Vol 10 No 8, August 2001
1009-1963/2001/10(08)/0679-04
Chinese Physics
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MASSIVE SCALAR PARTICLE IN THE GRAVITATIONAL FIELD OF A MICROCOSMIC BLACK HOLE*

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(Received 25 July 2000; revised manuscript received 2 April 2001)

The finite motion of a massive scalar particle in the gravitational field of a microcosmic black hole with weak relativistic approximation is discussed. In the Schwarzschild field, using the condition for balance σ=0, we obtain the relation between the produced and captured amplitudes for particles. In the Kerr field we show that the attenuation depends on the moment of the black hole and the attenuation process becomes an exciting one when $\omega < md_h$.

Keywords: Hawking effect, gravitation, microcosmic black hole
PACC: 0314

I. INTRODUCTION

The motion of the massive classical particle in the gravitational field has been discussed in many papers.[1] But near the microcosmic black hole, the gravitational radius is comparable with the de Broglie wavelength and the quantum character appears in the motion. In Ref.[2] the motion of the particle situated on a deeper quantum energy level (case $a = 0$ and $a \neq 0$ in the Kerr field) is studied. When $r_g \sim \lambda_{\rm Compton}$, near the horizon particles can come into being.[3]

In this paper we discuss the weak relativistic motion of a scalar particle, using the generalized Klein–Gordon equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}g^{\mu\nu}\frac{\partial \Phi}{\partial x^\nu}\right] + \mu^2 \Phi = 0, \qquad (1)$$

where μ is the mass of the particle.

II. SCHWARZSCHILD MICROCOSMIC BLACK HOLE

We now introduce the Eddington–Finkelstein transformation of coordinates

$$t = t_s + r_* - r, \quad r = r_s, \quad \theta = \theta_s, \quad \varphi = \varphi_s,$$

where t_s, r_s, θ_s and φ_s are Schwarzschild coordinates. The nonvanishing transformed component of metric $g^{\mu\nu}$ is then given by

$$g^{00} = 1 + \frac{r_g}{r}, \quad g^{10} = g^{01} = \frac{r_g}{r}, \quad g^{11} = -\left(1 - \frac{r_g}{r}\right),$$
$$g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta}. \qquad (2)$$

Substituting metric (2) into Eq.(1) and then separating the variables, we obtain

$$\Phi(t, r, \theta, \varphi) = \int \mathrm{e}^{-\mathrm{i}\tilde{\omega}t} R_m^l(r) Y_l^m(\theta, \varphi)\mathrm{d}\tilde{\omega}, \qquad (3)$$

where $\tilde{\omega} = \omega/r_g$, and

$$\xi(1-\xi)R'' + [1 - 2(1 - \mathrm{i}\omega)\xi]R' - [\omega^2\xi(\xi+1) - \mu^2\xi^2 - \mathrm{i}\omega - l(l+1)]R = 0, \qquad (4)$$

where $\xi = r/r_g$, and ξ, ω are dimensionless variables.

It is convenient to use the Eddington–Finkelstein coordinates, because one still can discuss the motion of the particle in both regions $r > r_g$ and $r < r_g$.

Near the horizon of the microcosmic black hole we have[3]

$$|\omega^2\xi(\xi+1) - \mu^2\xi^2| \ll (l+1). \qquad (5)$$

Thus, Eq.(4) becomes

$$\xi(\xi-1)R'' + [1 - 2(1-\mathrm{i}\omega)\xi]R' + [l(l+1) + \mathrm{i}\omega]R = 0. \qquad (6)$$

We obtain the linearly independent solutions of Eq.(6) as follows:

$$R_1 = F(-l - \mathrm{i}\omega,\ 1 + l - \mathrm{i}\omega;\ l;\ \xi),$$
$$R_2 = R_1 \ln\xi + \sum_{k=1}^{\infty} C_k \xi^k; \quad \text{when } \xi < 1 \qquad (7)$$
$$\tilde{R}_1 = \tilde{F}(-l - \mathrm{i}\omega, l + l - \mathrm{i}\omega;\ 1 - 2\mathrm{i}\omega;\ 1 - \xi),$$
$$\tilde{R}_2 = (1-\xi)^{2\mathrm{i}\omega} F(-l + \mathrm{i}\omega, l + l + \mathrm{i}\omega, 1 + 2\mathrm{i}\omega; 1 - \xi),$$
$$\text{when } \ |\xi - 1| < 1. \qquad (8)$$

*Project supported by the National Science Foundation of China (Grant No. 19975018).

