechanical systems.[1−17] Until now, however, all the studies have been limited to non-holonomic mechanical systems with the Chetaev-type ideal constraints. In this paper, we study further the symmetries of non-holonomic mechanical systems with the unilateral Vacco-type ideal constraints. Also, two problems of Lie symmetries for the systems are put forward and solved, i.e. finding the conserved quantity from a Lie symmetric transformation of the system and finding the corresponding Lie symmetry from an integral of the system.

2.Lie symmetries and conserved quantities of the systems

We suppose that the configuration of a mechanical system is determined by n generalized coordinates q_s (s=1, 2,$\cdots$, n). The motion of the system is subjected to the unilateral non-holonomic constraints of the Vacco-type

$$\varphi_\beta(t,q,\dot{q}) \geq 0 \qquad (\beta = 1,2,\cdots,g). \tag{1}$$

Then, the differential equations of motion of the system can be expressed in the form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s}\\ =&Q''_s - \sum_{\beta=1}^{g}\lambda_\beta\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_s} - \frac{\partial \varphi_\beta}{\partial q_s}\right) - \sum_{\beta=1}^{g}\dot{\lambda}_\beta\frac{\partial \varphi_\beta}{\partial \dot{q}_s},\\ &(s=1,2,\cdots,n),\\ &\lambda_\beta \geq 0, \quad \lambda_\beta\varphi_\beta = 0 \quad (\beta=1,2,\cdots,g),\end{aligned} \tag{2}$$

where L is the Lagrangian function of the system, Q''_s is the generalized non-potential force, and λ_β are the constraint multipliers. It is noticeable that the connecting conditions along the constraint hypersurface must be considered in order to make Eqs.(2) closed. For example, one can assume that the constraint hypersurface is absolutely smooth and the collision is completely elastic.

We consider the infinitesimal transformations of the one-parameter Lie group of transformations

$$\begin{cases} t^* = t + \Delta t, \\ q_s^* = q_s + \Delta q_s \end{cases} \qquad (s=1,2,\cdots,n) \tag{3}$$

or

$$\begin{cases} t^* = t + \varepsilon\xi_0(t,q,\dot{q}), \\ q_s^* = q_s + \varepsilon\xi_s(t,q,\dot{q}) \end{cases} \qquad (s=1,2,\cdots,n), \tag{4}$$

where ε is an infinitesimal parameter, and ξ_0 and ξ_s are generators of the infinitesimal transformations. We introduce the infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{s=1}^{n} \xi_s \frac{\partial}{\partial q_s}. \tag{5}$$

The first extended infinitesimal generator is

$$X^{(1)} = X^{(0)} + \sum_{s=1}^{n} (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}, \tag{6}$$

and the second is

$$X^{(2)} = X^{(1)} + \sum_{s=1}^{n} (\ddot{\xi}_s - 2\ddot{q}_s \dot{\xi}_0 - \dot{q}_s \ddot{\xi}_0) \frac{\partial}{\partial \ddot{q}_s}. \tag{7}$$

Then, we study an invariance of Eqs.(2) under the infinitesimal transformations (4).

If the system is constrained, i.e. constraints (1) take the sign of equality, we have

$$\varphi_\beta(t, q, \dot{q}) = 0 \qquad (\beta = 1, 2, \cdots, g). \tag{8}$$

λ_β can be expressed as explicit functions of t before integrating the differential equations of motion. Thus, Eqs.(2) can be written as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} \\ =& Q_s'' - \sum_{\beta=1}^{g} \lambda_\beta \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \varphi_\beta}{\partial \dot{q}_s} - \frac{\partial \varphi_\beta}{\partial q_s} \right) - \sum_{\beta=1}^{g} \dot{\lambda}_\beta \frac{\partial \varphi_\beta}{\partial \dot{q}_s} \\ & (s = 1, 2, \cdots, n). \end{aligned} \tag{9}$$

By constructing the generalized Lagrangian function of the system[18]

$$\mathcal{L} = L + \sum_{\beta=1}^{g} \lambda_\beta \varphi_\beta, \tag{10}$$

then Eqs.(9) change into

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} - \frac{\partial \mathcal{L}}{\partial q_s} = Q_s'' \qquad (s = 1, 2, \cdots, n). \tag{11}$$

Expanding Eqs.(11), we can seek all the generalized accelerations as

$$\ddot{q}_s = A_s(t, q, \dot{q}) \qquad (s = 1, 2, \cdots, n). \tag{12}$$

The necessary and sufficient conditions under which Eqs.(12) remain invariant are

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 A_s = X^{(1)}(A_s), \quad \text{for} \quad \varphi_\beta = 0. \tag{13}$$

Since the system is in the constraints (8), the restriction conditions of the constraints under the vector field (6) are

$$X^{(1)}[\varphi_\beta(t, q, \dot{q})] = 0, \quad \text{for} \quad \varphi_\beta = 0. \tag{14}$$

