Vol 10 No 7, July 2001
1009-1963/2001/10(07)/0606-05

Chinese Physics

© 2001 Chin. Phys. Soc.
and IOP Publishing Ltd

SYNCHRONIZATION FOR A CLASS OF CHAOTIC SYSTEMS BASED UPON OBSERVER THEORY

Liu Feng(刘 锋)[a], Ren Yong(任 勇)[a],
Shan Xiu-ming(山秀明)[a], and Qiu Zu-lian(邱祖廉)[b]
[a] *Department of Electronics Engineering, Tsinghua University, Beijing* 100084, *China*
[b] *Department of Automatic Control, Xi'an Jiaotong University, Xi'an* 710049, *China*

(Received 17 October 2000; revised manuscript received 16 January 2001)

A new synchronization theorem for a class of chaotic systems is presented based on nonlinear observer theory. We take the first state variable of the drive system as the driving scalar signal. Its linear feedback gain is a function of a free parameter. It is proven that global synchronization can be attained through simple linear output error feedback. This approach is illustrated by the WGY hyperchaotic system and Chua's oscillator.

Keywords: synchronization, chaos, observer
PACC: 0545

I. INTRODUCTION

Since chaos synchronization was presented for the first time in 1990,[1] due to its promising applications in secure communication, information and biotic science, there has been considerable interest in it and its practical applications.

In most synchronization schemes, the driving signal can be taken as one of the state variables of the drive system.[2] The driving scalar signal is used to synchronize the response system in the sense that the trajectory of the response system tracks the trajectory of the drive system. The synchronization stability is proven by numerically computing the conditional Lyapunov exponents of the response system[3] or finding a Lyapunov function.[4]

Because the chaotic system is a nonlinear system and synchronization means that the response system tracks the drive system, we can use the observer theory of nonlinear systems to synchronize chaotic systems.[5]

Our main result is that, under certain technical assumptions (such as Lipschitz functions and observability), an exponential synchronization rule can be built by linear output error feedback for a class of chaotic systems, based upon nonlinear observer theory. The synchronization error can be made to have arbitrary exponential decay and to be particularly simple.

Our work is organized as follows. Section II describes the problem to be discussed and gives a linear output feedback synchronization controller. The stability of synchronization is proven in section III. Some simulation results are given in section IV, and the conclusion is given in section V.

II. DESCRIPTION OF THE PROBLEM AND CONTROLLER DESIGN

We consider the drive system described by the following state equations:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + F(\boldsymbol{x}), \;\; y = C\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_n)^{\mathrm{T}} \in R^n$ is the state vector, $y \in R$ is the output, A is a constant matrix. Thus we have

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \in R^{n\times n},$$

$$F(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix} \in R^n \tag{2}$$

$$C = (1\ 0\ \cdots\ 0) \in R^n. \tag{3}$$

In our synchronization scheme, we use a linear output error feedback approach to synchronize the drive and response systems. We take the first state

variable x_1 as the driving scalar signal. The state equation of the response system is

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + F(\hat{\boldsymbol{x}}) + K(y - \hat{y}),\ \ \hat{y} = C\hat{\boldsymbol{x}}, \qquad (4)$$

where $\hat{\boldsymbol{x}} \in R^n$ is the state variable , $\hat{y} \in R$ is the output of the response system and $K = (k_1, k_2, ..., k_n)^T \in R^n$ is the feedback gain.

In the system equations (1) and (4), we make some assumptions as follows:

A1: For matrix A, $a_{i,i+1} \neq 0 (i = 1, ..., n-1)$, and $a_{ij} = 0 (j > i+1)$.

A2: Denote $\boldsymbol{x}_i$ for the vector $(x_1 ... x_i 0 ... 0)^T \in R^n$, and require every element $f_i(\underline{\boldsymbol{x}}_i) = f_i(x_i, ..., x_n) (i = 1, ..., n)$ in $F(\boldsymbol{x})$ satisfies the Lipschitz condition.

Assumption A1 guarantees the observability of (C, A) and is more general than that of Ref.[6]. In Ref.[6], the form of matrix A is more strict. Assumption A2 is a general condition in most nonlinear systems. If $f_i(\boldsymbol{x}_i)$ is not global, it should be at least satisfied locally almost everywhere. For nonlinear Lur'e systems, the form of $F(\boldsymbol{x})$ and the Lipschitz condition are satisfied. Also assumption A2 is not strict for most chaotic systems.

The design procedure of K is stated as follows:

(1) Construct a constant lower-triangular matrix $T = (t_{ij}) \in R^{n\times n}$, where

$$t_{11} = 1, \quad t_{kl} = \sum_{j=l-1}^{k-1} t_{k-1,j} a_{jl},$$
$$k = 2, 3, ..., n, \quad l = 1, 2, ..., k \qquad (5)$$

and denote $t_{i0} = a_{0i} = 0 (i = 1, 2, ..., n)$.

