Vol 10 No 8, August 2001
1009-1963/2001/10(08)/0694-04

Chinese Physics

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NEW EXPLICIT AND EXACT TRAVELLING WAVE SOLUTIONS FOR A COMPOUND KdV–BURGERS EQUATION*

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(Received 17 May 2000; revised manuscript received 22 January 2001)

In this paper, new explicit and exact travelling wave solutions for a compound KdV–Burgers equation are obtained by using the hyperbola function method and the Wu elimination method, which include new solitary wave solutions and periodic solutions. Particularly important cases of the equation, such as the compound KdV, mKdV–Burgers and mKdV equations can be solved by this method. The method can also solve other nonlinear partial differential equations.

Keywords: compound KdV–Burgers equation, travelling wave solution, Wu elimination method
PACC: 0340K, 0290

I. INTRODUCTION

In this paper we consider the compound KdV–Burgers equation

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \qquad (1)$$

where p, q, r and s are constants, which can be thought of as a composition of the KdV, mKdV and Burgers equations, involving nonlinear, dispersion and dissipation effects. Equation (1) contains the following particularly important cases:

(a) For $p \neq 0, q \neq 0, r = 0, s \neq 0$: Eq.(1) becomes the compound KdV equation

$$u_t + puu_x + qu^2u_x - su_{xxx} = 0. \qquad (2)$$

(b) For $p = 0, q \neq 0, r \neq 0, s \neq 0$: Eq.(1) becomes the mKdV–Burgers equation

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = 0. \qquad (3)$$

(c) In Eq.(3) if $r = 0$, then we have the mKdV equation

$$u_t + qu^2u_x - su_{xxx} = 0. \qquad (4)$$

In a recent paper, Wang *et al.*[1] found some exact solutions of Eq.(1) by using the homogeneous balance method.[2] In this paper we gain many new travelling wave solutions of Eq.(1) by use of a hyperbola function method[3] and the Wu elimination method.[4,5] The exact solutions of Eqs.(2)–(4), as the particular cases of the solutions for Eq.(1), are also obtained.

The main idea of the algorithm is as follows.

We give a partial differential equation, say, in two variables

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \qquad (5)$$

where P is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. Eqs.(1)–(4) are all of the type of Eq.(5). By using the travelling wave transformation, Eq.(5) has the following form of solution

$$u(x,t) = \phi(\xi), \quad \xi = \lambda(x - kt + c_0), \qquad (6)$$

where k and λ are constants to be determined later and c_0 is an arbitrary constant. Substituting Eq.(6) into Eq.(5) yields an ordinary differential equation for ϕ

$$P(\phi, \phi', \phi'', \cdots) = 0, \qquad (7)$$

where " $'$ " denotes $\mathrm{d}/\mathrm{d}\xi$.

According to the hyperbola function method, we suppose that Eq.(7) has a travelling wave solution of the following form

$$\varphi(\xi) = \sum_{i=1}^{n} \sinh^{i-1} w(B_i \sinh w + A_i \cosh w) + B_0, \qquad (8)$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}\xi} = \sinh w \quad \text{or} \quad \frac{\mathrm{d}w}{\mathrm{d}\xi} = \cosh w, \qquad (9)$$

*Project supported by the National Key Basic Research Development project of China (Grant No. 1998030600), the National Natural
Science Foundation of China (Grant No. 10072013) and Educational Committee of Liaoning Province (Grant No. 990421093).

where A_i and B_i (i=0,1,2,$\cdots n$) are constants to be determined later, and the value of n is obtained by equating the highest nonlinear terms and the highest-order partial derivative term in Eq.(5) (see Ref.[2] for details).

Step 1 Equating the highest-order nonlinear term and highest-order linear partial derivative term in Eq.(5) yields the value of n.

Step 2 Substituting Eqs.(8) and (9) into Eq.(7) yields a polynomial equation about $\cosh w \sinh^i w$, $\sinh^i w (i = 0, 1, 2, \cdots)$. This step can be carried out by computer.

Step 3 Setting the constant term and coefficients of $\sinh w$, $\cosh w$, $\sinh w \cosh w$, $\sinh^2 w, \cdots$ in the equations obtained in step 2 to zero, yields a system of algebraic equations about the unknown numbers $k, \lambda, B_0, A_i, B_i (i = 1, \cdots, n)$.