If the system is separated from the constraints, that is, the sign of inequality in constraints (1) hold strictly, we have

$$\varphi_\beta(t, q, \dot{q}) > 0 \qquad (\beta = 1, 2, \cdots, g). \tag{15}$$

Then Eqs.(2) become

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s'' \qquad (s = 1, 2, \cdots, n). \tag{16}$$

Expanding Eqs.(16), we obtain

$$\ddot{q}_s = B_s(t, q, \dot{q}) \qquad s = 1, 2, \cdots, n). \tag{17}$$

The necessary and sufficient conditions under which Eqs.(17) remain invariant under the infinitesimal transformations (4) are

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 B_s = X^{(1)}(B_s), \quad \text{for} \quad \varphi_\beta > 0. \tag{18}$$

The result discussed above can be summed up in the following proposition 1.

Proposition 1. For the Vacco systems (1) and (2) with unilateral constraints, if the infinitesimal transformations (4) satisfy the determining Eqs.(13) and (18) and the restriction conditions (14), then they are Lie symmetrical transformations. Hence, we have the following proposition.

Proposition 2. For the Lie symmetrical transformations (4) of the Vacco system with unilateral constraints, if there is a gauge function G satisfying the structure equations

$$\begin{aligned} &\mathcal{L}\dot{\xi}_0 + X^{(1)}(\mathcal{L}) + \sum_{s=1}^{n} Q_s''(\xi_s - \dot{q}_s \xi_0) + \dot{G} = 0, \\ &\text{for} \quad \varphi_\beta = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} &L\dot{\xi}_0 + X^{(1)}(L) + \sum_{s=1}^{n} Q_s''(\xi_s - \dot{q}_s \xi_0) + \dot{G} = 0, \\ &\text{for} \quad \varphi_\beta > 0, \end{aligned} \tag{20}$$

then the system has the conserved quantity

$$I = \mathcal{L}\xi_0 + \sum_{s=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \xi_0) + G = \text{const.} \tag{21}$$

Proof: We have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\dot{\mathcal{L}}\xi_0+\mathcal{L}\dot{\xi}_0+\sum_{s=1}^{n}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0-\ddot{q}_s\xi_0)\frac{\partial\mathcal{L}}{\partial\dot{q}_s}\\&+\sum_{s=1}^{n}(\xi_s-\dot{q}_s\xi_0)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_s}+\dot{G}\\=&\mathcal{L}\dot{\xi}_0+X^{(1)}(\mathcal{L})\\&+\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_s}-\frac{\partial\mathcal{L}}{\partial q_s}\right)(\xi_s-\dot{q}_s\xi_0)+\dot{G}.\end{aligned}$$

If the system is in the constraint, then by utilizing Eqs.(11) and the structure equation (19), we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\mathcal{L}\dot{\xi}_0+X^{(1)}(\mathcal{L})+\sum_{s=1}^{n}Q''_s(\xi_s-\dot{q}_s\xi_0)+\dot{G}\\&+\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_s}-\frac{\partial\mathcal{L}}{\partial q_s}-Q''_s\right)(\xi_s-\dot{q}_s\xi_0)=0.\end{aligned}$$

If the system is divorced from the constraint, then by utilizing Eqs.(16) and the structure equation (20), we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&L\dot{\xi}_0+X^{(1)}(L)+\sum_{s=1}^{n}Q''_s(\xi_s-\dot{q}_s\xi_0)+\dot{G}\\&+\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q''_s\right)(\xi_s-\dot{q}_s\xi_0)=0.\end{aligned}$$

Therefore, the system has the conserved quantity (21).

3. Inverse problem of Lie symmetries of the systems

For a Vacco system with unilateral constraints, the inverse problem of Lie symmetries is to discover the corresponding Lie symmetries according to the known conserved quantity.

We suppose that the system has a known integral

$$I=I(t,q,\dot{q})=\text{const.}\tag{22}$$

Differentiating it with respect to t, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\sum_{s=1}^{n}\frac{\partial I}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial I}{\partial\dot{q}_s}\ddot{q}_s=0.\tag{23}$$

Multiplying Eqs.(2) by

$$\overline{\xi}_s=\xi_s-\dot{q}_s\xi_0,\tag{24}$$

and taking the summation for s, we obtain

$$\sum_{s=1}^{n}\overline{\xi}_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_s}-\frac{\partial\mathcal{L}}{\partial q_s}-Q''_s\right)=0.\tag{25}$$

Adding the formulae (23) and (25), separating the terms containing $\ddot{q}_k$ and taking their coefficients as zeros, we obtain

$$-\frac{\partial I}{\partial\dot{q}_k}+\sum_{s=1}^{n}\frac{\partial^2\mathcal{L}}{\partial\dot{q}_s\partial\dot{q}_k}\overline{\xi}_s=0\qquad(k=1,2,\cdots,n).\tag{26}$$

