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FRUSTRATION EFFECT ON SYNCHRONIZATION AND CHAOS IN COUPLED OSCILLATORS*

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Synchronization dynamics in an array of coupled periodic oscillators with quenched natural frequencies are discussed in the presence of homogeneous phase shifts (frustrations). Frustration-induced desynchronization and chaos are found. The torus-doubling route to chaos, toroidal chaos and torus crisis are investigated.

Keywords: synchronization, frustration, torus doubling
PACC: 0545

A significant phenomenon observed in various fields is that a large population of interacting oscillators with distributed natural frequencies can spontaneously organize themselves to synchronize to a common frequency.[1] Such behaviour has been found in laser arrays, Josephson-junction arrays, magnetic resonance processes, and a variety of chemical and biological systems.[2] A promising strategy in understanding this feature is to introduce phase models, i.e. coupled limit cycles, where individual subsystems possess periodic motion in the absence of interactions. Recently, the notion of synchronization has been extended to chaotic oscillators.[3] One of the remarkable findings is the numerical and experimental observations of phase synchronization (PS) behaviour in systems of coupled non-identical chaotic oscillators.[4] This phenomenon is analogous to synchronization of periodic oscillators where only the phase locking is important, while their amplitudes remain chaotic and noncorrelated.

Even though studies on synchronizations have been extended to chaotic oscillators, it is still instructive to further study the intrinsic dynamics of coupled periodic oscillators, especially when the interactions among oscillators are nonlinear. Recently, this topic has been extensively studied, and abundant behaviours were revealed.[5,6] A subject of great interest but still unclear is the effect of frustration on the synchronization process and dynamics of the system.[7] Frustration is an important parameter in studies of phase transitions of the Heisenberg XY model,[8] the Frenkel–Kontorova lattices,[9] and other systems that have potential applications. For example, in coupled Josephson-junction arrays or ladders, the frustration plays an important role in determining the depinning threshold of vortex and transport properties.[10] The frustration can be experimentally implemented by adding a magnetic field. The frustration may also be significant in governing the synchronization (phase locking) and spatiotemporal patterns of coupled oscillators. This is the topic of the present paper. We will show that frustration can induce the desynchronization of synchronized oscillators, and this desynchronization accordingly leads to complicated dynamics and even chaos. We shall also investigate the route to chaos by varying the frustration. The torus-doubling route and the intermittency route to chaos are presented. Toroidal chaos and torus crisis are also presented.

For simplicity, the model we adopt is the following array of N sinusoidally coupled phase oscillators with different natural frequencies $\{\omega_i\}$ and with nearest coupling

$$\dot{\theta}_i = \omega_i + K \sum_{j=i\pm1} \sin(\theta_j - \theta_i - \alpha), \qquad (1)$$

where $i = 1, 2, ..., N$, and K, θ_i and $\dot{\theta}_i$ are, respectively, the coupling strength, the ith instantaneous phase and the corresponding phase velocity. $\alpha \in [0, 2\pi)$ is the phase shift, called the frustration. We will see that α plays a significant role in affecting the synchronization and dynamics of nonlinear sys-

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tems. For the case $\alpha = 0$, the phase-locking dynamics has been fully investigated.[2,5,6] For a given N and a distribution of $\{\omega_i\}$, $i = 1, 2, ..., N$, there exists a critical coupling K_c, when $K > K_c$ all oscillators can be phase locked to each other. One may define an average frequency of the ith oscillator as

$$\Omega_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{\theta}_i\,(t)\mathrm{d}t. \tag{2}$$

Synchronization implies the establishment of the relation $\Omega_i = \Omega_j$ or $|\theta_i - \theta_i| <$ const for $i \neq j$, and then one says that the oscillators i and j belong to the same cluster. For $K < K_c$, a cascade of clustering process can be found. Moreover, the process of synchronization is accompanied by the transitions from high-dimensional to low-dimensional quasi-periodic motions, interrupted by chaos.

