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SIMILARITY REDUCTIONS OF THE (2+1)-DIMENSIONAL BURGERS SYSTEM*

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In this paper, using the direct method of the (2+1)-dimensional multi-component Burgers system, some types of similarity reductions are obtained. The corresponding group explanations of the reductions, Virasoro integrability and soliton solutions of Burgers system are also discussed.

Keywords: Burgers system, similarity reduction, direct method, group explanation
PACC: 0340, 0230, 0365

I. INTRODUCTION

In this paper, we investigate the similarity so1-utions of the (2+1)-dimensional Burgers equations

$$\psi_{iy} = -\psi_{ixx} - 2\psi_{ix}u\ , \qquad i = 1, 2\cdots N\ , \quad (1)$$

$$u_z = \sum_{i=1}^{N} \psi_{ix}\ . \quad (2)$$

For convenience, we rewrite Eqs.(1) and (2) as:

$$H_1 = \psi_{iy} + \psi_{ixx} + 2\psi_{ix}u = 0\ , \qquad i = 1, 2\cdots N\ , \quad (3)$$

$$H_2 = u_z - \sum_{i=1}^{N} \psi_{ix} = 0\ . \quad (4)$$

The (2+1)-dimensional Burgers equation was first derived by Chen and Lou[1] when they used the inner parameter dependent symmetry constraint on the Lax pairs of the well-known modified Kadovtsev–Petviashvilli (mKP) equation. Chen and Lou have given a type of (2+1)-dimensional multi-component integrable extension of the Burgers equation, and the Painlevé property of the (2+1)-dimensional multi-component Burgers equation was proven by means of the standard singularity analysis.[1]

The traditional method for finding similarity reductions of a given partial differential equation (PDE) or a given PDE system is to use the classical Lie approach. However, the method cannot be used to obtain all the similarity solutions. Recently, Clarkson *et al.*[4] and Lou *et al.*[5] presented a direct and simple method of finding all the similarity solutions of the Boussineq equation. It is shown that the direct method is very useful for finding the exact solutions of all nonlinear PDEs, no matter whether they are integrable or not.[6–10]

In section II of this paper, we will use the direct method to reduce the (2+1)-dimensional Burgers equation and in section III, we give the group explanation of the reductions. In sections IV and V, we obtain the Virasoro integrability and the soliton solutions of the Burgers system. The last section is a short summary and discussion.

II. SIMILARITY REDUCTION OF THE (2+1)-DIMENSIONAL BURGERS EQUATION VIA THE DIRECT METHOD

As in the cases studied in Refs.[4–10] it is easy to prove that all the two-dimensional similarity reductions of the Burgers equations (1) and (2) in (2+1)-dimensions can be written in the simple forms

$$\psi_i = \alpha_i(x, y, z) + \beta_i(x, y, z)P_i(\xi, \zeta), \quad (5)$$

$$u = a(x, y, z) + b(x, y, z)Q(\xi, \zeta), \quad (6)$$

with

$$\xi = \xi(x, y, z),\ \zeta = \zeta(x, y, z) \quad (7)$$

instead of the most general forms

$$\psi_i = F_{1i}(x, y, z, P_i(\xi, \zeta)), \quad (8)$$

$$u = F_2(x, y, z, Q(\xi, \zeta)). \quad (9)$$

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Substituting Eqs.(5) and (6) into Eqs.(3) and (4) yields

$$\delta_0+\delta_1P_i+\delta_2P_{i\xi}+\delta_3P_{i\zeta}+\delta_4P_{i\xi\xi}+\delta_5P_{i\xi\zeta}+\delta_6P_{j\zeta\zeta}+\delta_7Q+\delta_8QP_i+\delta_9QP_{i\xi}+\delta_{10}QP_{i\zeta}=0, \tag{10}$$

$$\sum_{i=0}^{N}(\gamma_{0i}+\gamma_{1i}P_i+\gamma_{2i}P_{i\xi}+\gamma_{3i}P_{i\zeta}+\gamma_{4i}Q+\gamma_{5i}Q_\xi+\gamma_{6i}Q_\zeta=0, \tag{11}$$