Letting

$$\det\left|\frac{\partial^2\mathcal{L}}{\partial\dot{q}_s\partial\dot{q}_k}\right|\neq0,\tag{27}$$

from Eqs.(26), we can obtain

$$\overline{\xi}_s=\sum_{s=1}^{n}\tilde{\omega}_{sk}\frac{\partial I}{\partial\dot{q}_k}\qquad(s=1,2,\cdots,n),\tag{28}$$

where

$$\tilde{\omega}_{sk}\omega_{kr}=\delta_{sr},\quad\omega_{kr}=\frac{\partial^2\mathcal{L}}{\partial\dot{q}_k\partial\dot{q}_r}.\tag{29}$$

We let the integral (22) be equal to the conserved quantity (21), i.e.

$$\mathcal{L}\xi_0+\sum_{s=1}^{n}\frac{\partial\mathcal{L}}{\partial\dot{q}_s}\overline{\xi}_s+G=I.\tag{30}$$

Hence, the solution of the inverse problem of Lie symmetries for the Vacco system with unilateral constraints (1) and (2) may be summed up by the following proposition.

Proposition 3. If the integral (22) of the Vacco system with unilateral constraints (1) and (2) is given, then the infinitesimal group transformations determined by Eqs.(28) and (30) must be the Lie symmetrical transformations, once they satisfy the determining equations (13) and (18) and the restriction condition (14).

4. Example

We suppose that the Lagrangian function of a material point with unit mass is

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2),\tag{31}$$

and the motion of the material point is subjected to the unilateral Vacco constraint

$$\varphi=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-c\geq0.\tag{32}$$

We study the Lie symmetries and the conserved quantities of this system.

Firstly, we study the direct problem. The differential equations of motion of the system can be expressed as follows:

$$\begin{aligned}&(1+\lambda)\ddot{q}_1+\dot{\lambda}\dot{q}_1=0,\\&(1+\lambda)\ddot{q}_2+\dot{\lambda}\dot{q}_2=0,\\&(1+\lambda)\ddot{q}_2+\dot{\lambda}\dot{q}_3=0,\\&\text{for }\lambda\geq0,\quad\lambda\varphi=0.\end{aligned}\tag{33}$$

If the system is in the constraint, i.e.

$$\varphi = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - c = 0, \tag{34}$$

by differentiating Eq.(34) with respect to t, we have

$$\dot{q}_1\ddot{q}_1 + \dot{q}_2\ddot{q}_2 + \dot{q}_3\ddot{q}_3 = 0. \tag{35}$$

Substituting Eqs.(33) into Eq.(35), we obtain

$$\lambda = \lambda_0(\text{const}). \tag{36}$$

Then Eqs.(33) become

$$(1+\lambda_0)\ddot{q}_1 = 0, \quad (1+\lambda_0)\ddot{q}_2 = 0, \quad (1+\lambda_0)\ddot{q}_3 = 0. \tag{37}$$

If the system is separated from the constraint, i.e.

$$\varphi = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - c > 0, \tag{38}$$

we obtain λ=0, and Eqs.(33) become

$$\ddot{q}_1 = 0, \quad \ddot{q}_2 = 0, \quad \ddot{q}_3 = 0. \tag{39}$$

Secondly, we study the Lie symmetries and the conserved quantities of the system. The determining equations (13) and (18) give

$$\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 = 0, \quad \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 = 0, \quad \ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 = 0,$$
$$\text{for} \quad \varphi = 0, \tag{40}$$

$$\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 = 0, \quad \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 = 0, \quad \ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 = 0,$$
$$\text{for} \quad \varphi > 0, \tag{41}$$

and the restriction conditions (14) give

$$(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\dot{q}_1 + (\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)\dot{q}_2 + (\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)\dot{q}_3 = 0,$$
$$\text{for} \quad \varphi = 0. \tag{42}$$

We let the infinitesimal transformation be

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_1, \quad \xi_2 = \dot{q}_2, \quad \xi_3 = \dot{q}_3. \tag{43}$$

Obviously, formulae (40), (41) and (42) can all be satisfied, therefore, formula (43) corresponds to the Lie symmetrical transformation. Substituting Eq.(43) into the structure equations (19) and (20), we obtain the gauge function

$$G = -\frac{1+\lambda_0}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2). \tag{44}$$

Substituting Eqs.(43) and (44) into Eq.(21), we obtain the conserved quantity of the system

$$I = \frac{1+\lambda_0}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) = \text{const}. \tag{45}$$

Finally, we study the inverse problem of the Lie symmetries. Supposing that the integral (45) is given, then using formulae (28) and (30) we obtain

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_1, \quad \xi_2 = \dot{q}_2, \quad \xi_3 = \dot{q}_3. \tag{46}$$

The generators ξ_0, ξ_1, ξ_2 and ξ_3 correspond to Noether's symmetrical transformation of the system (31) and (32). It is easy to verify that for the gauge function G given by Eq.(44), the generators given by Eq.(46) satisfy the determining equations (13) and (18) and the restriction condition (14). Therefore, the generators (46) correspond to the Lie symmetrical transformation of the unilateral constraint Vacco system under study.

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