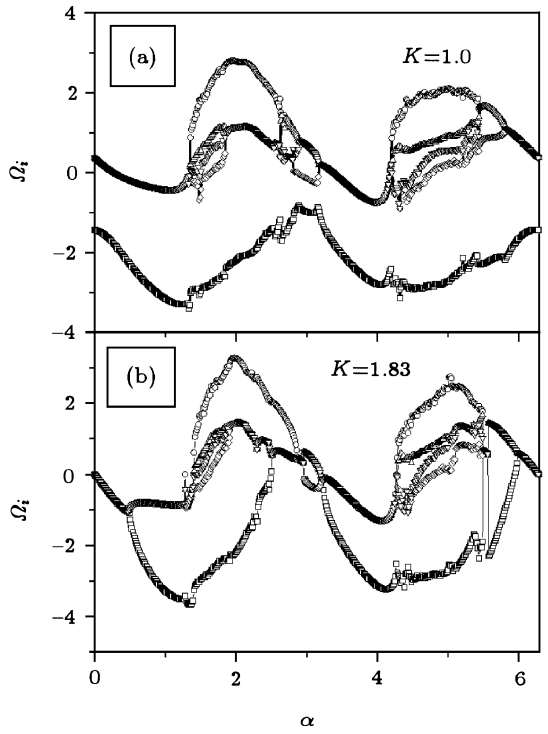


Fig.1. The synchronization tree $\{\Omega_i\}$ versus K for $N = 5$: (a) $K = 1.0$ (two-cluster state for α=0); (b) $K = 1.83$ (full synchronized state in the absence of the frustration). Here $\{\omega_i\}$=$\{1.955, 0.718, -2.862, -0.126, 0.315\}$. The frustration can induce the desynchronization of oscillators.

Now let us investigate the effect of frustration. In Fig.1, we show the synchronization tree $\{\Omega_i\}$ versus K for $N = 5$ and different coupling strengths $K = 1.83$ (fully synchronized state in the absence of the frustration) and $K = 1.0$ (two-cluster state for $\alpha = 0$), where $\{\omega_i\} = \{1.955, 0.718, -2.862, -0.126, 0.315\}$. One may find from Fig.1(a) that in a large interval of α the two-cluster state is preserved, and the only change is that the average winding numbers are shifted. At $\alpha \approx 1.25$, the two-cluster state is destroyed and the fully desynchronized states are respectively achieved. Three- and four-cluster states can also be observed. This result indicates that the frustration can induce the desynchronization of oscillators.

An intuitive understanding of the mechanism of the frustration-induced desynchronization can simply be made for the case of two coupled oscillators with different natural frequencies. For the case $N = 2$, one has

$$\dot{\theta}_{1,2} = \omega_{1,2} + K\sin(\theta_{2,1} - \theta_{1,2} - \alpha). \tag{3}$$

By introducing $\phi(t) = \theta_1(t) - \theta_2(t)$ and $\Delta = \omega_1 - \omega_2$ (assuming $\Delta > 0$ without losing generality), one has

$$\dot{\phi}\,(t) = \Delta + 2K\cos\alpha\sin\phi. \tag{4}$$

When $2K\,|\cos\alpha| \geq \Delta$, i.e.,

$$K \geq K_c(\alpha, \Delta) = \frac{\Delta}{2\,|\cos\alpha|}, \tag{5}$$

Eq.(3) has the fixed-point solution, $\dot{\phi}\,(t) = \dot{\theta}_1\,(t) - \dot{\theta}_2\,(t) = 0$, implying the phase-locking between $\theta_1(t)$ and $\theta_2(t)$. It can be obviously found from (5) that a larger coupling strength is needed for $\alpha \neq n\pi$ $(n = 0, 1, 2, ...)$, i.e., $K_c(\alpha \neq n\pi, \Delta) > K_c(n\pi, \Delta)$. Moreover, phase locking is inaccessible when $\alpha = (n + 1/2)\pi$, because $K_c \to \infty$ as $\alpha \to (n + 1/2)\pi$. In Fig.2(a), the critical lines $K_c \sim \alpha$ are given for $\Delta = 0.1$, 0.5 and 1.0. Above the critical line, a phase-locked state can be found. For $N > 2$, K_c has a complicated dependence on the frustration and natural frequencies. However, one can find similar behaviour, although the critical coupling is hard to work out analytically. This explains the above desynchronization behaviour. We performed numerical simulations for $N = 3$ and 4. In Figs.2(b) and 2(c), the synchronization diagrams are given for $N = 3$ ($\{\omega_i\} = \{-0.563, 0.102, 0.461\}$) and $N = 4$ ($\{\omega_i\} = \{-1.035, -0.27, -0.012, 1.417\}$). The shaded region corresponds to the fully phase-locked state. It can be found that the boundary is usually complicated, with a distortion of Eq.(5). For $\alpha = (n+1/2)\pi$, synchronization is always inaccessible.

