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PLANE ELASTICITY PROBLEM OF TWO-DIMENSIONAL OCTAGONAL QUASICRYSTALS AND CRACK PROBLEM*

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The plane elasticity theory of two-dimensional octagonal quasicrystals is developed in this paper. The plane elasticity problem of quasicrystals is reduced to a single higher-order partial differential equation by introducing a displacement function. As an example, the exact analytic solution of a Mode I Griffith crack in the material is obtained by using the Fourier transform and dual integral equations theory, then the displacement and stress fields, stress intensity factor and strain energy release rate can be calculated. The physical significance of the results relative to the phason and the difference between the mechanical behaviours of the crack problem in crystals and quasicrystals are figured out. These provide important information for studying the deformation and fracture of the new solid phase.

Keywords: plane elasticity, two-dimensional octagonal quasicrystals, crack, displacement function
PACC: 6140M, 6170G

I. INTRODUCTION

Quasicrystals as a new structure of solid matter were discovered around 1984.[1−3] The discovery has brought a significant breakthrough for condensed matter physics in recent years. The quasiperiodic symmetry of solids presents great theoretical significance and encourages considerable theoretical and experimental studies on the physical and mechanical properties of materials, including the elasticity theory of quasicrystals.

The defectiveness of quasicrystalline materials has been observed.[4] The defects greatly influence the physical and mechanical properties of solid materials including quasicrystals. Experiments have shown that quasicrystals are quite brittle,[5] and brittle materials are sensitive to defects. Among them, cracks are one type of defect, their existence and propagation are the basic cause of the failure of brittle materials. The study of the crack problem of quasicrystalline materials is meaningful both in theoretical and practical applications. Recently, Fourier analysis and function theory methods in point group 10 mm two-dimensional (2D) decagonal and Laue class $\mathrm{b/m_h}$ mm one-dimensional hexagonal quasicrystals have been developed,[6] and new findings were obtained. This paper extends previous work in a founding formulation for the plane elasticity problem of point group 8 mm 2D octagonal quasicrystals. The exact analytic solutions of the Mode I crack problem are achieved, and the displacement and stress fields are determined, as are the stress intensity factor and strain energy rate of the problem. This work reveals the physical significance of the results relative to the phason and the difference between the mechanical behaviours of the crack problem in crystals and quasicrystals.

II. PLANE PROBLEM AND DISPLACEMENT FUNCTION

The elasticity equations, including stress–strain relations, deformation geometry equations and equilibrium equations, of octagonal quasicrystals are the most complicated among 2D quasicrystals.[7−9] This is very difficult to solve in a straightforward way. Reference [10] suggests that the equations can be simplified in many cases. For example, if field variables are independent of z both geometrically and physically for some problems, i.e., ∂_z=0, and the z-axis is the periodic-arrangement direction, xy-plane the quasiperiodic plane, then the elasticity problem of the quasicrystal described by the basic equations can be turned into a superposition of two problems. One is the plane elasticity problem, in which the stress–strain relations are

$$\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + R(w_{xx} + w_{yy}), \qquad (1a)$$

$$\sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} - R(w_{xx} + w_{yy}), \qquad (1b)$$

$$\sigma_{zz} = C_{13}(\varepsilon_{xx} + \varepsilon_{yy}), \qquad (1c)$$

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$$\sigma_{xy} = \sigma_{yx} = 2C_{66}\varepsilon_{xy} + R(w_{yx} - w_{xy}), \quad (1d)$$

$$H_{xx} = K_1 w_{xx} + K_2 w_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}), \quad (1e)$$

$$H_{yy} = K_1 w_{yy} + K_2 w_{xx} + R(\varepsilon_{xx} - \varepsilon_{yy}), \quad (1f)$$

$$H_{xy} = (K_1 + K_2 + K_3)w_{xy} + K_3 w_{yx} - 2R\varepsilon_{xy}, \quad (1g)$$

$$H_{yx} = (K_1 + K_2 + K_3)w_{yx} + K_3 w_{xy} + 2R\varepsilon_{xy}; \quad (1h)$$

the deformation geometry equations are

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad w_{ij} = \partial_i w_j; \quad (2)$$

the equilibrium equations in the absence of body forces are

$$\partial_j \sigma_{ij} = 0, \quad \partial_j H_{ij} = 0. \quad (3)$$

The other is an anti-plane elasticity problem, which is a classical one, but we do not consider it here.

