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COHERENT INFORMATION ON THERMAL RADIATION NOISE CHANNEL: AN APPROACH OF INTEGRAL WITHIN ORDERED PRODUCT OF OPERATORS*

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An analytical expression is given to the coherent information of the thermal radiation signal transmitted over the thermal radiation noise channel, one of the most essential quantum Gaussian channels. Focusing on the single normal mode of the thermal radiation signal and noise, we resolve the entangled state density operator, which characterizes quantum information transmission, into a direct product of two parts, with each part being a thermal radiation density operator. The calculation is aided by the technique known as "integral within ordered product of operators".

Keywords: coherent information, exchange information, thermal radiation channel, integral within ordered product of operators

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I. INTRODUCTION

One of the most fundamental results of the communication theory is the noisy channel coding theorem. Half a century ago, Shannon proved the classical version of the theorem when he first founded a classical information theory. Recently, accompanying the fast development of the general theory of quantum information,[1,2] the quantum channel (noiseless, noisy) coding theorem has been proven.[3,4] Because the quantum channel can be used to transmit classical information as well as quantum information, the quantum version of the theorem was divided into two parts: one for classical information, and the other for quantum information. In the theorem, the most important quantity is the channel capacity, which limits the ultimate transmission rate of the channel. As is known, the capacity of the additive white Gaussian channel with an average power-limited input plays an important role in classical information theory. Similarly, the capacity of the quantum Gaussian channel is very important to the development of quantum information theory. For the classical capacity of the quantum Gaussian channel, an explicit calculation has recently been given.[5] While the problem of how to calculate the quantum capacity of the quantum Gaussian channel remains unsolved. Defined via the Gaussian form of its quantum characteristic function, quantum Gaussian noise not only includes thermal radiation noise as a special case, but also includes a wide class of states such as coherent states and also squeezed states.[5] However, thermal radiation may consist of many normal modes. To extricate the main idea of the quantum information transmission process, we shall restrict ourselves to the case of the single normal mode of the thermal radiation signal or noise, since the coherent information can be used to represent the quantum capacity of the channel without considering encoding,[6−9] and the thermal radiation noise channel is one of the most essential quantum Gaussian channels. The calculation of the coherent information of a thermal radiation signal transmitted over the thermal radiation noise channel should be considered as the first necessary step towards the goal of obtaining the capacity of a general quantum Gaussian channel.

II. QUANTUM GAUSSIAN CHANNEL

A density operator is termed Gaussian when its quantum characteristic function has some definite form.[5] Thermal radiation noise is Gaussian. Its Gaussian character is clearly seen when any of its modes is expressed in Glauber's P representation.[5]

$$\rho(0) = (\pi N)^{-1} \int \exp(-|\alpha|^2/N)|a\rangle\langle\alpha|\mathrm{d}^2\alpha. \qquad (1)$$

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Here N is the average number of photons in the mode, $|\alpha\rangle$ is the coherent state and is the eigenstate of the annihilation operator a. The mode we have considered is described by a, a^+ and the canonical commutation relation $[a, a^+]$=1. Now we ask what will the final state be when an initial quantum state is transmitted over a channel with quantum noise. The general description of this question is the superoperator,[6] a mapping of the initial operator ρ to the final operator ρ'. The superoperator ε has a normalized operator sum representation, that is

$$\rho' = \varepsilon(\rho) = \sum_\mu A_\mu \rho A_\mu^+. \tag{2}$$

The operators A_μ satisfy a normalization condition

$$\sum_\mu A_\mu^+ A_\mu = \text{identity operator}. \tag{3}$$

The operator sum representation can further be simplified to a channel matrix representation. We can do this by expanding A_μ with displacement operators $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$, that is

$$A_\mu = \pi^{-1} \int A_\mu(\alpha) D(\alpha) \mathrm{d}^2\alpha. \tag{4}$$

Let $B'(\alpha, \beta) = \sum_\mu A_\mu(\alpha) A_\mu(\beta)$, thus we have

$$\rho' = \pi^{-2} \iint B'(\alpha, \beta) D(\alpha) \rho D^+(\beta) \mathrm{d}^2\alpha \mathrm{d}^2\beta. \tag{5}$$