where δ_i $(i=0,1,...,10)$ and γ_{ji} $(j=0,1,2,...6, i=1,2,...,N)$ are given by

$$\begin{aligned}\delta_0&=\alpha_{iy}+\alpha_{ixx}+2a\alpha_{ix},\\ \delta_1&=\beta_{iy}+\beta_{ixx}+2a\beta_{ix},\end{aligned} \tag{12}$$

$$\begin{aligned}\delta_2&=\beta_i\xi_y+2\beta_{ix}\xi_x+\beta_i\xi_{xx}+2a\beta_i\xi_x,\\ \delta_3&=\beta_i\zeta_y+2\beta_{ix}\zeta_x+\beta_i\zeta_{xx}+2a\beta_i\zeta_x,\end{aligned} \tag{13}$$

$$\begin{aligned}\delta_4&=\beta_i\xi_x^2, \qquad \delta_5=2\beta_i\xi_x\zeta_x,\\ \delta_6&=\beta_i\zeta_x^2, \qquad \delta_7=2b\alpha_{ix},\end{aligned} \tag{14}$$

$$\delta_8=2b\beta_{ix}, \quad \delta_9=2b\beta_i\xi_x, \quad \delta_{10}=2b\beta_i\zeta_x. \tag{15}$$

For γ_{ji}, if j is fixed, we can take γ_j=$(\gamma_{j0},\gamma_{j1},\gamma_{j2},\cdots,\gamma_{jN})$. So

$$\begin{aligned}\gamma_0&=(\gamma_{00},\gamma_{01},\gamma_{02},\cdots,\gamma_{0N})\\ &=(a_z,-\alpha_{1x},-\alpha_{2x},\cdots,-\alpha_{Nx}),\end{aligned} \tag{16}$$

$$\begin{aligned}\gamma_1&=(\gamma_{10},\gamma_{11},\gamma_{12},\cdots,\gamma_{1N})\\ &=(0,-\beta_{1x},-\beta_{2x},\cdots,-\beta_{Nx}),\end{aligned} \tag{17}$$

$$\begin{aligned}\gamma_2&=(\gamma_{20},\gamma_{21},\gamma_{22},\cdots,\gamma_{2N})\\ &=\xi_x(0,-\beta_1,-\beta_2,\cdots,-\beta_N),\end{aligned} \tag{18}$$

$$\begin{aligned}\gamma_3&=(\gamma_{30},\gamma_{31},\gamma_{32},\cdots,\gamma_{3N})\\ &=\zeta_x(0,-\beta_1,-\beta_2,\cdots,-\beta_N),\end{aligned} \tag{19}$$

$$\gamma_4=(\gamma_{40},\gamma_{41},\gamma_{42},\cdots,\gamma_{4N})=b_z(1,0,0,\cdots,0), \tag{20}$$

$$\gamma_5=(\gamma_{50},\gamma_{51},\gamma_{52},\cdots,\gamma_{5N})=b\xi_z(1,0,0,\cdots,0), \tag{21}$$

$$\gamma_6=(\gamma_{60},\gamma_{61},\gamma_{62},\cdots,\gamma_{6N})=b\zeta_z(1,0,0,\cdots,0). \tag{22}$$

Eqs.(10) and (11) are PDEs of P and Q with respect to ξ and ζ only for the ratios of the coefficients of different derivatives and with the powers of P and Q being functions of ξ and ζ. That is to say, the constrained conditions

$$\delta_i=\delta_j\Delta_i(\xi,\zeta), \qquad (i=0,1,...10), \tag{23}$$

$$\begin{aligned}&\gamma_{ji}=\gamma_{j'i'}\Gamma_{ji}(\xi,\zeta),\\ &(i=1,2,...,N \text{ for } j=1,2,...,8)\end{aligned} \tag{24}$$

for some fixed non-zero δ_j and $\gamma_{j'i'}$ must be satisfied, where Δ_i and Γ_{ji} are some functions of ξ and ζ to be determined later.

To determine the non-fixed functions $\{\alpha_i,\ \beta_i,\ P_i,\ a,\ b, Q,\ \xi,\ \zeta,\ \Delta_i,\ \Gamma_{ji}\}$, we can use some remarks to simplify the calculations:

Remark 1. If $\alpha_i(x,y,z)$ (or $a(x,y,z)$) has the form $\alpha_i=\beta_i(x,y,z)\Xi_i(\xi,\zeta)+\alpha_{0i}(x,y,z)$ (or $a=b(x,y,z)\Xi(\xi,\zeta)+a_0(x,y,z)$), then we can take $\Xi_i\equiv 0$ (or $\Xi=0$) (by substituting $P_i(\xi,\zeta)\to P_i(\xi,\zeta)-\Xi_i(\xi,\zeta)$, or $Q(\xi,\zeta)\to Q(\xi,\zeta)-\Xi(\xi,\zeta)$).