In the above, σ_{ij}, H_{ij}, ε_{ij} and w_{ij} are the stress and strain tensors associated with phonon and phason fields, respectively; u_i and w_i are the phonon and phason displacement fields; C_{ij} are the elastic constants in the phonon field, in which $C_{66} = (C_{11} - C_{22})/2$, and K_1, K_2, K_3 denote the elastic constants in the phason field; R is the phonon–phason coupling elastic constant.

For the plane problem (1)–(3), if we introduce a displacement function $F(x, y)$, such as

$$\begin{aligned} u_x =& \{[C_{66}\alpha\Pi_1 + C_{11}\beta\Pi_2]\nabla^2 \\ &+ 4\omega(K_2 + K_3)[C_{66}\partial_x^2 \\ &+ (C_{12} + C_{66})\partial_y^2]\Lambda^2\}\partial_x\partial_y F, \end{aligned} \quad (4a)$$

$$\begin{aligned} u_y =& \{[C_{66}\alpha\partial_y^2\Pi_1 - C_{11}\beta\partial_x^2\Pi_2]\nabla^2 \\ &- 4\omega(K_2 + K_3)(C_{12} + C_{66})\partial_x^2\partial_y^2\Lambda^2\}F, \end{aligned} \quad (4b)$$

$$w_x = [\omega(\alpha - \beta)\Pi_1\Pi_2 - 4\omega^2(K_2 + K_3)\Lambda^2\Lambda^2]\partial_x\partial_y F, \quad (4c)$$

$$\begin{aligned} w_y =& [-\omega(\alpha\partial_y^2\Pi_1^2 + \beta\partial_x^2\Pi_2^2) \\ &- 8\omega^2(K_2 + K_3)\partial_x^2\partial_y^2\Lambda^2]F, \end{aligned} \quad (4d)$$

where $\alpha = C_{11}R - \omega(K_1 + K_2 + K_3), \beta = C_{66}R - \omega K_1, \omega = C_{11}C_{66}/R$,

$$\Pi_1 = 3\partial_x^2 - \partial_y^2, \Pi_2 = 3\partial_y^2 - \partial_x^2,$$

then Eqs.(1)–(3) can be simplified into the following single higher partial differential equation

$$(\nabla^2\nabla^2\nabla^2\nabla^2 - 4\varepsilon\nabla^2\nabla^2\Lambda^2\Lambda^2 + 4\varepsilon\Lambda^2\Lambda^2\Lambda^2\Lambda^2)F = 0, \quad (5)$$

where ∇^2, Λ^2 are differential operators of second order,

$$\nabla^2 = \partial_x^2 + \partial_y^2, \Lambda^2 = \partial_x^2 - \partial_y^2,$$

and ε represents a material constant

$$\varepsilon = \frac{R^2(C_{12} + C_{66})(K_2 + K_3)}{[C_{66}(K_1 + K_2 + K_3) - R^2][C_{11}K_1 - R^2]}, \quad (6)$$

which is a very important parameter. The range of its value controls the characteristics of the solution, e.g. if $K_2 + K_3 = 0$, then ε=0, and the material reduces to the point group 10 mm 2D decagonal quasicrystal, and Eq.(5) reduces to the governing equation of plane elasticity of decagonal quasicrystals.[6] For an octagonal quasicrystal, $K_2 + K_3 \neq 0$, hence $\varepsilon \neq 0$, but three cases exist: (i) $0 < \varepsilon < 1$, (ii) $\varepsilon < 0$, (iii) $\varepsilon > 1$.