We consider how to generate a quantum noise of the type of Eq.(1). From the viewpoint of transmission, the noise in classical theory is equal to the signal appearing at the receiving end when the sender sends nothing. This is also true for quantum theory. The difference is that in quantum theory "nothing" means a vacuum state, and we set

$$B'(\alpha, \beta) = \frac{1}{4} \exp\left(-\frac{N}{4} |\alpha - \beta|^2 \right) \tag{6}$$

for the thermal noise channel. At least, when the initial density operator is of the thermal noise type, which contains a vacuum state as a special case of zero average photon number, it can be proven, by using the integral technique within ordered product (IWOP),[10] that an equivalent expression of Eq.(5) exists for the thermal noise channel.

$$\rho' = \pi^{-1} \int B(\alpha) D(\alpha) \rho D^+(\alpha) \mathrm{d}^2\alpha, \tag{7}$$

with $B(\alpha) = \frac{1}{N} \exp(-|\alpha|^2/N)$. The main character of IWOP is that when the operators a, a^+ in the integrand are arranged in a normal form, that is, all the generation operators are on the left side of the annihilation operators, then the integral can be taken as if the operators are c-numbers. The denotation for normal form is : : , and a useful formula often encountered in IWOP is

$$|0\rangle\langle 0| =: \exp(-a^+ a) : . \tag{8}$$

When ρ is a thermal noise signal with its average photon number N_s, then ρ' will be a thermal noise signal with its average photon number $N_s + N$ (with the application of IWOP). Thus, the additive property of Gaussian signals is reserved in the quantum theory, and the von Neumann entropy of ρ is $S(\rho) = g(N_s)$, with $g(x) = (x+1)\ln(x+1) - x \ln x$ being a monotonically increasing convex function.

III. ENTANGLED STATE OPERATOR

Coherent information is defined[6] as $I_e = S(\rho^{Q'}) - S(\rho^{RQ'}) = S(\rho^{Q'}) - S_e$, where the index Q denotes the quantum system to be transmitted and R is a "reference" system. Initially, the joint system RQ is prepared in a pure entangled state $|\Psi^{RQ}\rangle$. The system R is dynamically isolated and has a zero internal Hamiltonian, while the system Q undergoes an interaction with the channel noise. The final state of RQ is possibly mixed and described by the density operator $\rho^{RQ'}$. An initial pure entangled state can be written, using the Schmit decomposition,[6]

$$|\Psi^{RQ}\rangle = \sum_k \sqrt{\lambda_k} |\lambda_k^R\rangle \otimes |\lambda_k^Q\rangle, \tag{9}$$

where λ_k and $|\lambda_k^Q\rangle$ are eigenvalues and eigenstates of ρ^Q. With no loss of generality, we set system R to have the same initial state as Q, that is, $|\lambda_k^R\rangle = |\lambda_k^Q\rangle = |\lambda_k\rangle$. Such a choice of R will not affect the coherent information of Q, for R is not intrinsic to Q in the transmission of quantum information, as mentioned in Ref.[6]. Then the final state is

$$\rho^{RQ'} = \sum_{kk'} \sqrt{\lambda_k \lambda_{k'}} \left(|\lambda_k^R\rangle\langle\lambda_{k'}^R| \right) \otimes \left(\int B(\alpha) D(\alpha) |\lambda_k^Q\rangle\langle\lambda_{k'}^Q| D^+(\alpha) \mathrm{d}^2\alpha/\pi \right). \tag{10}$$