Step 4 Using the Wu elimination method, the algebraic equations in step 3 are solved by computer. Then combining Eqs.(6)–(9) yields the solitary wave solutions and periodic solutions for the system (5).

Using the algorithm, we find that the form of solution not only includes the terms csch ξ or coth ξ (csc ξ or cotξ), but also has their combination term. This characteristic is different from that of other methods, such as the homogeneous balance method, etc.

II. NEW EXPLICIT AND EXACT TRAVELLING WAVE SOLUTION FOR EQ.(1)

We set

$$u(x,t) = \varphi(\xi), \qquad \xi = \lambda(x - kt + c_0), \tag{10}$$

where λ and k are constants to be determined later and c_0 is an arbitrary constant.

Substituting Eq.(10) into Eq.(1) yields

$$k\varphi' - p\phi\phi' - q\varphi^2\varphi' - \lambda r\varphi'' + s\lambda^2\varphi''' = 0, \tag{11}$$

where " $'$ " denotes $\mathrm{d}/\mathrm{d}\xi$.

According to step 1 in the introduction, we suppose that Eq.(2) has the following form of solution

$$\phi(\xi) = B_0 + A_1 \cosh w + B_1 \sinh w, \tag{12}$$

and the target equation

$$\frac{\mathrm{d}w}{\mathrm{d}\xi} = \sinh w. \tag{13}$$

With the aid of Mathematica or Maple, from Eqs.(12) and (13), we obtain

$$\begin{aligned}
&k\varphi' - p\phi\phi' - q\varphi^2\varphi' - \lambda r\varphi'' + s\lambda^2\varphi''' \\
=&(-A_1B_1p - 2A_1B_1B_0q - B_1r\lambda)\sinh w \\
&+ (B_1k - B_1B_0p - A_1^2B_1q \\
&- B_1B_0q + B_1\lambda s^2)\cosh w \sinh w \\
&+ (A_1k - A_1B_0p - A_1^3q - 2A_1B_1^2q \\
&- A_1B_0^2q + 4A_1s\lambda^2)\sinh^2 w \\
&+ (-A_1^2p - B_1^2p - 2A_1^2B_0q - 2B_1^2B_0q \\
&- 2A_1r\lambda)\cosh w \sinh^2 w \\
&+ (-2A_1B_1p - 4A_1B_1B_0q - 2B_1r\lambda)\sinh^3 w \\
&+ (-3A_1^2B_1q - B_1^3q + 6B_1s\lambda^2)\cosh w \sinh^3 w \\
&+ (-A_1^3q - 3A_1B_1^2q + 6A_1s\lambda^2)\sinh^4 w = 0.
\end{aligned}$$

Setting the coefficients of $\sinh^j w \cosh^i w$ (i=0,1, j=0,1,2,3,4) to zero, we have

$$A_1B_1p + 2A_1B_1B_0q + B_1r\lambda = 0, \tag{14}$$

$$B_1k - B_1B_0p - A_1^2B_1q - B_1B_0q + B_1\lambda s^2 = 0, \tag{15}$$

$$A_1k - A_1B_0p - A_1^3q - 2A_1B_1^2q - A_1B_0^2q + 4A_1s\lambda^2 = 0, \tag{16}$$

$$A_1^2p + B_1^2p + 2A_1^2B_0q + 2B_1^2B_0q + 2A_1r\lambda = 0, \tag{17}$$

$$(-2A, B_1p - 4A_1B_1B_0q - 2B_1r\lambda) = 0 \tag{18}$$

$$3A_1^2B_1q + B_1^3q - 6B_1s\lambda^2 = 0, \tag{19}$$

$$A_1^3q + 3A_1B_1^2q - 6A_1s\lambda^2 = 0. \tag{20}$$

Using the Wu algebraic elimination method to solve the system of the over-determined equations (14)–(19) with respect to $A_1, B_1, B_0, \lambda, k$, yields

Case 1: $2qr^2 = 3p^2s, \quad qs > 0,$

$$B_1 = B_0 = 0, \quad A_1 = \pm\lambda\sqrt{\frac{6s}{q}}, \quad k = 2s\lambda^2.$$