Remark 2. If $\beta_i(x,y,z)$ (or $b(x,y,z)$) has the form $\beta_i=\beta_{0i}(x,y,z)\Xi_i(\xi,zeta)$ (or $b=b_0(x,y,z)\Xi(\xi,\ \zeta)$), then we can take $\Xi_i\equiv C_i$ =constant (or $\Xi\equiv C$ =constant) (by substituting $P_i(\xi,\zeta)\to P_i(\xi,\zeta)C_i/\Xi_i(\xi,\zeta)$, or $Q(\xi,\zeta)\to Q(\xi,\zeta)C/\Xi(\xi,\zeta)$).

Remark 3. If $\xi=\xi(\xi_0(x,y,z),\zeta)$ (or $\zeta=\zeta(\xi,\zeta_0(x,y,z))$, then we can take $\xi\equiv\xi_0$ (or $\zeta\equiv\zeta_0$) (by substituting $\{P_i(\xi(\xi_0,\zeta),\zeta),\ Q(\xi(\xi_0,\zeta),\zeta)\}\to\{P_i(\xi_0,\zeta),\ Q(\xi_0,\zeta)\}$ or $\{P_i(\xi,\zeta(\xi,\zeta_0)),\ Q(\xi,\zeta(\xi,\zeta_0))\}\to\{P_i(\xi,\zeta_0),\ Q(\xi,\zeta_0)\}$).

Remark 4. If $\xi(x,y,z)$ (or $\zeta(x,y,z)$) has the form $\Xi(\xi)=\xi_0(x,y,z)$ (or $H(\zeta)=\zeta_0(x,y,z)$), where $\Xi(\xi)$ (or $H(\zeta)$) is any invertible function, then we can take $\Xi(\xi)\equiv\xi$ (or $H(\zeta)\equiv\zeta$) (by taking $\xi=\Xi^{-1}(\xi)$ or $\zeta=H^{-1}(\zeta)$).

To discuss further, three cases should be considered: (i) $\xi_x\neq 0$, (ii) $\xi_x=\zeta_x=0$, $\xi_y\neq 0$ and (iii) $\xi_x=\zeta_x=0$, $\xi_y=\zeta_y=0$. Only the first two cases will be discussed in detail because the last case is quite trivial.

1. $\xi_x\neq 0$.

In this case, we can choose $\delta_j=\delta_9=2b\beta_i\xi_x$ and $\gamma_{j'i'}=\gamma_{60}=b\zeta_z$ as common factors. Using remarks 1–4 to fix the freedoms in the determination of $\{\alpha_i,\ \beta_i,\ P_i,\ a,\ b,\ Q,\ \xi,\ \zeta,\ \Delta_i,\ \Gamma_{ji}\}$, and analysing Eqs.(23) and (24) with Eqs.(12)–(22) carefully, we can obtain three possible solutions of Eqs.(23) and (24).

(1) $\zeta_z\neq 0$.

$$\begin{aligned}\alpha_i&=\alpha_{0i}(y,z)=C_{0i}(y,z),\\ \beta_i&=\beta_{0i}(y,z)=C_{1i}(y,z),\end{aligned} \tag{25}$$

$$a=-\frac{(\zeta_y^{1/2})_y+C_{2y}}{2\zeta_y^{1/2}},\quad b=\frac{\xi_x}{2}=\frac{1}{2}\zeta_y^{1/2},\tag{26}$$

$$\xi=\zeta_y^{1/2}x+C_2(y,z),\quad \zeta=\zeta(y,z)=C_3(y,z),\tag{27}$$

where $C_{0i}(y,z)$, $C_{1i}(y,z)$, $C_2(y,z)$ and $C_3(y,z)$ are arbitrary functions of the indicated variables.