When Q is a single-mode thermal radiation system, if we set $|\lambda_k\rangle = |k\rangle$, where $|k\rangle$ is a number state of k-photons, then $\lambda_k = (1 - \nu_s)\nu_s^k$, where $\nu_s = N_s/(N_s + 1)$.[6] We now express Eq.(10) in the generation and annihilation operators in order to obtain

eigenvalues of $\rho^{RQ'}$, and the values of S_e, I_e. To reach this purpose we expand the states $|k\rangle$ in a coherent state basis $|\gamma\rangle$. By making use of the definition $|\gamma\rangle = D(\gamma)|0\rangle$ and Eq.(8), we obtain

$$|k\rangle\langle k'| = \int \frac{(\gamma^*)^k \delta^{k'}}{\sqrt{k!k'!}} : \exp(-|\gamma|^2 - |\delta|^2 + \gamma a^+ + \delta^* a - a^+ a) : \frac{\mathrm{d}^2\gamma \mathrm{d}^2\delta}{\pi^2}. \quad (11)$$

The summation on k and k' of Eq.(10) can be taken before the integral. With the help of denotations of $a_1 = a \otimes 1, a_2 = 1 \otimes a$, the direct product in Eq.(10) will be easily dealt with, and a_1, a_1^+ commute with a_2, a_2^+. By using IWOP, we have finally

$$\rho^{RQ'} = w : \exp[(\nu_s \nu - 1)a_1^+ a_1 + (\nu - 1)a_2^+ a_2 + \tau(a_1^+ a_1^+ + a_1 a_2)] :, \quad (12)$$

where $\tau = \sqrt{\nu_s}(1-\nu), w = (1-\nu)(1-\nu_s)$.

IV. EIGENVALUES OF ENTANGLED STATE OPERATOR

To obtain the eigenvalues of $\rho^{RQ'}$, we must first remove the normal order in Eq.(12). By using the formula : $\exp[(\nu-1)a^+ a] := \nu^{a^+ a}$,[10] the result reads

$$\rho^{RQ'} = w \exp(\tau a_1^+ a_2^+)(\nu\nu_s)^{a_1^+ a_1} \nu^{a_2^+ a_2} \exp(\tau a_1 a_2). \quad (13)$$

We rewrite the eigenvalue equation $\rho^{RQ'}|\Psi\rangle = \lambda|\Psi\rangle$ in a double-coherent-state representation

$$\pi^{-2} \int \langle\alpha_1\alpha_2|\rho^{RQ'}|\beta_1\beta_2\rangle\langle\beta_1\beta_2|\Psi\rangle \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2 = \lambda\langle\alpha_1\alpha_2|\Psi\rangle, \quad (14)$$

where $|\beta_1\beta_2\rangle$ are common eigenstates of a_1 and a_2 with eigenvalues β_1 and β_2 respectively. We suppose $\langle\beta_1\beta_2|\Psi\rangle = \exp\left[K\tau\beta_1^*\beta_2^* - \frac{1}{2}|\beta_1|^2 - \frac{1}{2}|\beta_2|^2\right] f(\beta_1^*, \beta_2^*)$, then Eq.(14) becomes

$$\pi^{-2} w \int f(\beta_1^*, \beta_2^*) \exp\Omega \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2 = \lambda f(\alpha_1^*, \alpha_2^*), \quad (15)$$

with $\Omega = (K\beta_1^*\beta_2^* + \beta_1\beta_2)\tau - |\beta_1|^2 - |\beta_2|^2 + \nu_s\nu\alpha_1^*\beta_1 + \nu\alpha_2^*\beta_2 + (1-K)\tau\alpha_1^*\alpha_2^*$. We integrate with respect to β_2 with the help of the following integration formula

$$\pi^{-1} \int \mathrm{d}^2\alpha \varphi(\alpha^*) \exp(-|\alpha|^2 + \alpha\sigma) = \varphi(\sigma). \quad (16)$$

The eigenvalue equation is reduced to

$$\pi^{-1} w \int \mathrm{d}^2\beta_1 f(\beta_1^*, \tau\beta_1 + \nu\alpha_2^*) \exp\Omega_1 = \lambda f(\alpha_1^*, \alpha_2^*), \quad (17)$$

with

$$\Omega_1 = -(1 - K\tau^2)|\beta_1|^2 + \nu_s\nu\alpha_1^*\beta_1 + K\tau\nu\alpha_2^*\beta_1^* + (1-K)\tau\alpha_1^*\alpha_2^*.$$