In general, case (ii) cannot appear physically. The following derivation will show that case (iii) leads to a divergent solution, which is meaningless. The unique meaningful case in physics and mathematics is case (i), i.e. $0 < \varepsilon < 1$.

The stress expressions of the phonon and phason fields in terms of $F(x, y)$ are easily obtained through the generalized Hooke's law, i.e. Eqs.(1); however, they are omitted here. It can be proven that the stress expressions and displacement expressions (4) provide the complete solutions of the plane problem.

Ⅲ. PROBLEM AND SOLUTION OF GRIFFITH CRACK

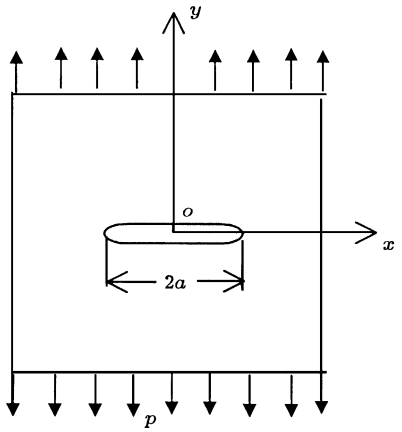


Fig.1. Griffith crack in octagonal quasicrystals.

We consider a Griffith crack with a length $2a$ in the x-direction, but extending throughout the z-direction, in the centre of a 2D quasicrystal of octagonal symmetry. At places far from the crack, the quasicrystal is subjected to a tension p (Fig.1). The

problem is equivalent to the surface of the crack being subjected to a pressure p and the far places being stress-free in the sense of the Griffith theory (shown in Fig 2). We now consider the latter in the following.

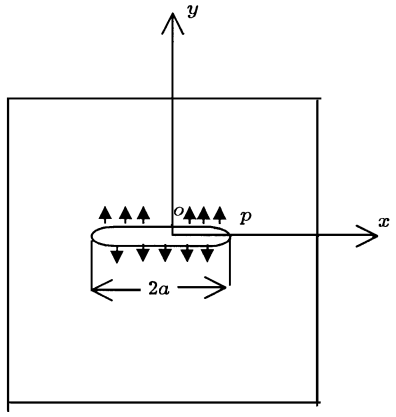


Fig.2. The problem is equivalent to that given by Fig.1.

Due to the symmetry of the problem, it is enough to study the upper half plane $y > 0$. The size of the crack is far smaller than that of the material, so that the size of the material can be considered as infinite. In this case, the boundary conditions can be described by

$$\sqrt{x^2+y^2} \to \infty : \sigma_{ij} = H_{ij} = 0, \tag{7a}$$

$$y = 0, |x| \le a : \quad \sigma_{yy} = -p, \sigma_{xy} = 0; H_{yy} = H_{yx} = 0, \tag{7b}$$

$$y = 0, |x| > a : \quad \sigma_{xy} = 0, u_y = 0; H_{yx} = 0, w_y = 0. \tag{7c}$$

The determination of the stress and displacement of the Griffith crack lies in solving a boundary-value problem (7) of the partial differential equation (5)

Performing the Fourier transform for Eq.(5) yields

$$\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \xi^2\right)^4 - 4\varepsilon\left(\frac{\mathrm{d}^4}{\mathrm{d}y^4} - \xi^4\right)^2 + 4\varepsilon\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \xi^2\right)^4\right]\hat{F} = 0, \tag{8}$$

in which $\hat{F}(\xi, y) = \int_{-\infty}^{+\infty} F(x, y)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x$.