Substituting Eqs.(12)–(22) into Eqs.(10) and (11), we obtain

$$\Delta_2=\Delta_5=\Delta_6=\Delta_7=\Delta_8=\Delta_{10}=0,\tag{28}$$

$$\Delta_3=\Delta_4=\Delta_9=1,\ \Delta_1=\frac{\beta_{iy}}{\beta_i\zeta_y},\ \Delta_0=\frac{\alpha_{iy}}{\beta_i\zeta_y},\tag{29}$$

$$\begin{aligned}
&\Gamma_0=\left(\frac{2a_z}{\zeta_y^{1/2}\zeta_z},0,0,\cdots,0\right),\\
&\Gamma_1=(0,0,0,\cdots,0),\qquad(30)\\
&\Gamma_2=-\frac{2}{\zeta_z}(0,\beta_1,\beta_2,\cdots,\beta_N),\\
&\Gamma_3=(0,0,0,\cdots,0),\qquad(31)\\
&\Gamma_4=\left(\frac{\zeta_{yz}}{2\zeta_y\zeta_z},0,0,\cdots,0\right),\\
&\Gamma_5=\left(\frac{\xi_z}{\zeta_z},0,0,\cdots,0\right),\\
&\Gamma_6=(1,0,0,\cdots,0).\qquad(32)
\end{aligned}$$

Substituting Eqs.(23) and (24) together with Eqs.(28)–(32) into Eqs.(10) and (11), we obtain the first type of similarity reduction of the (2+1)-dimensional Burgers equations (1) and (2):

$$\Delta_0(\zeta)+\Delta_1(\zeta)P_i+P_{i\zeta}+P_{i\xi\xi}+QP_{i\xi}=0,\tag{33}$$

$$\Gamma_{00}+\sum_{i=1}^{N}\Gamma_{2i}(\zeta)P_{i\xi}+\Gamma_{40}(\zeta)Q+\Gamma_{50}Q_\xi+Q_\zeta=0.\tag{34}$$

From

$$\Delta_0=\frac{\alpha_{iy}}{\beta_i\zeta_y},\qquad \Delta_1=\frac{\beta_{iy}}{\beta_i\zeta_y},\tag{35}$$

we can obtain

$$\beta_i=C_{4i}(z)\mathrm{e}^{\int^{\zeta}\Delta_1(\zeta_1)\mathrm{d}\zeta_1},\tag{36}$$

$$\alpha_i=C_{4i}(z)\int^{\zeta}\Delta_0(\zeta_2)\mathrm{e}^{\int^{\zeta_2}\Delta_1(\zeta_1)\mathrm{d}\zeta_1}\mathrm{d}\zeta_2+C_{5i}(z).\tag{37}$$

Using remarks 1 and 2 with Eqs.(36) and (37) yields

$$\begin{aligned}
&\alpha_i=\alpha_{i0}(z),\qquad \beta_i=\beta_{i0}(z),\\
&\Delta_0=0,\qquad \Delta_1=0.
\end{aligned}\tag{38}$$

For Γ_{2i} and Γ_{50}, using remarks 1 and 2 we have

$$\begin{aligned}
&\beta_i=-\frac{1}{2}\zeta_z=-\frac{1}{2}\zeta_{1z},\qquad \zeta=\zeta_1(z)+\zeta_0(y),\\
&C_1=C_{10}(y),\qquad a=a(x,y),
\end{aligned}\tag{39}$$

$$\begin{aligned}
&\Gamma_{00}=0,\qquad \Gamma_{2i}=1\ (i=1,2,\cdots,N),\\
&\Gamma_{40}=0,\qquad \Gamma_{50}=0.
\end{aligned}\tag{40}$$

From the analysis above, the first type of similarity reduction is simplified to:

$$P_{i\zeta}+P_{i\xi\xi}+QP_{i\xi}=0,\tag{41}$$

$$\sum_{i=1}^{N}P_{i\xi}+Q_\zeta=0.\tag{42}$$

with

$$\xi=C_{3y}^{1/2}x+C_1(y),\qquad \zeta=C_2(z)+C_3(y),\tag{43}$$

The corresponding solution of the system (1) and (2) reads:

$$\psi_i=C_{0i}-\frac{1}{2}C_{2z}P_i,\tag{44}$$

$$u=\frac{1}{2}C_{3y}^{1/2}-\frac{xC_{3yy}+C_{1y}C_{3y}^{1/2}}{4C_{3y}}Q.\tag{45}$$

where $C_{0i}(z)$, $C_1(y)$, $C_2(z)$ and $C_3(y)$ are arbitrary functions of the indicated variables. $P_i(\xi,\zeta)$ and $Q(\xi,\zeta)$ can be solved from Eqs.(41) and (42). In Section V, we will discuss the solution in detail.

