Vol 10 No 9, September 2001
1009-1963/2001/10(09)/0787-03

Chinese Physics

© 2001 Chin. Phys. Soc.
and IOP Publishing Ltd

STATIC SPHERICALLY SYMMETRIC SOLUTION OF EINSTEIN GRAVITY COUPLED TO ELECTRO-MAGNETIC AND SCALAR FIELDS*

CHEN GUANG(陈 光)

College of Engineering, Shantou University, Shantou 515063 *China;*

East China Institute for Theoretical Physics, Shanghai 200237, *China*

(Received 6 February 2001; revised manuscript received 30 May 2001)

The static spherically symmetric solution of Einstein gravity coupled to electromagnetic and scalar fields is obtained under the consideration of the self-gravitational interaction of the electromagnetic and scalar fields, which is singularity-free and stable.

Keywords: gravitation, electromagnetic field, scalar field, self-gravitational interaction
PACC: 0420

The static spherically symmetric solutions of Einstein gravity coupled to electromagnetic or scalar fields have been discussed by many researchers.[1−11] However, most solutions are obtained without consideration of the self-gravitational interaction of electromagnetic or scalar fields, and they are singular and of infinite energy. In this paper, we discuss Einstein gravity coupled to electromagnetic and scalar fields and derive the static spherically symmetric solution under the consideration of the self-gravitational interaction of electromagnetic and scalar fields. The configuration of the solution is determined by electric charges and proves to be singularity-free and of minimal energy. As a stable gravitation bound state, it can be regarded as a special non-topological soliton[12,13] with minimal coupling and no self-interaction potential.

The system of Einstein gravity coupled to electromagnetic and scalar fields is

$$F^{\mu\nu}{}_{;\nu}=0, \quad F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}, \tag{1}$$

$$\phi^{,\mu}{}_{;\mu}=0, \tag{2}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-8\pi G T_{\mu\nu}, \tag{3}$$

$$T_{\mu\nu}=T_{e\mu\nu}+T_{s\mu\nu},$$

$$T_{e\mu\nu}=F_{\mu\alpha}F^{\alpha}{}_{\nu}+\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

$$T_{s\mu\nu}=\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}.$$

We consider a coherent configuration of A_μ, ϕ and $g_{\mu\nu}$. The lowest-energy state will have no electric currents, and therefore no magnetic field. Furthermore, the lowest-energy configuration will be spherically symmetric and stationary.[14] For this configuration one can choose a gauge such that $A_0=\varphi$, A_i=0 (i=1,2,3) and the spacetime metric is

$$\begin{aligned}\mathrm{d}s^2=&g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\\ =&B\mathrm{d}t^2-A\mathrm{d}r^2-r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\Psi^2).\end{aligned} \tag{4}$$

Then Eqs.(1)–(3) will be reduced to

$$\varphi''+\varphi'\left[\frac{2}{r}-\frac{1}{2}\left(\frac{B'}{B}+\frac{A'}{A}\right)\right]=0, \tag{5}$$

$$\phi''+\phi'\left[\frac{2}{r}+\frac{1}{2}\left(\frac{B'}{B}-\frac{A'}{A}\right)\right]=0, \tag{6}$$

$$\begin{aligned}R_{tt}=&-\frac{B''}{2A}+\frac{B'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{B'}{rA}=-4\pi G\frac{\varphi'^2}{A},\\ R_{rr}=&\frac{B''}{2B}-\frac{B'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{A'}{rA}\\ =&-8\pi G\left(\phi'^2-\frac{\varphi'^2}{2B}\right),\\ R_{\theta\theta}=&-1+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)+\frac{1}{A}=-4\pi G r^2\frac{\varphi'^2}{AB},\\ R_{\Psi\Psi}=&R_{\theta\theta}\sin^2\theta,\end{aligned} \tag{7}$$

where the primes denote the derivative with respect to r.

*Project supported by the National Natural Science Foundation of China (Grant No. 19875016).