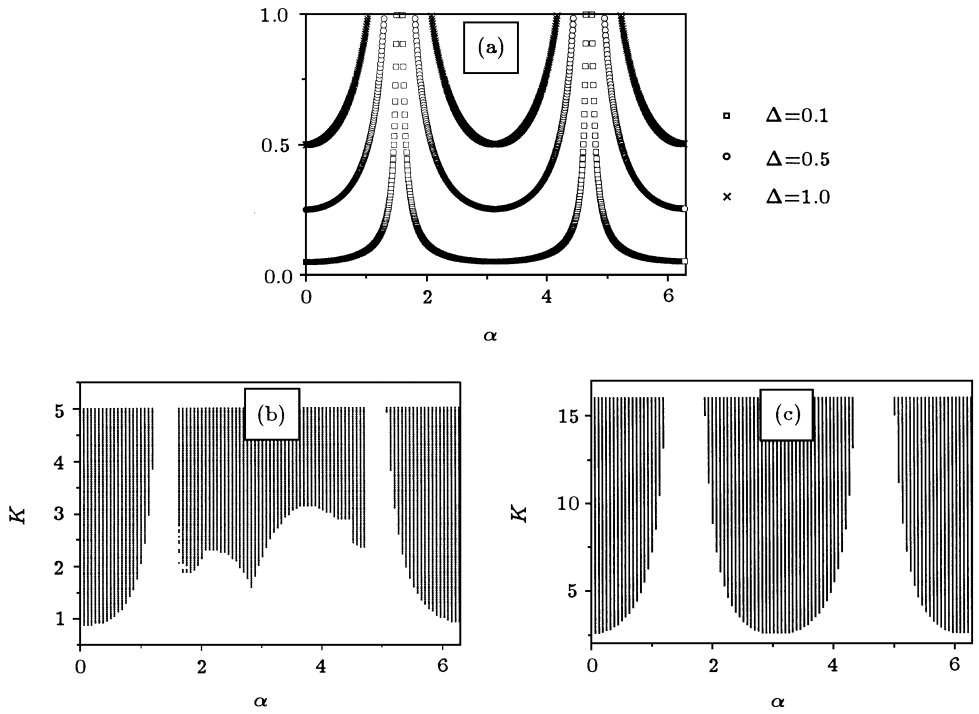


Fig.2. (a) The critical lines $K_c \sim \alpha$ for phase locking for $\Delta = 0.1, 0.5$ and 1.0. (b) The synchronization diagram for $N = 3$ ($\{\omega_i\} = \{-0.563, 0.102, 0.461\}$). (c) The synchronization diagram for $N = 4$ ($\{\omega_i\} = \{-1.035, -0.27, -0.012, 1.417\}$). The shaded region corresponds to the fully phase-locked state.

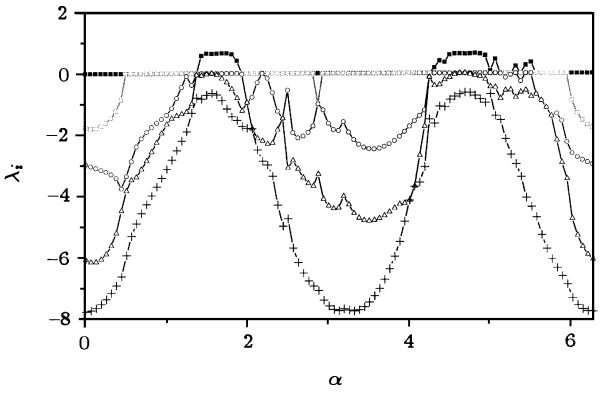


Fig.3. The Lyapunov exponent spectrum $\{\lambda_i\}$ for the same parameters as in Fig.1(b).

The frustration introduces desynchronization into the system, and it may also lead to complex dynamics. As we previously found, desynchronization is accompanied by the transition to high-dimensional quasi-periodicity. A higher-dimensional torus possesses lower topological stability. Therefore frustration may lead to chaotic motions of the coupled oscillators. In Fig.3, we compute the Lyapunov exponent spectrum $\{\lambda_i\}$ for the same parameters as in Fig.1(b). The numerical computation of these exponents is based on the Wolf algorithm. It can be clearly found that at $\alpha \approx 0.5$, the second Lyapunov exponent (negative) touches and remains zero, indicating a topological transition from periodic motion to quasi-periodic motion (T^2). This value corresponds to a desynchronization transition of the system, where the third particle deviates from the cluster. The quasi-periodic motion can become unstable and can be replaced by higher-dimensional tori or chaos. A chaotic regime can be found in Fig.3 by observing the positive maximum Lyapunov exponents.