From Eq.(30), we know that to get the first type of similarity solution, we need to use the assumption $\zeta_z\neq 0$. If $\zeta_z=0$, we may obtain the second type of similarity reduction:

(2) $\zeta_z=0,\ \xi_z\neq 0$.

In this case, the corresponding solution of the (2+1)-dimensional Burgers equation has the form

$$\psi_i=C_{0i}(z)-\frac{1}{2}C_{1z}P_i,\tag{46}$$

$$u=\frac{1}{2}Q,\tag{47}$$

where $C_{0i}(z)$ and $C_1(z)$ are arbitrary functions of z. The reduction equations for the reduction fields P_i and Q then read

$$P_{i\zeta}+P_{i\xi\xi}+QP_{i\xi}=0,\tag{48}$$

$$\sum_{i=1}^{N}P_{i\xi}+Q_\xi=0,\tag{49}$$

with

$$\xi=x+C_1(z),\qquad \zeta=y.\tag{50}$$

(3) $\zeta_z=0,\ \xi_z=0$.

In this simple case, the related reduction solution for the (2+1)-dimensional Burgers system (3) and (4) has the form

$$\psi_i = C_{0i}(z) + C_{1i}(z)P_i, \tag{51}$$

$$u = -\frac{1}{2}C_{2y} + \frac{1}{2}Q, \tag{52}$$

while P_i and Q are determined by

$$P_{i\zeta} + P_{i\xi\xi} + QP_{i\xi} = 0, \tag{53}$$

$$\sum_{i=1}^{N} C_i P_{i\xi} = 0. \tag{54}$$

with

$$\xi = x + C_2(y), \qquad \zeta = y, \tag{55}$$

where $C_{0i}(z)$, $C_{1i}(z)$ and $C_2(y)$ are arbitrary functions of the indicated variables and $C_i = \dfrac{\beta_i}{\beta_1} = \text{const.}$

2. $\xi_x = 0,\ \zeta_x = 0,\ \xi_y \neq 0$**.**

In this case, we can choose $\delta_j = \delta_2 = \beta_i\xi_y$ and $\gamma_{j'i'} = \gamma_{60} = b\zeta_z$. Now, using remarks 1–4, we obtain the only two possible independent solutions of Eqs.(23) and (24) with $\xi_x = 0,\ \zeta_x = 0$ and $\xi_y \neq 0$:
(1) $\alpha_{iy} = 0,\ \beta_{ix} \neq 0$.

The reduction equations in this case have the forms:

$$P_{i\xi} + QP_i = 0, \tag{56}$$

$$\sum_{i=1}^{N} P_i + Q_\zeta = 0. \tag{57}$$

with

$$\xi = y,\ \zeta = z. \tag{58}$$

The corresponding special solution of Eqs.(1) and (2) reads:

$$\psi_i = C_{0i}(z) + \left[-\frac{1}{8}x^2 + C_{1i}(y)x - 2C_{1i}^2(y)\right] P_i(y,z), \tag{59}$$

$$u = \left[\frac{1}{4C_{1i}(y) - x} + 4C_{1iy}\right] + \left[\frac{1}{4}x - C_{1i}(y)\right] Q(y,z). \tag{60}$$

where $C_{0i}(z)$, $C_{1i}(y)$ and $C_{2i}(x,y)$ are the arbitrary functions of the indicated variables, and $P_i(y,z)$ and $Q(y,z)$ can be solved from Eqs.(56) and (57).
(2) $\beta_{ix} = 0$.

The reduction equations in this case have the forms:

$$P_{i\xi} + Q = 0, \tag{61}$$

$$Q_\zeta = 0. \tag{62}$$

with

$$\xi = y, \qquad \zeta = z. \tag{63}$$

The reduction equations (61) and (62) can be solved easily. The results read:

$$P_i = -\int^y B(y_1)\mathrm{d}y_1 + D(z), \qquad Q(y) = B(y), \tag{64}$$

where $B(y)$ and $D(z)$ are the arbitrary functions of y and z, respectively.