As mentioned in the previous section, we focus our attention on the case $0 < \varepsilon < 1$. In this case, the solution of Eq.(8) under the condition of Eq.(7a) is

$$\begin{aligned}\hat{F}(\xi, y) =& A_1(\xi)\mathrm{e}^{-\lambda_1|\xi|y}\cos(\lambda_2|\xi|y) \\ &+ A_2(\xi)\mathrm{e}^{-\lambda_1|\xi|y}\sin(\lambda_2|\xi|y) \\ &+ A_3(\xi)\mathrm{e}^{-\lambda_3|\xi|y}\cos(\lambda_4|\xi|y) \\ &+ A_4(\xi)\mathrm{e}^{-\lambda_3|\xi|y}\sin(\lambda_4|\xi|y),\end{aligned} \tag{9}$$

in which

$$\begin{aligned}\lambda_1 &= [(1+\sqrt{\varepsilon})^{1/2} + \varepsilon^{1/4}](1-\sqrt{\varepsilon})^{1/2}, \\ \lambda_2 &= [(1+\sqrt{\varepsilon})^{1/2} + \varepsilon^{1/4}]\varepsilon^{1/4}, \\ \lambda_3 &= [(1+\sqrt{\varepsilon})^{1/2} - \varepsilon^{1/4}](1-\sqrt{\varepsilon})^{1/2}, \\ \lambda_4 &= [(1+\sqrt{\varepsilon})^{1/2} - \varepsilon^{1/4}]\varepsilon^{1/4},\end{aligned}$$

and $A_j(\xi)(j$=1,2,3,4) are to be determined.

From Eqs.(7b) and (7c), we obtain the dual integral equations as follows,

$$\begin{aligned}&\int_{-\infty}^{+\infty} A_j|\xi|^7\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}\xi = b_j p\pi, |x| < a, \\ &\int_{-\infty}^{+\infty} A_j\xi^6\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}\xi = 0, |x| > a,\end{aligned} \tag{10}$$

in which $b_j(j$=1,2,3,4) are known constants composed of the elastic constants.

The solutions of the dual integral equations are

$$A_j = b_j pa\pi\xi^{-7}J_1(a\xi)/2, (j = 1, 2, 3, 4), \tag{11}$$

where $J_1(a\xi)$ is the Bessel function of the first kind, first order.

The problem is thus solved. By taking the Fourier inversion transform and substituting it into Eqs.(4) and (1), the displacement and stress fields are determined. From the above results, it is easy to obtain

$$\sigma_{yy}(x, 0) = \begin{cases} -p, x \le a, \\ p(x - \sqrt{x^2 - a^2})/\sqrt{x^2 - a^2}, x > a. \end{cases} \tag{12}$$

This shows that the stress exhibits -1/2 order singularity in the vicinity of the crack tip.

The stress intensity factor K_I can be defined as follows,[11]

$$K_\mathrm{I} = \lim_{x \to a^+} \sqrt{2\pi(x-a)}\sigma_{yy}(x, 0). \tag{13}$$

The calculation shows that $K_\mathrm{I} = p\sqrt{a\pi}$, and it is independent of the elasticity constants, but we can prove that the strain energy release rate G_I is dependent on all the material constants, including the elastic constants C_{ij}, K_i in phonon and phason fields, and the phonon–phason coupling elastic constant R. Due to the limitation of space, we omit the calculation here.

In general, we have $C_{66}(K_1 + K_2 + K_3) > R^2, C_{11}K_1 > R^2$. The case $\varepsilon < 0$ does not appear in physics. Meanwhile, in the case $\varepsilon > 1$, the solution is divergent, which is meaningless. Thus all of the solutions that have physical meanings have been discovered.

