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# A NONLINEAR ANALYTICAL AND NUMERICAL STUDY FOR AN OPEN SYSTEM WITH COHERENT OFF-RESONANCE FIELDS*

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An exact nonlinear analytical solution of an open V-type inversionless lasing system with two incoherent pump fields, off-resonant driving and probe fields is given. Through qualitative and quantitative comparison of the gain, dispersion and population differences of the linear and nonlinear cases, some conclusions concerning the Rabi frequencies of the driving and the probe fields are drawn and discussed in detail.

**Keywords:** lasing without inversion, nonlinearity, gain, dispersion
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## I. INTRODUCTION

Quantum coherence and interference in atomic systems have a number of important consequences, including lasing without inversion (LWI),[1] the subrecoil cooling of atoms,[2] potential for sensitive measurements of magnetic fields,[3] the electromagnetically-induced transparency[4] and so on. In particular, LWI has attracted much more attention,[5,6] because it can bring about laser actions in the short wavelength (e.g. X-ray, even $\gamma$-ray) regions[6] where lasing is impractical with conventional pumping schemes. Also, it has interesting statistical properties, such as narrower intrinsic line widths due to reduced spontaneous emission noises,[7] which could be advantageous in, for example, optical communication networks.[8] Recently a considerable amount of theoretical work has been directed towards LWI in open systems.[9−13] Yelin *et al.*[11] studied the linear gain without inversion of an open V-type system with two incoherent pump fields, coherent driving and probe fields detuned from resonance. However, the linear approximation contains no saturation terms and consequently is valid only for a weak probe field. In this paper, we give for the first time a general exact nonlinear analytical solution of the system, and discuss the difference between the linear and nonlinear solutions.

## II. MOTION EQUATIONS AND EXACT NONLINEAR ANALYTICAL SOLUTION

The semiclassical density matrix equations of motion for an open three-level V-type system with two incoherent pump fields, coherent driving and probe fields detuned from resonance (Fig.1) under the electric dipole and the rotating wave approximations can be written as[11]

$$\begin{aligned}
\mathrm{d}\rho_{aa}/\mathrm{d}t =& -(r_{a-b}+r_{a-c}+r_0)\rho_{aa} \\
&+ r\rho_{bb}+r'\rho_{cc}+\mathrm{i}(\varepsilon\rho_{ba}-\varepsilon^*\rho_{ab}), && (1a)\\
\mathrm{d}\rho_{bb}/\mathrm{d}t =& r_{a-b}\rho_{aa}-(r+r_0)\rho_{bb}+r_{b-c}\rho_{cc}+\mathrm{i}(\Omega^*\rho_{cb} \\
&- \Omega\rho_{bc})+\mathrm{i}(\varepsilon^*\rho_{ab}-\varepsilon\rho_{ba})+r_b, && (1b)\\
\mathrm{d}\rho_{cc}/\mathrm{d}t =& r_{a-c}\rho_{aa}-(r_{c-b}+r'+r_0)\rho_{cc} \\
&+ \mathrm{i}(\Omega\rho_{bc}-\Omega^*\rho_{cb})+r_c, && (1c)\\
\mathrm{d}\rho_{ab}/\mathrm{d}t =& -\Gamma_{ab}\rho_{ab}-\mathrm{i}\varepsilon(\rho_{aa}-\rho_{bb})-\mathrm{i}\Omega\rho_{ac}, && (1d)\\
\mathrm{d}\rho_{ac}/\mathrm{d}t =& -\Gamma_{ac}\rho_{ac}-\mathrm{i}\Omega^*\rho_{ab}+\mathrm{i}\varepsilon\rho_{bc}, && (1e)\\
\mathrm{d}\rho_{cb}/\mathrm{d}t =& -\Gamma_{cb}\rho_{cb}+\mathrm{i}\Omega(\rho_{bb}-\rho_{cc})-\mathrm{i}\varepsilon\rho_{ab}, && (1f)
\end{aligned}$$

along with the equations of their complex conjugates. Here $\rho_{ii}$ denotes the population of level $|i\rangle$, $\rho_{ij}$ the coherence between levels $|i\rangle$ and $|j\rangle$, $r_{i-j}$ the radiative decay from level $|i\rangle$ to $|j\rangle$, and $r_0$ the decay out of the system. The atoms are pumped into the lower levels $|b\rangle$ and $|c\rangle$ with rates $r_b$ and $r_c$. We also assume that the number of interacting atoms is constant, which means that $r_b+r_c=r_0$. $r$ and $r'$ are incoherent pump rates from $|b\rangle$ to $|a\rangle$ and $|c\rangle$ to $|a\rangle$, respectively. $\Omega$ and $\varepsilon$ are drive and probe Rabi frequencies, respectively. The $\Gamma_{ij}$'s are the complex coherence decay rates defined as

$$\begin{aligned}
&\Gamma_{ab}=r_{ab}-\mathrm{i}\Delta, \quad \Gamma_{ac}=r_{ac}+\mathrm{i}\Delta_0-\mathrm{i}\Delta, \\
&\Gamma_{cb}=r_{cb}-\mathrm{i}\Delta_0, && (2)
\end{aligned}$$

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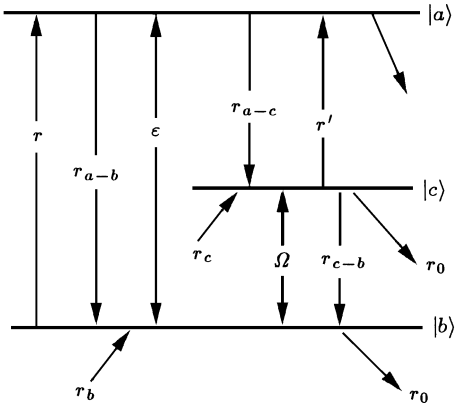