For the particle with an arbitrary moment moving in the gravitational field of the microcosmic black hole, the existence of the logarithmic singular ($\xi \to 0$) is a characteristic behaviour of the wave function. The characteristic behaviour is independent of the long wave approximation, and it will appear even if $\lambda \ll r_g$. The reason for this process is that the particle captured by the microcosmic black hole and falling into the centre singularity is independent of the moment of the particle.

In the region far from the microcosmic black hole, where $\omega^2\xi^2 \gg l(l+1)$, we can use the weak relativity approximation

$$d^2 \equiv \mu^2 - \omega^2 \ll \mu^2. \tag{9}$$

Then the asymptotic solution of Eq.(4) can be described by a degenerative super-geometric series

$$R_\infty(\xi) = \mathrm{e}^{-\mathrm{d}x}\xi^{l+\mathrm{i}\omega}\,\Psi\left(1+l-\frac{\omega^2}{2d}, 2+2l; 2\mathrm{d}\xi\right). \tag{10}$$

Using the solution and boundary conditions (in the centre and at the boundary), we can obtain the energy spectrum of the weak relativistic particle in the Schwarzschild field.

The Schwarzschild surface r_g is a unidirectional membrane for classical particles. If the property is still true for the quantum particle, we can choose $\tilde{R}_1$ which corresponds to the particle captured by the black hole on the surface r_g. In the region $\xi \gg 1$, the solution possesses the following asymptotic behaviour

$$\begin{aligned}\tilde{R}_1 \sim{}& \xi^{1+\mathrm{i}\omega}\frac{\Gamma(1+l)}{\Gamma(1+l-\mathrm{i}\omega)} \\ &+ \frac{\mathrm{i}}{\omega}\xi^{-1-l-\mathrm{i}\omega}\frac{\Gamma(1+l+\mathrm{i}\omega)}{\Gamma(2+2l)|\Gamma(-l-\mathrm{i}\omega)|^2}.\end{aligned} \tag{11}$$

When $\mathrm{d}\xi \ll 1$, the solution R_∞ takes a similar form. Neglecting the logarithmic term, the function R_∞ can be expressed as

$$\begin{aligned}R_\infty \sim{}& \xi^{l+\mathrm{i}\omega}\,\Psi\left(1+l-\frac{\omega^2}{2d}\right) \\ &\cdot\left[\Gamma(2+2l)\Gamma\left(-l-\frac{\omega^2}{2d}\right)\right]^{-1} \\ &+ \xi^{-1-l+\mathrm{i}\omega}\left(\frac{1}{2d}\right)^{2l+1}\Gamma(1+2l) \\ &\cdot\left[\Gamma\left(1+l-\frac{\omega^2}{2d}\right)\right]^{-1}\end{aligned} \tag{12}$$

Comparing the coefficients for terms ξ^l and ξ^{-1-l} in (11) and (12) respectively, we obtain the equation for determining the characteristic energy spectrum in the Schwarzschild field:

$$\begin{aligned}&\left(1-l-\frac{\omega^2}{2d}\right)\Gamma\left(1+l-\frac{\omega^2}{2d}\right) \\ &\cdot\left[\Gamma\left(-l-\frac{\omega^2}{2d}\right)\right]^{-1}(2d)^{1+2l} \\ =&\mathrm{i}\omega[\Gamma(1+2l)\Gamma(2+2l)]^2 \\ &\cdot|\Gamma(-l-\mathrm{i}\omega)|^2|\Gamma(1+l-\mathrm{i}\omega)|^{-2}.\end{aligned} \tag{13}$$