Apparently, $f(\beta_1^*, \beta_2^*) = \beta_1^{*m}$ are the solutions of Eq.(17), in which K satisfies $\nu_s\nu^2 K/(1 - K\tau^2) = K - 1$. Letting $u = 1 - K\tau^2$, then we have

$$u^2 - (1 - \nu_s + 2\nu\nu_s)u + \nu^2\nu_s = 0. \quad (18)$$

Equation (18) has two roots. We take the root with the plus sign for further use, and drop the root with the minus sign, because it will lead to probabilities larger than one or less than zero. Then the eigenvalues of Eq.(17) are

$$\lambda_{m0} = \frac{(1-\nu_s)(1-\nu)}{u}\left(\frac{\nu_s\nu}{u}\right)^m. \quad (19)$$

Similarly, $f(\beta_1^*, \beta_2^*) = \beta_2^{*n}$ are also the solutions of Eq.(17), with the eigenvalues

$$\lambda_{0n} = \frac{(1-\nu_s)(1-\nu)}{u}\left(\frac{\nu}{u}\right)^n. \quad (20)$$

We suppose $f(\beta_1^*, \beta_2^*) = \sum_{n=0}^{\infty} c_n^{(l)} \beta_1^{*n+l} \beta_2^{*n}$, with l being a positive integer, then each term of the f function on the left-hand side of Eq.(17) can be rewritten as

$$w \int \pi^{-1} \mathrm{d}^2\beta_1 \beta_1^{*n+l} (\tau\beta_1 + \nu\alpha_2^*)^n \exp\Omega_1 = \sum_{m=0}^{n} M_{mn}^{(l)} \alpha_1^{*m+l} \alpha_2^{*m}. \quad (21)$$

Thus Eq.(17) can be written as $\sum_{n=0}^{\infty} M_{mn}^{(l)} c_n^{(l)} = \lambda c_m^{(l)}$ with $M_{mn}^{(l)}$=0 for $m < n$, and $M^{(l)}$ are the upper triangle matrices. Their eigenvalues are their diagonal elements, which can be easily obtained. The general form of eigenvalues of $\rho^{RQ'}$ is

$$\lambda_{mn} = \frac{(1-\nu_s)(1-\nu)}{u}\left(\frac{\nu_s\nu}{u}\right)^m\left(\frac{\nu}{u}\right)^n = \left(1 - \frac{\nu_s\nu}{u}\right)\left(1 - \frac{\nu}{u}\right)\left(\frac{\nu_s\nu}{u}\right)^m\left(\frac{\nu}{u}\right)^n. \quad (22)$$

The second equality is due to Eq.(18), and $\sum_{m,n=0}^{\infty} \lambda_{mn}$=1.

V. COHERENT INFORMATION

Let $N_1 = \dfrac{\nu_s \nu}{u - \nu_s \nu}$ and $N_2 = \dfrac{\nu}{u - \nu}$. The exchange information of the transmission, i.e. the entropy of the density operator $\rho^{RQ'}$, is expressed as

$$S_e = -\mathrm{Tr}\rho^{RQ'} \ln \rho^{RQ'} = g(N_1) + g(N_2). \qquad (23)$$

From Eq.(5) we know that the final state is a thermal noise state of the same mode with its average photon number being $N_s + N$. Hence the coherent information is given by

$$I_e = S(\rho^{Q'}) - S_e = g(N_s + N) - g(N_1) - g(N_2). \quad (24)$$

VI. CONCLUSION

We have analysed the quantum transmission of a thermal radiation signal over a thermal noise quantum channel. The input, a thermal radiation field of single mode, is purified to an entangled state. In the purification process the auxiliary "reference" system is chosen to be exactly the same as the input. The final state of the entangled state is critical in calculating the coherent information of the quantum transmission. We find that after interaction with the channel, the entangled state can be resolved to a direct product of two parts. Each part is a thermal radiation field of a single mode, which is characterized by an average photon number. Then the exchange information of the transmission can be expressed in terms of these average photon numbers, as can the coherent information.

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