**Fig.1.** V-type LWI system with off-resonant driving and probe fields.

where $r_{ij}$'s are the coherence decay rates on the corresponding transitions $|i\rangle \to |j\rangle$, $\Delta$ $(= \omega_{ab} - \omega_p)$ and $\Delta_0 (= \omega_{cb} - \omega_d)$ are the detunings of the probe field (frequency $\omega_p$) and driving field (frequency $\omega_d$), respectively. For the convenience of calculation, $\Omega$ and $\varepsilon$ will be chosen to be real, which does not affect the generality of the following discussion. The response of the atomic system on the probe field can be expressed by the complex susceptibility

$$\chi = \chi' + \mathrm{i}\chi'' = \frac{\mu^2 N \rho_{ab}}{\varepsilon_0 \hbar \varepsilon}, \tag{3}$$

where $N$ is the atomic density, $\mu$ is the electric dipole matrix element of the optical transition $|a\rangle \to |b\rangle$, $\rho_{ab}$ is the corresponding element of the density matrix, and $\varepsilon_0$ is the dielectric constant in vacuum. Here $\varepsilon = \mu\hbar/E$, where $E$ is the amplitude of the probe field. The real part of the susceptibility shows the enhancement of the refractive index whereas the imaginary part gives the gain or loss in lasing without inversion. In our notation, if $\mathrm{Im}\rho_{ab} < 0$ $(\chi'' < 0)$ the system exhibits gain for the probe field, that is, lasing can be established on the transition $|b\rangle \leftrightarrow |a\rangle$; if $\mathrm{Im}\rho_{ab} > 0$ $(\chi'' > 0)$, the probe laser field is attenuated. Furthermore, the dispersion is determined by $\mathrm{Re}\rho_{ab}$. $\mathrm{Re}\rho_{ab} > 0$ $(\chi' > 0)$ corresponds to the redshift of the frequency of the probe field, and $\mathrm{Re}\rho_{ab} < 0$ $(\chi' < 0)$ to the blueshift.[14]

We solve Eqs.(1) in the steady state and obtain the following general exact nonlinear analytical solutions for the polarization $\rho_{ab}$ and population differences $\rho_{aa} - \rho_{bb}$ and $\rho_{cc} - \rho_{bb}$ corresponding to the transitions coupled by the probe and driving fields, respectively:

$$\mathrm{Im}\rho_{ab} = -\frac{\varepsilon(D_2\varepsilon^2 + D_0)}{\Psi}, \tag{4a}$$

$$\mathrm{Re}\rho_{ab} = -\frac{\varepsilon(F_2\varepsilon^2 + F_0)}{\Psi}, \tag{4b}$$

$$\rho_{aa} - \rho_{bb} = \frac{M_4\varepsilon^4 + M_2\varepsilon^2 + M_0}{\Psi}, \tag{4c}$$

$$\rho_{cc} - \rho_{bb} = \frac{R_4\varepsilon^4 + R_2\varepsilon^2 + R_0}{\Psi}, \tag{4d}$$

where

$$\Psi = r_0(A_4\varepsilon^4 + A_2\varepsilon^2 + A_0). \tag{5}$$

The expressions of $D_i$, $F_i (i = 0, 2)$ and $M_j$, $R_j$, $A_j$ $(j = 0, 2, 4)$ in these equations are given in the appendix. Combining Eq.(3) with Eqs.(4a) and (4b), we have

$$\chi' = -\frac{\mu_{ab}^2 N}{\varepsilon_0 \hbar} \frac{(F_2\varepsilon^2 + F_0)}{\Psi}, \tag{6a}$$

$$\chi'' = -\frac{\mu_{ab}^2 N}{\varepsilon_0 \hbar} \frac{(D_2\varepsilon^2 + D_0)}{\Psi}, \tag{6b}$$

Eqs.(6a) and (6b) give the nonlinear dispersion and gain (or loss) of the probe field, respectively. When $\rho_{aa} - \rho_{bb} < 0$ and $\chi'' < 0$ $(-\mathrm{Im}\rho_{ab} > 0)$, the probe field has gain without inversion.

## Ⅲ. DISCUSSION OF THE NONLINEAR ANALYTICAL SOLUTION

Eq.(6b) shows that the numerator and the denominator are second-order and fourth-order functions, respectively, of the Rabi frequency, $\varepsilon$, of the probe field. So the lasing process would be saturated as the probe laser intensity increases (in a complicated way).

When dropping all the second-order and fourth-order terms of $\varepsilon$ in the nonlinear solution, Eqs.(4), we obtain the linear solution of the system:

$$\mathrm{Im}\rho_{ab} = -\frac{\varepsilon E_0}{r_0 A_0}, \quad \mathrm{Re}\rho_{ab} = -\frac{\varepsilon F_0}{r_0 A_0}, \tag{7a}$$

$$\rho_{aa} - \rho_{bb} = \frac{M_0}{r_0 A_0}, \quad \rho_{cc} - \rho_{bb} = \frac{R_0}{r_0 A_0}. \tag{7b}$$

Similarly, from the nonlinear dispersion and gain, Eqs.(6a) and (6b), we obtain the linear dispersion and gain:

$$\chi' = \frac{\mu_{ab}^2 N}{\varepsilon_0 \hbar} \frac{F_0}{r_0 A_0},$$
$$\chi'' = \frac{\mu_{ab}^2 N}{\varepsilon_0 \hbar} \frac{D_0}{r_0 A_0}. \tag{8}$$

The second equation in (8) corresponds to the linear gain expression (6) in Ref.[11] and the latter is obtained by an approximation method. Comparing Eqs.(7b) and (8) with Eqs. (4c), (4d) and (6), considering simultaneously the expressions of $D_i$, $E_i (i = 0, 2)$ and $M_j$, $R_j$, $A_j$ $(j = 0, 2, 4)$, it is easy to find that there are two important differences between

the analytical linear and nonlinear solutions of the system. Firstly, in the linear solution case, the gain, dispersion and population differences depend only on the even (2,4,6)-order terms of the Rabi frequency of the driving field (RFDF) and are not related to the probe field. However, in the nonlinear solution case, these quantities depend on the even (2,4,6)-order and odd (1,3)-order terms of RFDF and the even (2,4)-order terms of the Rabi frequency of the probe field (RFPF). Moreover, the even (2,4)-order terms of RFPF include the cross terms of the driving and probe fields. This means that only the nonlinear solution can simultaneously reflect the effect of the driving field, the probe field and the interaction of the driving and probe fields on the system. Secondly, the nonlinear solution predicts that the gain would be saturated as the probe laser intensity increases, and this will turn out to be important for the stability of the system.