The corresponding special solution of Eqs.(1) and (2) reads:

$$\psi_i = C_{0i}(y)x + C_{1i}(y) + P_i, \tag{65}$$

$$u = -[C_{0iy}(y)x + C_{1iy}] + Q(y). \tag{66}$$

where $C_{0i}(y)$ and $C_{1i}(y)$ are arbitrary functions of y.

III. GROUP EXPLANATION OF THE REDUCTIONS

Usually, one obtains the similarity solutions by using the standard Lie group approach. Actually, one can easily prove that the first type of symmetry reduction 2.1 (Eqs.(1) and (2)) can also be used by means of the classical Lie approach.[12]

Using the standard Lie approach, one can easily prove that the (2+1)-dimensional Burgers equations (1) and (2) are invariant under the transformation:

$$y \to y + \epsilon Y, \quad Y \equiv Y(y) = \text{arbitrary function of } y, \tag{67}$$

$$x \to x + \epsilon X, \qquad X \equiv \frac{1}{2}Y'(y)x + X_0(y), \tag{68}$$

$$z \to z + \epsilon Z, \quad Z \equiv Z(z) = \text{arbitrary function of } z, \tag{69}$$

$$\psi_i \to \psi_i + \epsilon\,\Psi_i, \qquad \Psi_i \equiv -Z'(z)\psi_i - V_i(z), \tag{70}$$

$$u \to u + \epsilon U, \quad U \equiv -\frac{1}{2}uY'(y) + \frac{1}{4}Y''(y)x + \frac{1}{2}X_0'(y), \tag{71}$$

where $Y(y)$, $X_0(y)$, $Z(z)$ and $V_i(z)$ are the arbitrary functions of the indicated variables, and the superscript primes of Y and Z are the derivatives with respect to the corresponding arguments.

Now according to the standard Lie approach, we should solve the following characteristic equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}z}{Z} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}\psi_i}{\Psi_i} \tag{72}$$

to obtain the related similarity reductions.

To solve the above characteristic equations, we consider some special cases:
(3-1) the first case $\xi_x \neq 0,\ \zeta_z \neq 0$.

In this case, we use the solutions in 2.1(1) to obtain the results:

$$X(x,y) = \frac{1}{2}xC_{3y}^{-2}C_{3yy} + C_{1y}C_{3y}^{-\frac{3}{2}},$$

$$Y(y) = -C_{3y}^{-1},$$

$$Z(z) = C_{2z}^{-1},$$

$$U = \frac{1}{2}C_{3y}^{-2}C_{3yy}u + \frac{1}{4}x[2C_{3y}^{-3}C_{3yy}^{2} - C_{3y}^{-2}C_{3yyy}] + \frac{1}{2}[C_{1yy}C_{3y}^{-\frac{3}{2}} - \frac{3}{2}C_{1y}C_{3y}^{-\frac{5}{2}}C_{3yy}],$$

$$\Psi_i = C_{2z}^{-2}C_{2zz}\psi_i - C_{2z}^{-2}[C_{0i}C_{2zz} - C_{0iz}C_{2z}].$$

By comparison with Eqs.(67)–(71), it is easy to see that the result is totally the same as Eqs.(41)–(45) with the relations:

$$Y(y) = -C_{3y}^{-1},$$

$$Z(z) = C_{2z}^{-1},$$

$$X_0(y) = C_{1y}C_{3y}^{-\frac{3}{2}},$$

$$V_i(z) = C_{2z}^{-2}[C_{0i}C_{2zz} - C_{0iz}C_{2z}]. \qquad (73)$$

(3-2) the second case $\xi_x = \zeta_x = 0, \xi_y \neq 0$.

In this case, we use the solutions in 2.2(1) to obtain the results:

$$X(x,y) = (x - 4C_{1i}(y))^2,$$

$$Y(y) = 0,$$

$$Z(z) = 0,$$

$$U = (x - 4C_{1i}(y))u + 4C_{1i}(y)(x - 4C_{1i}(y)) + 2,$$

$$\Psi_i = -4(x - 2C_{1i}(y))\psi_i + 4(x - 2C_{1i}(y))C_{0i}(z).$$

By comparison with Eqs.(41) and (42) which are obtained by the classical Lie approach, we reach the conclusion that the solutions of the second case cannot be obtained by the classical Lie approach; that is to say, they can only be obtained by the non-classical Lie approach.