Case 2:

$$qs > 0, \quad B_0 = -\frac{r}{\sqrt{6qs}} - \frac{p}{2q}, \quad A_1 = B_1 = \pm\lambda\sqrt{\frac{3s}{2q}},$$

$$k = \frac{1}{2}\left(s\lambda^2 + \frac{r^2}{3s} - \frac{p^2}{2q}\right).$$

We take the target equation as

$$\frac{\mathrm{d}w}{\mathrm{d}\xi} = \cosh w. \tag{21}$$

Using the same method as above and solving the corresponding over-determined equations, we obtain

Case 3: $2qr^2 = 3sp^2, \quad ks < 0,$

$$B_0 = A_1 = 0, \quad B_1 = \pm\frac{2r}{p}\sqrt{\frac{-k}{2s}}, \quad \lambda = \pm\sqrt{\frac{-k}{2s}}.$$

Case 4: $2r^2q = 3sp^2, \quad kq < 0,$

$$B_0 = 0, \quad A_1 = B = \pm\sqrt{\frac{-3k}{q}}, \quad \lambda = \pm\frac{p}{r}\sqrt{\frac{-3k}{q}}.$$

Case 5:

$$qs > 0, \quad A_1 = B_1 = \pm\sqrt{\frac{-3s}{2q}}\lambda, \quad B_0 = \frac{p}{2q} \pm \frac{r}{\sqrt{6qs}},$$

$$k = \left(\frac{3p^2}{4q} \pm \frac{2pr}{\sqrt{6qs}} + \frac{r^2 - 3s^2\lambda^2}{6s}\right).$$

For the equation $\dfrac{\mathrm{d}w}{\mathrm{d}\xi} = \sin w$, by integrating it (taking the integration constant to be zero), we have

$$\sinh w = -\mathrm{csch}\xi, \tag{22}$$

$$\cosh w = -\coth\xi. \tag{23}$$

Also from Eq.(21), we can obtain

$$\sinh w = -\cot\xi, \tag{24}$$

$$\cosh w = \mathrm{csc}\xi. \tag{25}$$

According to Eqs.(12), (22)–(25) and cases 1–5, we can obtain the following solitary wave solutions (i)–(ii) and periodic solutions (iii)–(v) for Eq.(1).

(i) $qs > 0, \quad 2qr^2 = 3p^2s,$

$$u(x,t) = \pm\lambda\sqrt{\frac{6s}{q}}\coth\lambda(x - 2s\lambda t + c_0).$$

for $t \to +\infty$, $u(x,t) \to \infty$. This implies that the wave may have the blow-up phenomenon.

(ii) $qs > 0,$

$$u(x,t) = \pm\lambda\sqrt{\frac{3s}{2q}}(\mathrm{csch}\xi + \coth\xi) - \frac{r}{\sqrt{6qs}} - \frac{p}{2q},$$

where

$$\xi = \lambda(x - kt + c_0), k = \frac{1}{2}\left(s\lambda^2 + \frac{r^2}{3s} - \frac{p^2}{2q}\right).$$

(iii) $2qr^2 = 3sp^2, ks < 0,$

$$u(x,t) = \pm\frac{2r}{p}\sqrt{\frac{-k}{2s}}\cot\left(\pm\sqrt{\frac{-k}{2s}}(x - kt + c_0)\right),$$

(iv) $2r^2q = 3sp^2, kq < 0,$

$$u(x,t) = \pm\sqrt{\frac{-3k}{q}}(\mathrm{csc}\xi - \cot\xi),$$

where

$$\xi = \pm\frac{p}{r}\sqrt{\frac{-3k}{q}}(x - kt + c_0).$$

(v) $qs > 0,$

$$u(x,t) = \pm\sqrt{\frac{3s}{2q}}\lambda(\mathrm{csc}\xi - \cot\xi) + \frac{p}{2q} \pm \frac{r}{\sqrt{6qs}},$$

where

$$\xi = \lambda(x - kt + c_0), k = \left(\frac{3p^2}{4q} + \frac{2pr}{\sqrt{6qs}} + \frac{r^2 - 3s^2\lambda^2}{6s}\right).$$

Ⅲ. SPECIAL CASES OF EQ.(1) AND THEIR TRAVELLING SOLUTIONS

We omit the steps of solving Eq.(2)–(4).