Now we analyse the conditions of the gain without inversion of the nonlinear solution in two special cases.

(1) The strong driving field limit: $\Omega >> \varepsilon, r_{ij}, r_{i-j}, r_0, r_b, r_c$

In this case, all odd-order terms of RFDF, $\Omega$, are zero in Eqs.(4)–(6). We find that the gain condition is

$$\begin{aligned}&[r_{a-c}(r_{c-b}+r_0-r)+(r_{a-b}+r_0)(r'+r_{c-b}+r_0)\\&+2r_{ac}(r+r'-r_{a-b}-r_0-r_{a-c})](x+1)\\&>r_0(r+r'+2r_{a-b}+2r_0+2r_{a-c}),\end{aligned}\tag{9}$$

where $x{=}r_b/r_c$. The condition without inversion, i.e., $\rho_{aa}-\rho_{bb}<0$ is

$$r+r'<r_{a-b}+r_0+r_{a-c}.\tag{10}$$

If condition (9) is satisfied but condition (10) is not, the system may produce lasing with inversion; if both conditions (9) and (10) are simultaneously satisfied, then the system will produce lasing without inversion. In the strong driving field limit, $\rho_{cc}-\rho_{bb}$ is always negative; that is, inversion between levels $|b\rangle$ and $|c\rangle$ does not arise.

(2) The strong probe laser field limit: $\varepsilon \gg \Omega$, $r_{ij}, r_{i-j}, r_0, r_b, r_c$

In this case, the inversionless lasing conditions are

$$\begin{aligned}&[(r-s_3)(s_2+r')-r_{a-c}s_2](r_b+r_c)\\&+r_0(r_{a-c}+2r'-r+s_3)r_c<0,\end{aligned}\tag{11a}$$

$$\begin{aligned}&[s_2(s_3+r_{a-c})+r'(r+s_3)+r(r_{a-c}+s_2)\\&+2r_{ac}(r_{a-c}+2r'-r+s_3)]\Omega\\&+2[r_{ac}\Delta_0-r_{bc}(\Delta-\Delta_0)]\\&\cdot(r_{a-c}+2s_2+2r')>0,\end{aligned}\tag{11b}$$

and

$$r_{ac}\Delta_0-r_{bc}(\Delta-\Delta_0)<0;\tag{11c}$$

or

$$\begin{aligned}&[(r-s_3)(s_2+r')-r_{a-c}s_2](r_b+r_c)\\&+r_0(r_{a-c}+2r'-r+s_3)r_c>0,\end{aligned}\tag{12a}$$

$$\begin{aligned}&[s_2(s_3+r_{a-c})+r'(r+s_3)+r(r_{a-c}+s_2)\\&+2r_{ac}(r_{a-c}+2r'-r+s_3)]\Omega\\&+2[r_{ac}\Delta_0-r_{bc}(\Delta-\Delta_0)]\\&\cdot(r_{a-c}+2s_2+2r')<0,\end{aligned}\tag{12b}$$

and

$$r_{ac}\Delta_0-r_{bc}(\Delta-\Delta_0)<0.\tag{12c}$$

## IV. NUMERICAL ANALYSIS

Eqs.(4)–(8) are the basic equations for the following discussion. In order to gain a better insight into this problem, we use the numerical analysis method.

It has been shown[15] that when both probe and driving fields are resonant with the corresponding atomic transitions, in order to have LWI in a closed V system (let $r', r_0, r_b, r_c, r_{a-c}{=}0$ in Fig.1) the decay rate along the driving transition should be at least of the same order or higher as the decay along the lasing transition, i.e.,

$$r_{c-b}\geq r_{a-b}.\tag{13}$$

The condition will be naturally satisfied when the frequency of the driving field is higher than that of the lasing field (frequency down-conversion). For the specific case of frequency up-conversion, i.e. when the frequency of the probe field is much higher than that of the driving field, condition (13) is difficult to fulfil since the decay rate grows with the cube of the frequency of the corresponding transition. Yelin *et al.*[11] have proven that Eq.(13) is not a fundamental condition for LWI and that it is possible to get LWI in the frequency up-conversion regime in the system if