The complex value of energy satisfies Eq.(13), that is to say, the energy levels are not stationary. When a particle falls into the microcosmic black hole, the separation of quasi-stationary levels is decided by the energy and moment of the particle in nature. In flat space, the characteristic spectrum of the quasi-hydrogen atom is determined by the condition that the wave function is equal to zero at the boundary. In the above case, the difference between the Schwarzschild and Coulomb fields is very small in the local space. The contact energy is

$$d = \frac{\mu^2}{2n} + \Delta d - \mathrm{i}\sigma, \tag{14}$$

with

$$|\Delta d|, |\sigma| \ll \frac{\mu^2}{2n} \equiv d_0;$$

then we obtain

$$1+l-\frac{\omega^2}{2d} \approx 1+l-m\left(1-\frac{\Delta d}{d_0}-\mathrm{i}\frac{\sigma}{d_0}\right). \tag{15}$$

In the long wave approximation, the attenuation reduces quickly with the increase of the particle moment

$$\begin{aligned}\left(\frac{\sigma}{d_0}\right)_{n,l} =&\omega(d_0)^{1-2l}\prod_{s=1}^{l}\left(1+\frac{\omega^2}{s^2}\right)\frac{(2l+1)^2}{n2^{2l-1}} \\ &\cdot[(2l+1)!!]^{-4}\frac{\Gamma(n+l+1)}{\Gamma(n-1)}.\end{aligned} \tag{16}$$

Here we have assumed that the particle is classical near the black hole, but when $r_g \sim \lambda_{\mathrm{compton}}$ the production of particles is possible (Hawking process). In the Hawking process, produced particles may hold the quasi-stationary energy level. Produced anti-particles fall into the central singularity after finite time in commoving coordinates. According to Ref.[3], applying the analytical extensive method to the solution on the horizon, we now discuss the process in which the particle is produced.

In the region $\xi \gg 1$, the solution $\tilde{R}_2$ becomes

$$\begin{aligned}\tilde{R}_2 \sim{}& \xi^{l+\mathrm{i}\omega}\Gamma(1-2l)[\Gamma(1+l-\mathrm{i}\omega)]^{-1} \\ &-\frac{\mathrm{i}}{\omega}\xi^{-1-l+\mathrm{i}\omega}\Gamma(1+l+\mathrm{i}\omega) \\ &\cdot[\Gamma(2+2l)]^{-1}|\Gamma(-l-\mathrm{i}\omega)|^2,\end{aligned} \tag{17}$$

which is different from $\tilde{R}_1$ only by the symbol in front of i. This leads to a different symbol of the attenuation σ. Thus, the Hawking effect makes particles near the black hole collect to the quasi-stationary level. The process of the collection of particles and the process of capture by the black hole are independent of each other. Therefore, a dynamic balance can be expected in the physical system. In the balance state, using the condition for balance σ=0, we can obtain the relation between the produced and captured amplitudes for the particles.

A complete solution corresponding to the production of the particle–anti-particle pair in the region $|\xi - 1| < 1$ can be written as

$$\Phi \sim \{\theta(\xi-1)\tilde{R}_2(\xi-1) + \mathrm{e}^{2\pi\omega}\theta(1-\xi)\tilde{R}_2(1-\xi)\}\mathrm{e}^{-\mathrm{i}\omega t}Y_l^m(\theta,\varphi). \quad (18)$$

In this case, the solution near $\xi = 0$ is very interesting. It is obtained by the analytical extending method and can be shown by[4]

$$\Phi \sim R_1 \ln\xi + B_k\xi^k. \quad (19)$$

The solution has a singular term, which is different from the stationary solution in the Coulomb field of flat space.

III. KERR MICROCOSMIC BLACK HOLE

In the Boyer–Lindquist coordinates, the Kerr metric is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - 4Mar\Sigma \cdot \sin^2\theta\mathrm{d}t\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 - \sin^2\theta(r^2 + a^2 - 2Mar\sin^2\theta/\Sigma)\mathrm{d}\varphi^2; \quad (20)$$

$$\Delta = r^2 - 2Mr + a^2 = (r + r_+)(r - r_-),$$

$$\Sigma = r^2 + a^2\cos^2\theta.$$

Substituting the metric into the Klein–Gordon equation and separating the variables, we obtain[5]

$$\Phi(t,r,\theta,\varphi) = \int \mathrm{e}^{\mathrm{i}\omega}R^l_{\omega,m}(r)\Theta^m_l(\theta)\mathrm{e}^{2m\varphi}\mathrm{d}\omega. \quad (21)$$