1. Equation (2) has solitary and periodic solutions of the following form. These solutions differ from other solutions,[1,6,7]

$$u_1(x,t) = -\frac{p}{2q} \pm \lambda\sqrt{\frac{3s}{2q}}(\mathrm{csch}\xi + \coth\xi),$$

where

$$\xi = \lambda(x - kt + c_0), \quad k = \frac{1}{4}\left(2s\lambda^2 - \frac{p^2}{q}\right).$$

$$u_2(x,t) = \pm\frac{1}{2q}\sqrt{-3(p^2 + 4kq)}\cot\lambda(x - kt + c_0),$$

where

$$\lambda = \pm\frac{1}{2}\sqrt{\frac{-(p^2 + 4kq)}{2sq}}, \quad p^2 + 4kq < 0, \quad sq > 0.$$

$$u_3(x,t) = \pm\frac{1}{q}\sqrt{\frac{3}{2}(4kq - p^2)}\mathrm{csc}\left(\pm\frac{1}{2}\sqrt{\frac{4kq - p^2}{sq}} \cdot (x - kt + c_0)\right), sq > 0, 4kq - p^2 > 0.$$

$$u_4(x,t) = \pm\frac{1}{2q}\sqrt{-(3p^2 + 12kq)}(\mathrm{csc}\xi - \cot\xi) - \frac{p}{2q},$$

where

$$\xi = \pm\sqrt{\frac{-(p^2 + 4kq)}{2qs}}(x - kt + c_0), p^2 + 4kq < 0, qs > 0.$$

2. Equation (3) has solitary and periodic solutions of the following form

$$u_1(x,t) = \pm\lambda\sqrt{\frac{3s}{2q}}(\coth\xi + \mathrm{csch}\xi) \pm \frac{r}{\sqrt{6sq}},$$

where

$$\xi = \pm\frac{1}{s}\sqrt{\frac{2ks - r^2}{3}}(x - kt + c_0), sq > 0, 2ks - r^2 > 0.$$

$$u_2(x,t) = \pm\sqrt{\frac{r^2-6sk}{2sq}}\cot \cdot\left(\pm\frac{1}{2s}\sqrt{\frac{r^2-6sk}{3}}(x-kt+c_0)\right), \quad r^2-6sk>0, sq>0.$$

$$u_3(x,t) = \pm\sqrt{\frac{r^2-6sk}{2sq}}(\csc\xi-\cot\xi)\pm\frac{r}{\sqrt{6sq}},$$

where

$$\xi = \pm\frac{1}{s}\sqrt{\frac{r^2-6sk}{3}}(x-kt+c_0), sq>0, r^2-6sk>0.$$

3. Equation (4) has following solitary and periodic solutions

$$u_1(x,t) = \pm\lambda\sqrt{\frac{3s}{2q}}(\mathrm{csch}\xi+\coth\xi),$$

where

$$\xi = \lambda\left(x-\frac{1}{2}s\lambda^2t+c_0\right), qs>0.$$

$$u_2(x,t) = \pm\lambda\sqrt{\frac{6s}{q}}\mathrm{csch}(\lambda(x+\lambda s^2t+c_0)), sq>0.$$

$$u_3(x,t) = \pm\lambda\sqrt{\frac{6s}{q}}\coth(\lambda(x+2\lambda s^2t+c_0)), sq>0.$$

$$u_4(x,t) = \pm\lambda\sqrt{\frac{3s}{2q}}(\csc\xi-\cot\xi),$$

where

$$\xi = \lambda\left(x+\frac{1}{2}s\lambda^2t+c_0\right), sq>0.$$

$$u_5(x,t) = \pm\lambda\sqrt{\frac{6s}{q}}\csc\lambda(x-\lambda s^2t+c_0), qs>0.$$

$$u_6(x,t) = \pm\lambda\sqrt{\frac{6s}{q}}\cot\lambda(x+2\lambda s^2t+c_0), qs>0.$$

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