$$(r_{a-b}+r_{a-c})^2-r^2\leq rr_{a-c},\quad(\Delta_0=0)\tag{14a}$$

or

$$r>r_{a-b},\quad(\Delta_0\neq 0).\tag{14b}$$

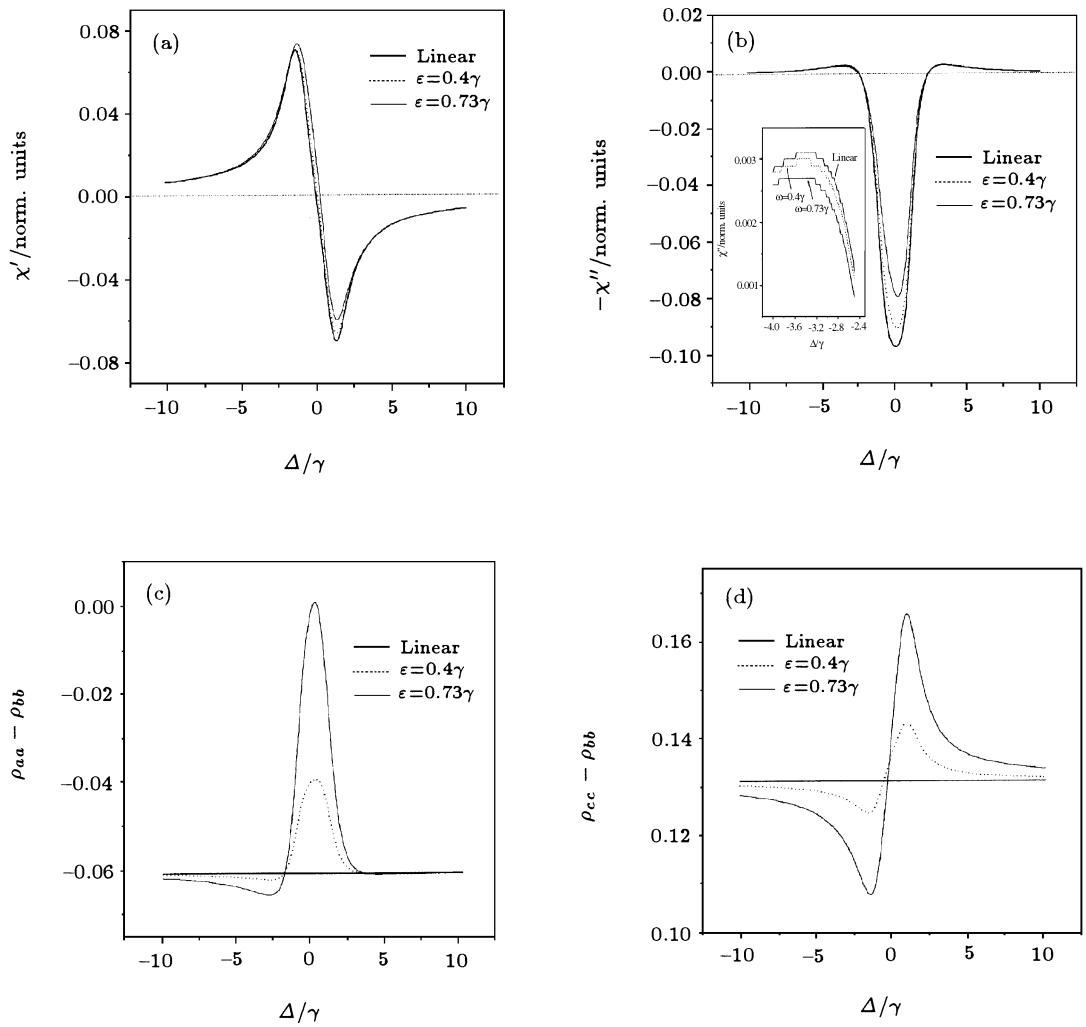



**Fig.2.** (a) $\chi'$, (b) $-\chi''$, (c) $\rho_{aa}-\rho_{bb}$ and (d) $\rho_{cc}-\rho_{bb}$ as functions of $\Delta$. The parameter values are $r_{ab}=r_{ac}=r_{bc}=\gamma$, $r_{a-c}=\gamma$, $r_b=0.1\gamma$, $r_c=0.3\gamma$, $r_{c-b}=0.05\gamma$, $r'=0.2\gamma$, $r_0=0.4\gamma$, $r_{a-b}=0.2\gamma$, $\Omega=\gamma$, $r=\gamma$, $\Delta_0=0$. The small figure on the left-hand side in Fig.2(b) is the amplification of the curves $-\chi''$ in the region from $\Delta/\gamma=-4.0$ to $\Delta/\gamma=-2.4$.

Especially, it is much easier to fulfil condition (14b) than (14a), particularly in the practically interesting case $r_{a-b}<r_{a-c}$. However, it is very difficult to give a simple explicit condition for getting LWI in the frequency up-conversion regime from the analytical nonlinear solution, Eqs.(4). So, now we compare the numerical results obtained from the analytical linear and nonlinear solutions using parameter values satisfying conditions (14a) and (14b). We also find the difference between both solutions; that is, the effects of the nonlinearities on the gain, dispersion and population differences.

We first examine the effect of nonlinearities on the gain, dispersion and population differences of the system when the probe field is weaker.

When the driving field is resonant ($\Delta_0$=0), the real and imaginary parts, $\chi'$ and $\chi''$, of the resulting susceptibility and the population differences, $\rho_{aa}-\rho_{bb}$ and $\rho_{cc}-\rho_{bb}$, as functions of the probe field detuning, $\Delta$, are illustrated in Fig.2. In Fig.2 and later related figures, $\chi'$ and $\chi''$ are plotted in normalized units of $\mu_{ab}^2N/(\varepsilon_0\hbar\gamma)$. In order to express more directly the gain and absorption of the probe field, we mark the vertical axis by $-\chi''$ in Fig.2(b) and later relative figures. From Fig.2 (a) we find that: the positive (redshift) maxima of the dispersion, $\chi'$, increase with increasing $\varepsilon$ and the linear analysis corresponds to the smallest gain; the negative (blueshift) maxima of $\chi'$ decrease with increasing $\varepsilon$, and the linear analysis corresponds to the biggest gain; the detuning of the zero dispersion shifts with $\varepsilon$; the high index of refraction does not correspond to the zero absorption (compared with Fig.2(b)). Some of these characters are very different from those in the $\Lambda$ system with

a coherent resonant driving field and a coherent off-resonant probe field.[16] In the latter case, both the positive and negative maxima of $\chi'$ decrease with increasing $\varepsilon$, the linear analysis gives the biggest gain, and there is a high index of refraction without absorption. Fig.2(b) shows that both the inversionless ($\rho_{aa} - \rho_{bb} < 0$) gain and absorption decrease with $\varepsilon$ increasing, the linear analysis gives the biggest gain, i.e. the nonlinearity does lead to the gain saturation (even though it is not evident here, when the probe field is stronger, e.g. in Fig.4(b), this phenomenon is considerable). We see that the effect of the nonlinearity on the maxima of absorption and dispersion is bigger than on the gain. Figs.2(a) and (b) tell us that the probe field detuning is a key parameter determining gain (absorption) and dispersion behaviour in both linear and nonlinear cases. Figs.2(c) and (d) indicate that the nonlinearity has a much larger effect on population differences, $\rho_{aa} - \rho_{bb}$ and $\rho_{cc} - \rho_{bb}$ than on dispersion and gain, $\chi'$ and $\chi''$. In the case of linear solution, $\rho_{aa} - \rho_{bb}$ and $\rho_{cc} - \rho_{bb}$ keep, respectively, a constant value with $\Delta$ increasing although the linear expression, (7b), of $\rho_{aa} - \rho_{bb}$ and $\rho_{cc} - \rho_{bb}$ does include parameter $\Delta$. Numerical calculations show that this conclusion is always correct for any parameter value of the system. The reason for this is that when $\Delta$ increases, the numerator and denominator of the expression of $\rho_{aa} - \rho_{bb}(\rho_{cc} - \rho_{bb})$ vary, respectively, but their ratio is invariable. Because the population differences are not related to the Rabi frequency of the probe field in the linear case (refer to Eq.(7b)), so now we can say that the probe field (including its amplitude and frequency) has no effect on the population distribution between levels coupled by it in the linear case. In the nonlinear case, the values of $\rho_{aa} - \rho_{bb}$ ($\rho_{cc} - \rho_{bb}$) oscillate obviously around the constant value of the linear solution with increasing $\Delta$. Moreover, the amplitudes of the oscillation increase with increasing $\varepsilon$.