IV. LINEAR ELASTIC FRACTURE MECHANICS OF QUASICRYSTALS

Based on the above solution, we can suggest two fracture criteria for octagonal quasicrystals: one is $K_{\rm I} = K_{\rm IC}$, the other is $G_{\rm I} = G_{\rm IC}$, in which $K_{\rm I}$ and $G_{\rm I}$ are given by the above results and $K_{\rm IC}$ and $G_{\rm IC}$ are the critical values of $K_{\rm I}$ and $G_{\rm I}$ measured through experiment, which can be understood as the generalized fracture toughness of quasicrystalline materials. One can find that the two criteria are equivalent to each other. Though $K_{\rm I}$ is formally similar to that of crystals, however the connotation of $K_{\rm IC}$ here is different from that of crystals, and $G_{\rm I}$ is quite different from that of crystals, which is evidently related to not only phonon elastic constants C_{ij}, but also phason elastic constants K_1, K_2, K_3 and phonon–phason coupling elastic constant R. Therefore, the linear elastic fracture mechanics (LEFM) of a conventional crystal material cannot be directly used in treating quasicrystal fracture, and the study of quasicrystal linear elastic fracture mechanics (QLEFM) is of importance. References [6,12] and the present study provide the exact analytic solutions for Mode I, Mode II and Mode III Griffith cracks of 2D decagonal and octagonal and one-dimensional hexagonal quasicrystals, offer the fracture parameters $K_{\rm I}$, ,$K_{\rm II}$ $K_{\rm III}$ and $G_{\rm I}, G_{\rm II}, G_{\rm III}$, and suggest the relevant fracture criteria and the frame of QLEFM. Some measurements of $K_{\rm IC}$ and $G_{\rm IC}$ were carried out by Meng,[5] while the systematic measurement work is underway in our laboratory.

V. DISCUSSION AND CONCLUSION

Calculations show that the stress distributions of phonons, as well as phasons are independent of material constants when the applied stresses at the crack surface form a self-equilibrium force system, e.g. given by Eqs.(7). In this case, the stress intensity factor of the Mode I crack is independent of material constants (this still holds for Mode II and Mode III cracks in our experience). Both phonon and phason displacement fields are dependent on all of the material constants (phonon, phason and phonon–phason coupling constants C_{ij}, K_i and R), so the crack strain energy and strain energy release rate are dependent on all of the material constants. The existence of the phason field strongly influences the deformation of the octagonal quasicrystals. In the xy-plane, the deformation of the phonon field is isotropic, while that of the phason field is anisotropic, which is identical to results given by finite element simulation.[13]

The basic equations of elasticity of octagonal quasicrystals are very complicated. When the equations are in a coupled crack boundary-value condition, the problem is fundamentally difficult to solve. Very few analytic solutions have been published so far. To construct an analytic solution, we have to use some special mathematical methods and make lengthy calculations. The finite element method[13] can obtain a numerical solution of the problem, but it is a discrete result. The analytic solution in closed form has many advantages compared with numerical solutions. The present method has been extended to solve the elasticity and defect problem of three-dimensional cubic quasicrystals, and analytic solutions have been obtained.[14,15] This shows that the analytic method is effective.

As mentioned above, all displacement components are related to the material constants; however, their expressions are in very complicated form, and need very lengthy algebraic manipulations. For simplicity, the displacement components and results related to these (e.g. strain energy and strain energy release rate) are omitted in the text.

With the stress intensity factor $K_{\rm I}$, we can set up a fracture criterion

$$K_{\rm I} = K_{\rm IC},$$

for the solid, where $K_{\rm IC}$ is the critical value of $K_{\rm I}$, a constant of quasicrystal, called fracture toughness of the material, and can be measured by experiment. It is easy to prove that this criterion is equivalent to the following fracture criterion

$$G_{\rm I} = G_{\rm IC},$$

in which $G_{\rm I}$ represents the strain energy release rate, $G_{\rm IC}$ the criterion value of $G_{\rm I}$, a constant of quasicrystalline material observed in experiments. Because of this equivalency, the stress intensity factor provides the necessary and sufficient information for studying the fracture of quasicrystals. The omitted discussion on the strain energy does not influence the completeness of the theory.

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