(2) For a given positive constant θ, calculate the feedback gain

$$K = T^{-1} P^{-1}(\theta) C^T, \qquad (6)$$

where

$$P(\theta) = (p_{ij}(\theta)) \in R^{n\times n},$$
$$\text{and } p_{ij}(\theta) = \frac{(-1)^{i+j} C_{i+j-2}^{j-1}}{\theta^{i+j-1}},$$
$$1 \leq i,\ j \leq n, \quad C_n^i = \frac{n!}{(n-i)!i!}. \qquad (7)$$

Remark: (1) Since $t_{11} = 1 \neq 0$, $t_{ii} = t_{i-1,i-1}\ a_{i-1,i} \neq 0$ (i=2,3,...,n), T is a nonsingular matrix, so T has inverse matrix T^{-1}. T is a lower-triangular matrix, as is T^{-1}.

(2) It is clear that $P(\theta) = \theta \cdot \mathrm{diag}\left(\frac{1}{\theta}, \frac{1}{\theta^2}, \cdots, \frac{1}{\theta^n}\right) \cdot P(1) \cdot \mathrm{diag}\left(\frac{1}{\theta}, \frac{1}{\theta^2}, ..., \frac{1}{\theta^n}\right)$, and if $P(\theta)$ is nonsingular, its inverse matrix is $P^{-1}(\theta) = \theta^{-1} \cdot \mathrm{diag}(\theta, \theta^2, ..., \theta^n) \cdot P^{-1}(1) \cdot \mathrm{diag}(\theta, \theta^2, ..., \theta^n)$.

Ⅲ. PROOF OF THE MAIN THEOREM

Now, we give our main theorem as follows. Firstly, we give two lemmas.

Lemma 1 For matrices A_0 and C, there exists a symmetric positive definite matrix $P(\theta) = (p_{ij}(\theta)) \in R^{n\times n}$ which satisfies the equation

$$\theta P + A_0^T P + P A_0 - C^T C = 0 \qquad (8)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \in R^{n\times n} \qquad (9)$$

Moreover, $p_{ij}(\theta)$ satisfies Eq.(7).

A direct computation leads to the following Lemma 2, which is useful for the proof of theorem 1.

Lemma 2 For matrix A in Eq.(2), and T in Eq.(5), there exist constants $a_1, a_2, ..., a_n \in R$ that satisfy the following equation

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2, & \cdots & \cdots & a_n \end{bmatrix}. \qquad (10)$$

Then we consider, when $A = A_0$, the feedback gain K of system (4). Gauthier[6] constructed an observer of system (1) by a linear output feedback, when $A = A_0$. We state their theorem 3 in Ref.[6] as our theorem 1.

Theorem 1 For systems (1) and (4), when $A = A_0$ and satisfies A2, there exists $\theta_0 > 0$, when $\theta > \theta_0$ and $K = S^{-1}(\theta) C^T$. Then system (4) is an observer of system (4), and for θ large enough, we have $\left\|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\right|| \leq k(\theta) \exp(-\theta t/3)\ \left\|\boldsymbol{x}(0) - \hat{\boldsymbol{x}}(0)\right\|$, where $k(\theta)$ is a constant about θ.

Then for the general form of A, we have the following theorem.

Theorem 2 For systems (1) and (4), which satisfy A1 and A2, there exists $\theta_0 > 0$, when $\theta > \theta_0$ and

$K = T^{-1}P^{-1}(\theta)C^T$. Then the drive system (1) and response system (4) are globally asymptotic synchronized in the sense that $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\| = 0$.

Proof: We consider the coordinate transformation $\boldsymbol{z} = T\boldsymbol{x}$, where $\boldsymbol{z} = (z_1, z_2, ..., z_n)^T \in R^n$. System (1) becomes

$$\dot{\boldsymbol{z}} = TAT^{-1}\boldsymbol{z} + TF(T^{-1}\boldsymbol{z}), y = CT^{-1}\boldsymbol{z} = C\boldsymbol{z}. \quad (11)$$

Using Eq.(10), we have

$$\dot{\boldsymbol{z}} = A_0\boldsymbol{z} + \left(0...0\sum_{i=1}^{n} a_i z_i\right)^T + TF(T^{-1}\boldsymbol{z}), y = C\boldsymbol{z}. \quad (12)$$

For sake of simplicity, we denote the lower-triangular matrix as $T^{-1} = (r_{ij}) \in R^{n\times n}, i, j = 1, 2, ..., n$.