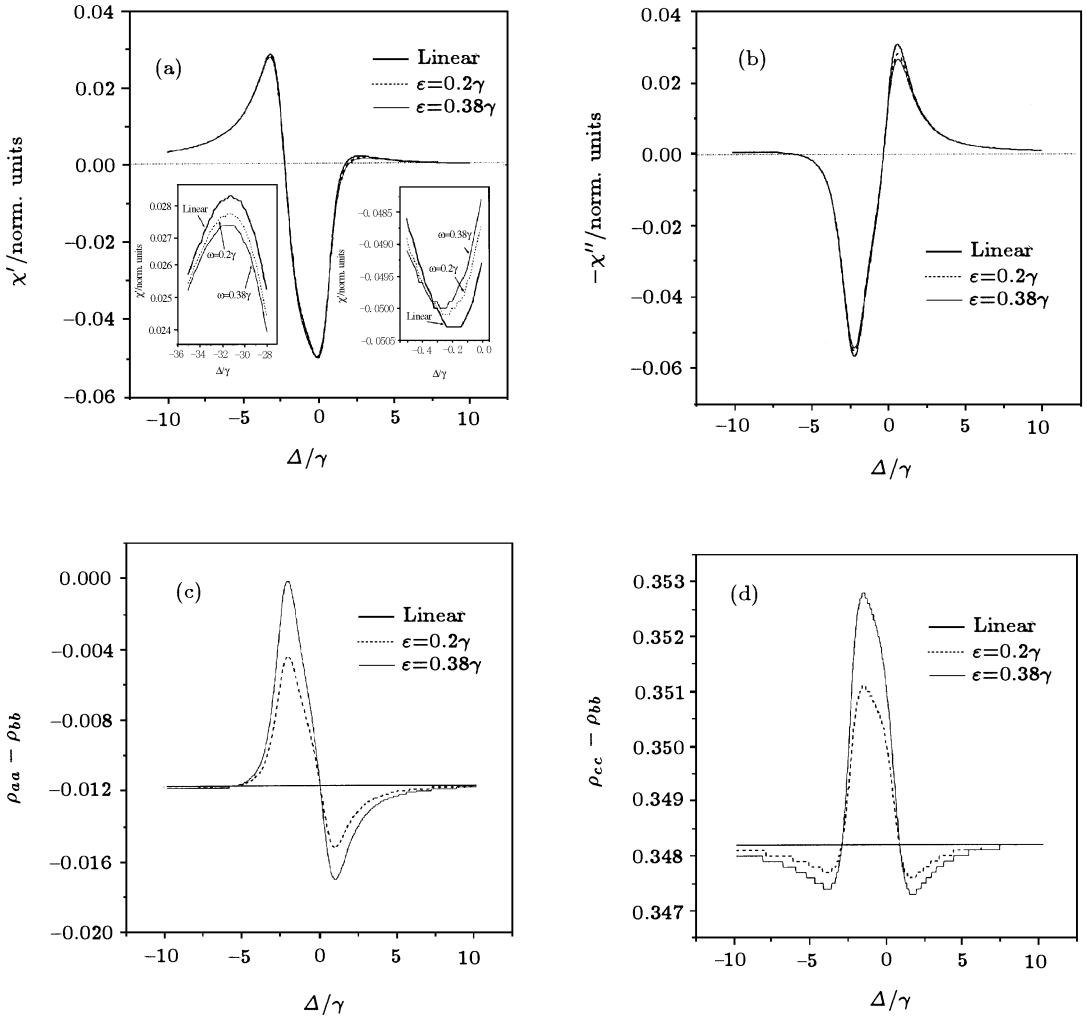



**Fig.3.** (a) $\chi'$, (b) $-\chi''$, (c) $\rho_{aa} - \rho_{bb}$ and (d) $\rho_{cc} - \rho_{bb}$ as functions of $\Delta$ with the same parameter values as those in Fig.2 except $\Delta_0 = -2$.