IV. VIRASORO INTEGRABILITY OF THE BURGERS SYSTEM

In Ref.[1], Chen and Lou have proven the Burgers integrability of the system. It is known that an integrable model may possess many other interesting properties. For instance, an integrable model may possess infinitely many symmetries and conserved quantities.[9] Especially, for all the known (2+1)-dimensional integrable models, there exists a common infinite dimensional Virasoro algebra

$$[\sigma(f_1), \sigma(f_2)] = \sigma(\dot{f}_1 f_2 - \dot{f}_2 f_1), \qquad (74)$$

where f_1 and f_2 are the arbitrary functions of one of the independent variables $\{x, y, z\}$[10] and a dot over the functions represents the derivative with respect to its independent argument. According to this property, Lou *et al.* have defined that if a model possesses the symmetry algebra (74), then the model is Virasoro integrable.

From the discussions in Section III, it is easy to see that the model (1) is also Virasoro integrable. For instance, from Eqs.(67) and (68), we know that

$$\sigma(Z(z)) = Z(z)\begin{pmatrix} \psi_{iz} \\ u_z \end{pmatrix} + \begin{pmatrix} -\dot{Z}(z) \\ 0 \end{pmatrix} \qquad (75)$$

is a symmetry of (1) and $\sigma(Z(z))$ satisfies the symmetry algebra (74).

V. SOLITON SOLUTIONS OF THE BURGERS SYSTEM

In Section II, we have given many similarity reductions. After solving these lower dimensional reduction equations, we may obtain many significant exact solutions. Here we give only a simple but important soliton solution for the first type of reduction equations (41) and (42). It is easy to verify that

$$P_i = \frac{kC_i\exp(k\xi - k^2\zeta)}{1 + \exp(k\xi - k^2\zeta)},$$
$$Q = \frac{2k\exp(k\xi - k^2\zeta)}{1 + \exp(k\xi - k^2\zeta)} \qquad (76)$$

where k and $\sum_{i=1}^{N} C_i = 2k$ are arbitrary constants, is a solution of Eqs.(41) and (42). Returning to the original field variables ψ_i and u shown by Eqs.(51) and (52), we obtain the related soliton-like solutions; that is, for ψ_i, we have

$$\psi_i = C_{0i} - \frac{k}{2}C_iC_{2z} \cdot \frac{\exp[k(C_{3y}^{1/2}x + C_1(y)) - k^2(C_2(z) + C_3(y))]}{1 + \exp[k(C_{3y}^{1/2}x + C_1(y)) - k^2(C_2(z) + C_3(y))]}. \qquad (77)$$

If we take y as "time" and select C_{2z} as constant and $C_{0i} = 0$, then Eq.(77) denotes the straight line soliton. If we select $C_{0i} = 0$ and C_{2z} as some multiple straight-line solitons parallel to the x-axis, then

ψ_{ix} with Eq.(77) denotes the so-called dromion soliton which decays exponentially in all directions.[12]

VI. SUMMARY AND DISCUSSION

In this paper, we apply the direct method to the (2+1)-dimensional Burgers system. Many types of (2+1)-dimensional similarity reductions are obtained. Only the first case (3-1) of these can also be obtained by means of the classical Lie approach while all other types of similarity reductions can only be obtained by means of the non-classical Lie approach.

In various cases, as shown in Refs.[4–10], the direct method can be used to discover all the possible similarity reductions. For the (2+1)-dimensional Burgers system (1) and (2), using a similar discussion as in Ref.[4], we can prove that the form (5) and (6) is sufficient to discover all possible similarity reductions in the form (8) and (9). But we only find the two cases of $\zeta_z \neq 0$ and $\zeta_z = 0$. $\xi_z \neq 0$ under the $\xi_x \neq 0$ circumstance can be obtained by the classical Lie approach under the $\xi_x \neq 0$ circumstance, so other cases surely can be obtained by the non-classical Lie approach. The (2+1)-dimensional multi-component Burgers system is not only Painlevé integrable but also Virasoro integrable. The (2+1)-dimensional Burgers system also possesses many types of localized excitations including dromion solitons. The other kinds of integrable properties of the (2+1)-dimensional Burgers system are worth further study.

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