Usually T^2 can become unstable when varying parameters and is replaced by chaos. In the present model, there are rich behaviours, such as high-dimensional quasi-periodicity, chaotic motion, and intermittency. It is an interesting issue to study the way of the transition from quasi-periodic motion to chaos. This model may serve as an ideal model for exploring this behaviour. There are many different ways for a torus to lose its stability and become chaotic. A typical way of this instability is the breaking of the torus through intermittency before the emergence of three-dimensional torus T^3.[2] A torus may also lose its stability by a series of doubling processes. We have observed this scenario in our present model. In Fig.4, we give such an example by plotting the Poincaré surface of sections (PSS) at different frustrations for $N = 5$ and $K = 1.667$. Each point on the section (θ_2, θ_3) is obtained when θ_1 passes π (with 2π modulo). In Fig.4(a) for $\alpha = 1.38$, the torus section is composed of two continuous lines, indicating

two-dimensional quasi-periodicity (two incommensurate frequencies). At $\alpha = 1.395$, each line in Fig.4(a) bifurcates into two crossed lines, signifying a topological change (folding or rotating) of the original torus. This bifurcation is just the so-called *torus doubling*. The torus in Fig.4(b) obviously becomes more complicated than that in Fig.4(a). The doubling process may persist many times, as shown in Figs.4(c)–(e). It is not clear whether the process is infinite or finite for the present system. Usually only finite times of torus doubling could be observed. We have observed as many as eight to nine doublings in Fig.4, where only some of them are shown. In Fig.4(f) for $\alpha = 1.4039$, the doubling process terminates and the torus becomes chaotic. This toroidal chaos occurs on the two-dimensional torus. The chaos region expands at $\alpha = 1.4044$ in 4(h), where the chaotic T^2 is totally broken via intermittency.

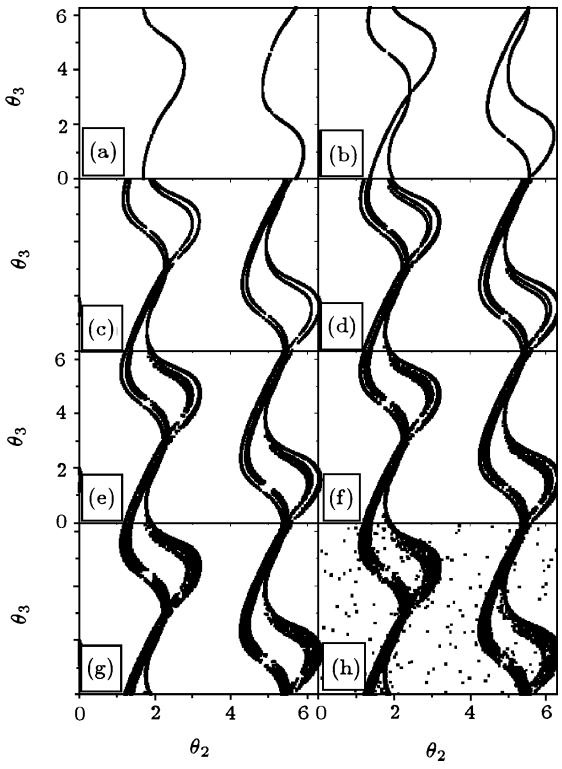


Fig.4. The Poincaré surface of sections (θ_2, θ_3) for $N = 5$, $K = 1.83$: $\alpha = 1.38$, 1.395, 1.4025, 1.4037, 1.4038, 1.4039, 1.4041 and 1.4044 from (a) to (h). Each point on the section is obtained when θ_1 passes π (with 2π modulo). *Torus-doubling* route to chaos can be observed.