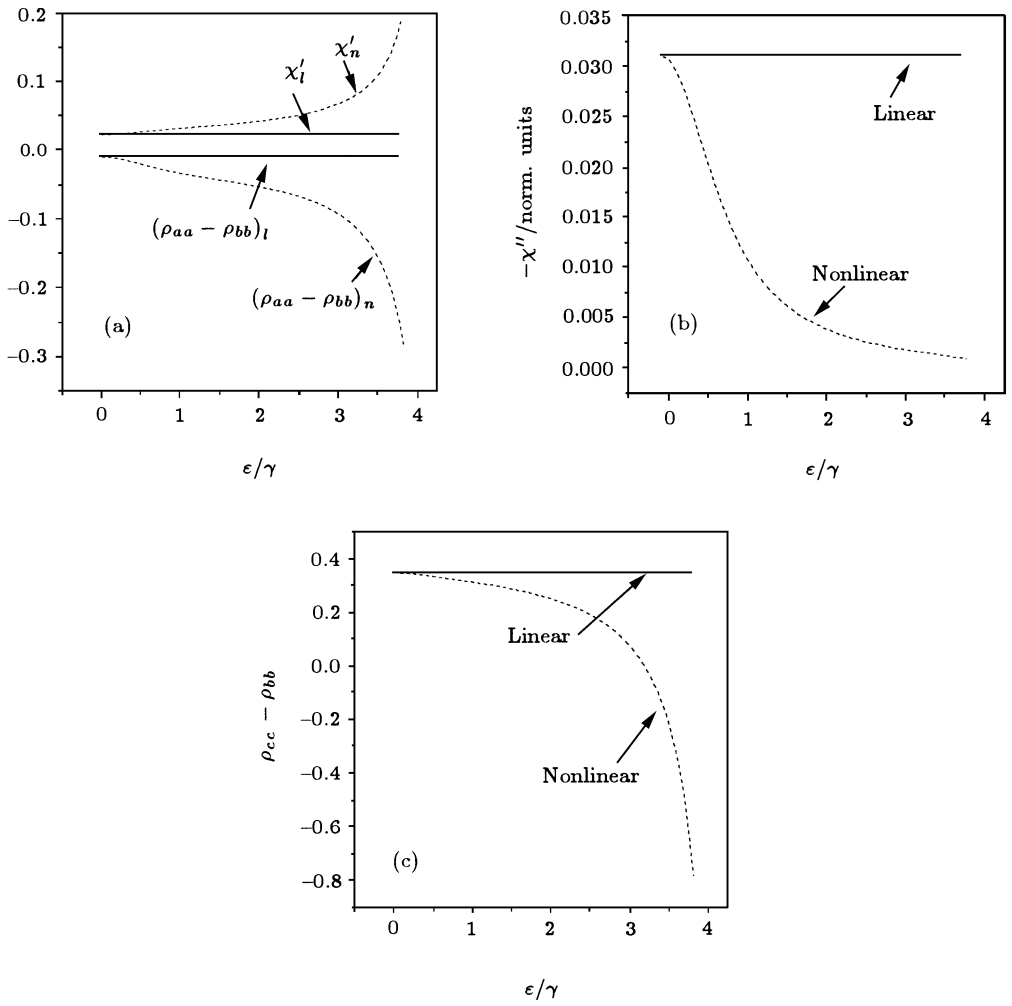



**Fig.4.** (a) $\chi'$ and $\rho_{aa}-\rho_{bb}$, (b) $-\chi''$, (c) $\rho_{cc}-\rho_{bb}$ as functions of $\varepsilon$ with the same parameter values as those in Fig.2 except $\Delta=-0.8$ and $\Delta_0=0.2$. The subscripts '$l$' and '$n$' represent 'linear' and 'nonlinear', respectively.

Figure 3 illustrates the gain, dispersion and population differences $(\rho_{aa}-\rho_{bb})$ as functions of the probe field detuning $\Delta$ when the driving field detuning exists $(\Delta_0=-2)$, other parameter values being the same as those in Fig.2. Different from that in the case $\Delta_0$=0, the redshift maxima also decrease with increasing $\varepsilon$ (see Fig.3(a)). The variation rules of maxima of the blueshift, $\chi''$ and population differences are similar to those in the case of $\Delta_0$=0, but the curves of the $\chi'$, $\chi''$ and population differences are very different; the largest value of $\varepsilon$ remaining inversionless is evidently smaller than that in the case of $\Delta_0$=0. This indicates that the effect of the driving field detuning on gain (absorption), dispersion and population distribution is considerable. In contrast to the case of $\Delta_0$=0, the nonlinearity has a larger effect on the maxima of gain than on the maxima of absorption and dispersion.

With the Rabi frequency (or amplitude) of the probe field increasing, the quantitative difference between the linear and nonlinear gain, dispersion and population differences will more obviously be seen. Fig.4 plots $\chi'$, $\chi''$, $\rho_{aa}-\rho_{bb}$ and $\rho_{cc}-\rho_{bb}$ as functions of $\varepsilon$. It shows that in the nonlinear case, with $\varepsilon$ increasing, the gain, $\rho_{aa}-\rho_{bb}$ and $\rho_{cc}-\rho_{bb}$ decrease while dispersion (redshift) increases. However, in the linear case, the gain, dispersion, $\rho_{aa}-\rho_{bb}$ and $\rho_{cc}-\rho_{bb}$ remain unvaried when $\varepsilon$ increases. Moreover, the linear analysis gives minimum of dispersion, maxima of gain and population differences. So when the probe laser field is stronger, to obtain a precise result of gain, dispersion and population differences of the system, we must use the nonlinear solution instead of the linear solution.

## V. CONCLUSION

To summarize, we give a general exact analytical

nonlinear solution of an open V-type system with two incoherent pump fields, coherent off-resonant driving and probe fields, and present the conditions of the gain without inversion in both the strong driving field limit and the strong probe field limit cases. We make a qualitative and quantitative comparison of the gain, dispersion and population differences in both the linear and nonlinear cases. We find that: (1) the nonlinear gain, dispersion and population differences depend on the even (2,4,6)-order and odd (1,3)-order terms of RFDF and the even (2,4)-order terms of RFPF. Moreover, the even (2,4)-order terms of RFPF include the cross terms of the driving and probe fields, but the linear ones depend only on the even (2,4,6)-order terms of RFDF and are not related to the probe field. This means that only the nonlinear solution can simultaneously reflect the effect of the driving field, the probe field and the interaction of the driving and probe fields on the system. (2) With RFPF increasing, the nonlinear maxima of the inversionless gain, absorption and blueshift decrease always, the linear analysis corresponding to the biggest ones; the nonlinear redshift maxima increase or decrease depending on the driving field detuning and the corresponding linear analysis gives the smallest or biggest one. (3) When the probe field is weaker, if the driving field is resonant, nonlinearity has a bigger effect on maxima of absorption and dispersion than gain, and the case is reversed if the driving field is off-resonant. (4) With the probe field detuning increasing, the linear population difference remains constant but the nonlinear population difference shows an obvious oscillation around the linear value. (5) The higher Rabi frequency (or amplitude) of the probe field, the more obvious the quantitative difference between the linear and nonlinear gain, dispersion and population differences.

Thus from the physical viewpoint, we must use the nonlinear solution to obtain the gain, dispersion and population differences of the system regardless of the probe field being weaker or stronger. Although from the mathematical viewpoint, only when the probe field is stronger and for obtaining precise results, do we need to use the nonlinear solution instead of the linear.