The radial equation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \{\Delta^{-1}[(r^2+a^2)\omega^2 - 4aMm\omega r + am^2] - \mu^2(r^2+a^2) - C\}R = 0, \quad (22)$$

where C=const. The function $\Theta(a\sqrt{\mu^2-\omega^2}, \cos\theta)$ satisfies the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[a^2(\mu^2-\omega^2)\sin^2\theta - \frac{m^2}{\sin^2\theta}\right]\Theta = 0. \quad (23)$$

We now discuss the long wave approximation $r_g \ll \lambda$ as we did in section I. The solution of Eq.(23) satisfying the boundary condition on the horizon of the Kerr black hole near r_+ is obtained in Ref.[6], which can be expressed as

$$R^{\mathrm{in}} = \left(\frac{\xi}{\xi+1}\right)^{\mathrm{iN}} F(-l, 1+l, 1+2iN; \xi+1), \quad (24)$$

where

$$N = (\omega - md_h)r_g r_+/(r_+ - r_-), \quad \xi = \frac{r-r}{r_+ - r_-},$$

and d_h is angular velocity of the black hole. In order to obtain the solution one has to assume:

$$a(\mu^2-\omega^2)^{1/2} \ll 1, \quad C \approx l(l+1).$$

In the region far from the black hole the asymptotic behaviour of the solution is as follows:

$$R^{\mathrm{in}} \sim (-1)^l\frac{(2l)!}{l!}\frac{\Gamma(1+2iN)}{\Gamma(1+l+2iN)}\xi^l + (-1)^{1+l}\frac{l!}{(1+2l)!}\frac{\Gamma(1+2iN)}{\Gamma(-l+2iN)}\xi^{-1-l}. \quad (25)$$

When $r \to \infty$ the solution of Eq.(22) tends to

$$R_\infty = \xi^l\mathrm{e}^{-ad\xi}\,\Psi(1+l-b, 2+2l; 2ad\xi), \quad (26)$$

where

$$d = \sqrt{\mu^2-\omega^2}r_+, \qquad a = (r_+ - r_-)/r_+,$$
$$b = \frac{\omega^2}{d}\left(1 - \frac{a}{2}\right).$$

For $\xi \to 0$ we have

$$R_\infty \approx \Psi(1+l-b)[\Gamma(-l-b)]^{-1}(2ad)^{1+2l}\xi^l + \Gamma(1+2l)\Gamma(2+2l)[\Gamma(1+l-b)]^{-1}\xi^{-1-l}. \quad (27)$$

Connecting the solutions (24) and (26) in the public region we obtain the following equation that can determine the characteristic energy spectrum in the Kerr field:

$$(1+l-b)\Gamma(1+l-b)[\Gamma(-l-b)]^{-1}(2ab)^{1+2l} = \left[\frac{(2l)!(1+2l)!}{l!}\right]^2\Gamma(-l+2iN)[\Gamma(1+l+2iN)]^{-1}. \quad (28)$$

Using the result and relations

$$d = d_0 + \mathrm{i}\sigma, \quad 1 + l - b \approx 1 + l - n\left(1 - \frac{\mathrm{i}\sigma}{d_0}\right), \quad (29)$$

we obtain

$$\left(\frac{\sigma}{d_0}\right)_{n,l,m} = N(ad_0)^{1+2l} \prod_{s=1}^{l}(1 + 4N^2/s^2) \cdot \frac{\Gamma(n+l+1)}{\Gamma(n-l)n} \cdot \frac{(1+2l)^2}{2^{2l-1}[(2l+1)!!]^4}, \quad (30)$$

which describes the attenuation of the quasi-stationary energy level in the Kerr field.

Obviously, the attenuation depends on the moment of the black hole. When $\omega < md_h$ [in Eq.(24)], we know from (30) that the sign of the attenuation can be changed, that is, the attenuation process becomes an exciting one. This condition strengthens the multipole wave being scattered by the microcosmic black hole.[7]

The results obtained with the methods of general relativity and quantum mechanics in this paper are possibly useful for revealing the gravitational properties of the microcosmic black hole (a kind of special celestial body) in astrophysics.

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