To give a clearer picture of the doubling process, we adopt the technique of high-dimensional PSS. In the present paper, we get each point on the section

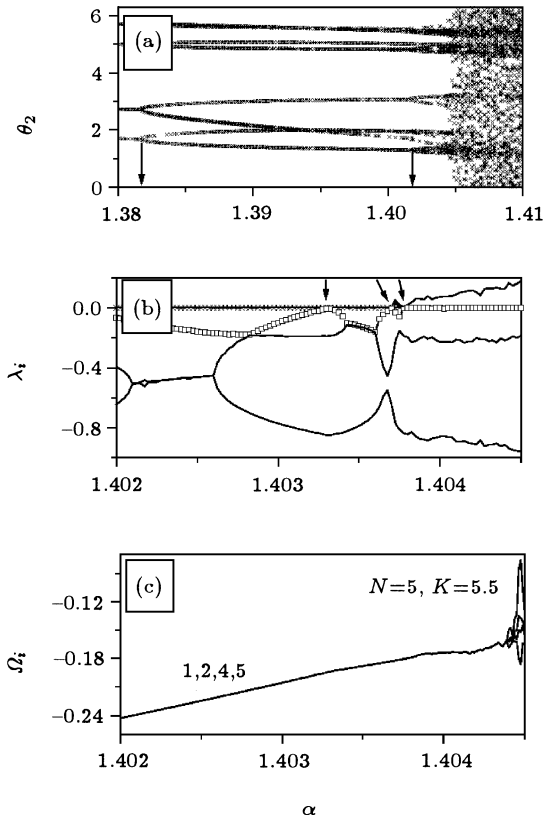


Fig.5. (a) The high-dimensional bifurcation diagram $\theta_2 \sim \alpha$ for the same parameters as Fig.4. The *torus crisis* can be found. (b) The Lyapunov exponent spectrum $\{\lambda_i\}$ against the frustration. (c) The bifurcation tree (only the cluster {1,2,4,5} is shown). the partial chaos synchronization is shown, which is destroyed at the torus crisis.

(θ_1, θ_2) when θ_3 and θ_4 simultaneously pass π. For convenience, in Fig.5(a) we plot the high-dimensional bifurcation diagram $\theta_2 \sim \alpha$ for the same parameters as Fig.4. One may clearly observe the bifurcation process (note that this is a torus-doubling diagram) at $\alpha = 1.382$ and 1.4018. The torus experiences a breaking at $\alpha \approx 1.4045$, where a local chaos expands to a full-range chaos. This is quite similar to the crisis in the circle map and other typical one-dimensional maps. We call this interesting behaviour the *torus crisis*. In Figs.5(b) and 5(c), we study the dynamics of the torus-doubling process and the synchronization. In Fig.5(b), we plot the Lyapunov exponent spectrum $\{\lambda_i\}$ against the frustration. For the T^2 torus $\lambda_{1,2} = 0$ and $\lambda_{3,4,5} < 0$. One may clearly observe that, at each torus-doubling point, the third (negative) Lyapunov exponent reaches zero (indicating a bifurcation). In Fig.5(b), only two bifurcations at $\alpha = 1.4033$ and 1.4037 are presented. In fact, many bifurcations can be found before the onset of chaos. $\lambda_1 > 0$ when $\alpha \geq 1.4037$, indicating chaos. In this case $\lambda_{2,3} = 0$,

indicating a toroidal chaos. In Fig.5(c) the bifurcation tree is shown, where only the cluster {1,2,4,5} is shown and the other cluster is {3}. One may find that in (1.4037,1.4044), the cluster still persists. Thus this is a partial chaos synchronization. This synchronization is destroyed when torus crisis occurs.

In summary, in this paper we study the effect of homogeneous frustration (phase shift) on synchronization and dynamics. We find that frustration can induce desynchronization, and also lead to chaotic motions of the system. The torus-doubling route to chaos is shown. Toroidal chaos, torus crisis and induced desynchronization are found. In fact, the present model provides a good way of exploring high-dimensional quasi-periodicity and chaos. In recent years, quasi-periodically driven systems were extensively studied and people found strange nonchaotic attractors (SNA), where the time evolution of the system is nonchaotic but the topology of the attractor is strange (i.e. fractal).[11] But all these studies were focused on non-autonomous systems, i.e. driven systems. It is an intriguing problem whether SNA can be observed in autonomous systems. Our model may serve as an ideal candidate in revealing this puzzle. Another important issue relates to the transition to chaos from the high-dimensional torus. For example, it is not clear whether $T^n (n > 2)$ can also experience a doubling scenario to chaos.[12]

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