## APPENDIX A

The expressions of $D_i$, $F_i(i=0,2)$ and $M_j$, $R_j$ $A_j$ $(j=0,2,4)$ in Eqs.(4) and (5) are given in the following:

$$
\begin{aligned}
&D_2 = (e_{12}r_b + d_{12}r_c)\Omega,\\
&D_0 = (e_4 r_b + d_4 r_c)\Omega^4 + (e_2 r_b + d_{24} r_c)\Omega^2 + (e_0 r_b + d_0 r_c);\\
&F_2 = (f_{32}r_b + h_{32}r_c)\Omega^3 + (f_{22}r_b + h_{22}r_c)\Omega^2 + (f_{12}r_b + h_{12}r_c)\Omega,\\
&F_0 = (f_4 r_b + h_4 r_c)\Omega^4 + (f_{20} r_b + h_{20} r_c)\Omega^2 + (f_0 r_b + h_0 r_c);\\
&M_4 = (k_{24}r_b + m_{24}r_c)\Omega^2,\\
&M_2 = (k_{42}r_b + m_{42}r_c)\Omega^4 + (k_{32}r_b + m_{32}r_c)\Omega^3 + (k_{22}r_b + m_{22}r_c)\Omega^2,\\
&M_0 = (k_{40}r_b + m_{40}r_c)\Omega^4 + (k_{20}r_b + m_{20}r_c)\Omega^2 + (k_0 r_b + m_0 r_c);\\
&R_4 = (l_{24}r_b + n_{24}r_c)\Omega^2 + (l_{14}r_b + n_{14}r_c)\Omega,\\
&R_2 = (l_{32}r_b + n_{32}r_c)\Omega^3 + (l_{22}r_b + n_{22}r_c)\Omega^2 + (l_{12}r_b + n_{12}r_c)\Omega + (l_{02}r_b + n_{02}r_c),\\
&R_0 = (l_4 r_b + n_4 r_c)\Omega^4 + (l_{20} r_b + n_{20} r_c)\Omega^2 + (l_{00} r_b + n_{00} r_c);\\
&A_4 = \alpha_{24}\Omega^2 + \alpha_{14}\Omega,\\
&A_2 = \alpha_{42}\Omega^4 + \alpha_{32}\Omega^3 + \alpha_{22}\Omega^2 + \alpha_{12}\Omega + \alpha_0,\\
&A_0 = \alpha_6\Omega^6 + \alpha_{40}\Omega^4 + \alpha_{20}\Omega^2 + \alpha_{00};
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_6 = 2r_{bc}t_0, && \alpha_{42} = 2r_{bc}[6r_{ac} - 2r_{a-c} - (r' + r_0 + r_{c-b})],\\
&\alpha_{40} = 4r_{bc}t_0c_0 + c_4t_1, && \alpha_{32} = 2(t_8c_1 - t_0c_2), && \alpha_{24} = -t_1 + 2r_{ac}t_8,\\
&\alpha_{22} = 2[r_{ac}t_7c_4 - (s_2 + r' + 2r_{a-c})(r_{bc}c_0 + \Delta_0 c_2) + 6r_{ab}r_{bc}c_5],\\
&\alpha_{20} = 2t_1c_0c_4 + 2r_{bc}t_0c_5c_6, && \alpha_{14} = -2c_1t_7, && \alpha_{12} = 2t_8c_3c_5,\\
&\alpha_{02} = \alpha_{00} = 2c_4c_5r_{ab}t_7, && \alpha_{00} = c_4c_5c_6t_1;\\
&e_4 = r_{bc}(t_5 + 2r_{ac}t_2), && e_2 = t_5(\Delta_0 c_2 + r_{bc}c_0) + 2r_{ab}r_{bc}c_5t_2 + r_{ac}c_4t_3,\\
&e_{12} = -t_3c_1, && e_0 = r_{ab}c_5c_4t_3;\\
&d_4 = r_{bc}(2r_{ac}t_2 - r_0t_0 + t_5), && d_2 = 2r_{bc}r_{ab}c_5t_2 + r_{ac}c_4t_4 - (r_{bc}c_0 + \Delta_0 c_2)(r_0t_0 - t_5),\\
&d_{12} = -t_4c_1, && d_0 = r_{ab}c_5c_4t_4;\\
&f_4 = \Delta_0 t_5 + 2r_{bc}Dt_2, && f_{32} = t_5 + 2r_{ac}t_2, && f_{22} = -2c_1t_6,\\
&f_{20} = Dc_4t_3 - r_{bc}c_2(t_5 + 2r_{ac}t_2) + 2r_{bc}Dc_0t_2 + \Delta_0 c_0t_5, && && f_{12} = t_3(r_{bc}r_{ac} + \Delta_0 D),
\end{aligned}
$$

$$f_0 = -\Delta c_4 c_5 t_3;$$
$$h_4 = \Delta_0 t_5 + 2r_{bc} D t_2 - \Delta_0 r_0 t_0, \quad h_{32} = t_5 + 2r_{bc} t_2 - r_0 t_0, \quad h_{22} = -2c_1(3r_0 + t_6),$$
$$h_{20} = -2r_{bc}\Delta c_5 t_2 + (r_{bc} c_2 - \Delta_0 c_0)(r_0 t_0 - t_5) + D c_4 t_4, \quad h_{12} = (r_{bc} r_{ac} + \Delta_0 D) t_4,$$
$$h_0 = -\Delta c_4 c_5 t_4;$$
$$k_6 = 2r_{bc} t_2, \quad k_{42} = 2r_{bc} t_6, \quad k_{40} = t_3 c_4 + 4r_{bc} t_2 c_0,$$
$$k_{32} = -2t_2 c_2, \quad k_{24} = -t_3, \quad k_{22} = 2(\Delta_0 c_2 + r_{bc} c_0) t_6$$
$$k_{20} = 2c_4 c_0 t_3 + 2r_{bc} t_2 c_5 c_6, \quad k_0 = c_4 c_5 c_6 t_3;$$
$$m_6 = k_6, \quad m_{42} = 2r_{bc}(3r_0 + t_6), \quad m_{40} = c_4 t_4 + 4r_{bc} c_0 t_2,$$
$$m_{32} = k_{32}, \quad m_{24} = -t_4, \quad m_{20} = 2c_4 c_0 t_4 + 2r_{bc} c_5 c_6 t_2,$$
$$m_{22} = 2(3r_0 + t^6)(r_{bc} c_0 + \Delta_0 c_2), \quad m_0 = c_4 c_5 c_6 t_4;$$
$$l_4 = -c_4 t_5, \quad l_{32} = -2c_1 t_2, \quad l_{24} = t_5 + 2r_{ac} t_2,$$
$$l_{22} = 2r_{ac} c_4 t_6, \quad l_{20} = -2c_4 c_0 t_5, \quad l_{14} = -2c_1 t_6,$$
$$l_{12} = -c_3 c_5 t_2, \quad l_{00} = -c_4 c_5 c_6 t_5;$$
$$n_4 = c_4(r_0 t_0 - t_5), \quad n_{32} = l_{32}, \quad n_{24} = -t_0 r_0 + 2r_{ac} t_2 + t_5,$$
$$n_{22} = 2r_{ac} c_4(3r_0 + t_6), \quad n_{20} = 2c_4 c_0(t_0 r_0 - t_5), \quad n_{14} = -2c_1(3r_0 + t_6),$$
$$n_{12} = -2c_5 c_3 t_2, \quad n_{02} = 2r_{ab}(3r_0 + t_6), \quad n_{00} = c_5 c_4 c_6(r_0 t_0 - t_5).$$