The formal solutions of Eqs.(5)–(7)can be written as

$$\varphi' = \frac{Q}{4\pi r^2}\exp\left(- \int_r^\infty 4\pi G l\phi'^2 \mathrm{d}l\right), \tag{8}$$

$$\phi' = \frac{C}{4\pi r^2}\exp\left(\int_r^\infty \frac{1}{l[1-\Delta(l)]}\left[\Delta(l) - \frac{GQ^2}{4\pi l^2}\right]\mathrm{d}l\right), \tag{9}$$

and

$$\begin{aligned} A(r) &= [1-\Delta(r)]^{-1}, \\ B(r) &= [1-\Delta(r)]\exp\left\{ - \int_r^\infty 8\pi G l\phi'^2 \mathrm{d}l\right\}, \\ \Delta(r) &= \frac{1}{r}\int_0^r 4\pi G\frac{l^2}{A}\left(\phi'^2 + \frac{\varphi'^2}{B}\right)\mathrm{d}l, \end{aligned} \tag{10}$$

where Q and C are integral constants. Equations (8)–(10) show that a self-gravitational interaction exists in the system of Einstein gravity coupled to electromagnetic and scalar fields. Moreover, it can be found that φ'^2 is a non-monotonic function with two minima at $r \to r_0$ and $r \to \infty$, as well as a maximum at $r = r_1$; while ϕ'^2 is a monotonic function with a maximum at $r \to r_0$ and a minimum at $r \to \infty$. From

$$\varphi'' = 0, \qquad \text{for} \quad r \to r_0 \quad \text{and} \quad r = r_1,$$

and

$$\phi'' = 0, \qquad \text{for} \quad r \to r_0,$$

we can deduce

$$r_0^2 = \frac{GQ^2}{8\pi}, \qquad \Delta(r_0) = 0, \tag{11}$$

$$C^2 = Q^2\exp\left(\int_{r_0}^\infty \frac{2\mathrm{d}l}{l[1-\Delta(l)]}\left[\frac{GQ^2}{4\pi l^2} - \Delta(l)\right]\right), \tag{12}$$

and

$$r_1^2 = \frac{GQ^2}{8\pi}\exp\left(\int_{r_9}^{r_1} \frac{2\mathrm{d}l}{l[1-\Delta(l)]}\left[\frac{GQ^2}{4\pi l^2} - \Delta(l)\right]\right). \tag{13}$$

Also, from Eqs.(10) and (11), one can deduce that, when $r < r_0$, $\varphi' = \phi' = 0$, i.e. $\varphi = \phi$=constant. Hence $r = r_0$ is a discontinuous point of φ' and ϕ'. This means that, on the sphere $r = r_0$, a distribution of charges exists, which constitutes the source of the static electric field. Therefore, Eqs.(8) and (9) are only the external solutions of Eqs.(5) and (6). As for the internal solutions, we can let

$$\varphi = \varphi_0 = \frac{Z}{4\pi r_0}, \qquad \text{for} \quad r < r_0, \tag{14}$$

$$\phi = \phi_0 = \frac{\Sigma}{4\pi r_0}, \qquad \text{for} \quad r < r_0. \tag{15}$$

Correspondingly, Eq.(10) becomes

$$\begin{aligned} A(r) &= 1, \\ B(r) &= B(r_0) = \exp\left\{ - \int_{r_0}^\infty 8\pi G l\phi'^2 \mathrm{d}l\right\}, \\ \Delta(r) &= 0, \qquad \text{for} \quad r < r_0, \end{aligned} \tag{16}$$

and thereby the spacetime of $r < r_0$ is flat. Integrating Eqs.(8) and (9), we derive

$$\begin{aligned} \varphi = &- \int_r^\infty \varphi' \mathrm{d}\tau = - \int_r^\infty \frac{Q}{4\pi\tau^2} \\ &\cdot \exp\left(- \int_r^\infty 4\pi G l\phi'^2 \mathrm{d}l\right)\mathrm{d}\tau, \quad \text{for} \quad r > r_0, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \phi = &- \int_r^\infty \phi' \mathrm{d}\tau = - \int_r^\infty \frac{C}{4\pi\tau^2} \\ &\cdot \exp\left(\int_r^\infty \frac{1}{l[1-\Delta(l)]}\left[\Delta(l) - \frac{GQ^2}{4\pi l^2}\right]\mathrm{d}l\right)\mathrm{d}\tau, \\ &\text{for} \quad r > r_0. \end{aligned} \tag{18}$$