In the above expressions, $D$, $s_2$, $s_3$, $c_i$ $(i = 0, 1, ..., 5)$ and $t_j$ $(j = 0, 1, ..., 8)$ are defined as follows:

$$D = \Delta - \Delta_0, \quad s_3 = r_{a-b} + r_0, \quad s_2 = r_{c-b} + r_0,$$
$$c_0 = r_{ab} r_{ac} - \Delta D, \quad c_1 = r_{ac}\Delta_0 - r_{bc} D, \quad c_2 = \Delta r_{ac} + r_{ab} D,$$
$$c_3 = r_{bc}\Delta + r_{ab}\Delta_0, \quad c_4 = \Delta_0^2 + r_{bc}^2, \quad c_5 = D^2 + r_{ac}^2,$$
$$c_6 = \Delta^2 + r_{ab}^2, \quad t_0 = r + r' + 2s_3 + 2r_{a-c},$$
$$t_1 = s_2(s_3 + r_{a-c}) + r'(r + s_3) + r(r_{a-c} + s_2), \quad t_2 = r + r' - s_3 - r_{a-c},$$
$$t_3 = (r - s_3)(s_2 + r') - r_{a-c} s_2, \quad t_4 = r'(2r_0 + r - s_3) + (r - s_3 - r_{a-c})(s_2 - r_0),$$
$$t_5 = r_{a-c}(s_2 - r) + s_3(r' + s_2), \quad t_6 = r_{a-c} - r' - s_2, \quad t_7 = r_{a-c} + 2s_2 + 2r',$$
$$t_8 = r_{a-c} + 2r' - r + s_3.$$

# References


[1] Kocharovskaya O and Khanin Ya I 1988 *JETP Lett.* **48** 630;
Harris S E 1989 *Phys. Rev. Lett.* **62** 1033;
Scully M O, Zhu S Y and Gavrielides A 1989 *Phys. Rev. Lett.* **62** 2813

[2] Asdpect A, Arimondo E and Kaiser R *et al* 1988 *Phys. Rev. Lett.* **61** 826;
Asdpect A, Arimondo E and Kaiser R *et al* 1989 *J. Opt. Soc. Am.* B **6** 2112

[3] Scully M O and Fleischhauer M 1992 *Phys. Rev. Lett.* **69** 1360

[4] Harris S E, Field J E and Imamouglu A 1990 *Phys. Rev. Lett.* **64** 1032;
Boller K J, Imamouglu A and Harris S E 1991 *Phys. Rev. Lett.* **66** 2593;
Field J E, Hahn K H and Harris S E 1991 *Phys. Rev. Lett.* **67** 3062

[5] For reviews, see Kocharovskaya O 1992 *Phys. Rep.* **219** 175;
Scully M O 1994 *Quantum Opt.* **6** 203;
Arimondo E 1996 *Progress in Optics XXXV* ed E Wolf (Amsterdam: Elsevier Science) p 257;
Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge Univesity Press); and also Ref.[6]

[6] Mandel P 1993 *Contemporary Physics* **34** 235
Kocharovskaya O 1997 *Hyperfine Interactions* **107** 187

[7] Agarwal G S 1991 *Phys. Rev. Lett.* **67** 980

[8] Bogar P and Bergou J A 1997 *Phys. Rev.* A **56** 1021

[9] Zhu S Y, Wang D Z and Gao J Y 1997 *Phys. Rev.* A **55** 1339

[10] Manassah J T and Gross B 1998 *Opt. Commun.* **148** 404

[11] Yelin S F, Lukin M D, Scully M O and Mandel P 1998 *Phys. Rev.* A **57** 3858

[12] Hu X M and Peng J S 1998 *Acta Phys. Sin.* **48** 1632 (in Chinese)

[13] Fan X J and Tian S F *et al* 1998 *OSA Annual Meeting/ ILS-XIV, TuQ19 (Baltimore, USA)*;
Fan X J and Tian S F *et al*1999 *OSA Annual Meeting/ ILS-XV, ThCC4 (Santa Clara, USA)*;
Fan X J and Tian S F *et al*1999 *Chin. J. Laser* **26** 609 (in Chinese);
Fan X J and Tian S F *et al*2000 *J. Optoelectron. Laser* **11** 101;
Fan X J and Tian S F *et al*2000 *Acta Phys. Sin.* **49** 1719 (in Chinese);
Fan X J and Tian S F *et al*2000 *Acta Photon. Sin.* 29 492;
Tian S F and Fan X J 2000 *OSA Annual Meeting/ ILS-XVI ThU28, ThU29 (Providence, USA)*

[14] Sargent III M, Scully M O and Lamb W L 1974 *Laser Physics* (New York: Addison-Wesley)

[15] Lukin M D, Scully M O and Welch G R *et al* 1996 *Laser Phys.* **6** 436

[16] Rathe U, Freischhauer M and Zhu S Y *et al* 1993 *Phys. Rev.* A **47** 4994