Comparing Eqs.(14) and (15) with Eqs.(17) and (18), when $r \to r_0$, we obtain

$$Z = -r_0 Q\int_{r_0}^\infty \frac{\mathrm{d}r}{r^2}\exp\left(- \int_r^\infty 4\pi G l\phi'^2 \mathrm{d}l\right), \tag{19}$$

and

$$\begin{aligned} \Sigma = &- r_0 C\int_{r_0}^\infty \frac{\mathrm{d}r}{r^2} \\ &\cdot \exp\left(\int_r^\infty \frac{1}{l[1-\Delta(l)]}\left[\Delta(l) - \frac{GQ^2}{4\pi l^2}\right]\mathrm{d}l\right). \end{aligned} \tag{20}$$

We know that the electric charge Z is a conservation quantity, which determines Q, r_0, r_1, C and Σ through Eqs.(11)–(13), as well as Eqs.(19) and (20).

Furthermore from

$$\Delta'(r)\big|_{r=r_2} = -\frac{\Delta(r)}{r} + \frac{4\pi G r}{A}\left(\phi'^2 + \frac{\varphi'^2}{B}\right)\Big|_{r=r_2} = 0,$$

we can deduce the maximum of $\Delta(r)$

$$\Delta_{\max} = \frac{4\pi G r_2^2\left(\phi'^2(r_2) + \dfrac{\varphi'^2(r_2)}{B(r_2)}\right)}{1 + 4\pi G r_2^2\left(\phi'^2(r_2) + \dfrac{\varphi'^2(r_2)}{B(r_2)}\right)}, \tag{21}$$

and therefore,

$$0 < \Delta_{\max} < 1. \tag{22}$$

Then by Eqs.(4) and (10), this means that the spacetime is singularity-free.

Finally, the total energy of the electromagnetic and scalar fields is

$$E = \int T_0^0 \sqrt{-g}\mathrm{d}^3x = \int \frac{1}{2A}\left(\phi'^2 + \frac{1}{B}\varphi'^2\right)\sqrt{-g}\mathrm{d}^3x. \qquad (23)$$

By varying E with respect to φ and ϕ, we have

$$\delta E = 0, \qquad \delta^2 E \geq 0. \qquad (24)$$

This means that the solution obtained above is of minimal energy and stable.

References

[1] Graves J C and Brill D R 1960 *Phys. Rev.* **120** 1507
[2] Bergmann O and Leipnik L 1957 *Phys. Rev.* **107** 1157
[3] Buchdahl H A 1959 *Phys. Rev.* **115** 1325
[4] Yilimaz H 1958 *Phys. Rev.* **111** 1417
[5] Wyman M 1981 *Phys. Rev.* D **24** 839
[6] Agnese A G and Camera M L 1985 *Phys. Rev.* D **31** 1280
[7] Xanthopoulos B C 1989 *Phys. Rev.* D **40** 2564
[8] Zhang J Y 1997 *Acta Phys. Sin.* **46** 2294 (in Chinese)
[9] Zhang J Y 1999 *Acta Phys. Sin* **48** 2158 (in Chinese)
[10] Chen G 1999 *Acta Phys. Sin.* **48** 992 (in Chinese)
[11] Li X Z 2000 *Acta Phys. Sin.* **49** 1035 (in Chinese)
[12] Li X Z, Ni Z X and Zhang J Z 1994 *J. Phys. A: Math. Gen.* **27** 507
[13] Li X Z and Zhang J Z 1992 *Phys. Rev.* D **45** 2888
[14] Lee K, Stain Schabes J A, Watkin R and Widrow L M 1989 *Phys. Rev.* D **39** 1665