Obviously the term $\sum_{j=1}^{i} t_{ji} f_j\left(\sum_{j=1}^{i} r_{ji} z_j\right)$, the ith element of $TF(T^{-1}\boldsymbol{z})$, a function of $z_1, ..., z_i$, can be denoted as $f_i'(\boldsymbol{z}_i)$. Denoting l_i as the Lipschitz constant of $f_i(\underline{\boldsymbol{x}}_i)$, then we have

$$\begin{aligned}
&\|f_i'(\underline{\boldsymbol{z}}_i) - f_i'(\hat{\underline{\boldsymbol{z}}}_i)\| \\
=&\left\|\sum_{j=1}^{i} t_{ji}\cdot(f_i(\underline{\boldsymbol{x}}_i) - f_i(\hat{\underline{\boldsymbol{x}}}_i))\right\| \\
\le&\left(\sum_{j=1}^{i}|t_{ji}|\cdot l_i\right)\cdot\|\underline{\boldsymbol{x}}_i - \hat{\underline{\boldsymbol{x}}}_i\| \\
=&\left(\sum_{j=1}^{i}|t_{ji}|\cdot l_i\right)\|T^{-1}(\underline{\boldsymbol{z}}_i - \hat{\underline{\boldsymbol{z}}}_i)\| \\
\le&\left(\sum_{j=1}^{i}|t_{ji}|\cdot l_i\right)\|T^{-1}\|\ \|\underline{\boldsymbol{z}}_i - \hat{\underline{\boldsymbol{z}}}_i\|.
\end{aligned}$$

So $f_i'(\underline{\boldsymbol{z}}_i)$ satisfies the Lipschitz condition. It is obvious that $\sum_{i=1}^{n} a_i z_i$ is a function about $\boldsymbol{z}$ and satisfies the Lipschitz condition, therefore every element of $\left(0...0\sum_{i=1}^{n} a_i z_i\right)^T + TF(T^{-1}\boldsymbol{z})$ satisfies the Lipschitz condition and A2 is satisfied.

Similarly, considering coordinate transformation $\hat{\boldsymbol{z}} = T\hat{\boldsymbol{x}}$, where $\hat{z} \in R^n$, then from Eq.(4) we have

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}} =& A_0\hat{\boldsymbol{z}} + \left(0...0\sum_{i=1}^{n} a_i\hat{\boldsymbol{z}}_i\right)^T + TF(T^{-1}\hat{\boldsymbol{z}}) \\
&+ TK(y - \hat{y}), \quad \hat{y} = C\hat{\boldsymbol{z}}. \quad (13)
\end{aligned}$$

A2 is also satisfied.

Then, for systems (12) and (13), from theorem 1, there exists $\theta_0 > 0$ that $\|\boldsymbol{z}(t) - \hat{\boldsymbol{z}}(t)\| \le k(\theta)\exp(-\theta t/3)\|\boldsymbol{z}(0) - \hat{\boldsymbol{z}}(0)\|$ when $\theta > \theta_0$ and $K = T^{-1}P^{-1}(\theta)C^T$. Therefore, $\lim_{t\to\infty} \|\boldsymbol{z}(t) - \hat{\boldsymbol{z}}(t)\| = 0$. Accordingly, for systems (1) and (4), there exists $\theta_0 > 0$ that $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\| = 0$ when $\theta > \theta_0$ and $K = T^{-1}P^{-1}(\theta)C^T$

This ends the proof of theorem 2.

Compared with Wang's work,[7] the application range of theorem 2 is larger. It can be used to deal with the synchronization of some chaotic systems in complex form, including hyperchaotic systems, in which Wang's theorem cannot be effective.

IV. SIMULATION

(1) WGY system

The state equation for the WGY system[8] is given by

$$\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = -ax_2 + bx_3 - f(x_1), \\
\dot{x}_3 = -cx_2 + dx_3 - ex_4, \\
\dot{x}_4 = fx_3,
\end{cases} \quad (14)$$

$$y = x_1, \quad (15)$$

where $f(x_1) = x + |x_1 + 1| + |x_1 - 1|$. The system parameters are selected to be a=0.05402, b=0.5369, c=0.3725, d=0.03536, e=0.589, f=0.8489.

The initial states of the drive and the response systems are $\boldsymbol{x}(0) = (-1.0, 0.1, -1.5, 0.05)^T$ and $\hat{\boldsymbol{x}}(0) = (2.0, -1.0, 1.0, 1.0)^T$. We have T by Eq.(5), and the feedback gain is $K = T^{-1}P^{-1}(\theta)C^T$.

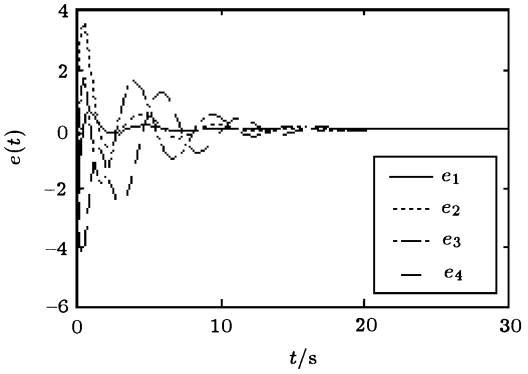


Fig.1. Synchronization error in two WGY systems performed with θ=1.0.

We find that the synchronization occurs when $\theta \ge 1.0$. When θ=1.0, we have K=$(4.0, 6.0, 8.05, -7.16)^T$, and the error variables varying with time are shown by Fig 1. It is found that the maximal absolute error is less than 10^{-3} when t >24s. The feedback gain K varying with θ is shown in Fig.2, and it is obvious that K is a power function of θ from Eq.(6).

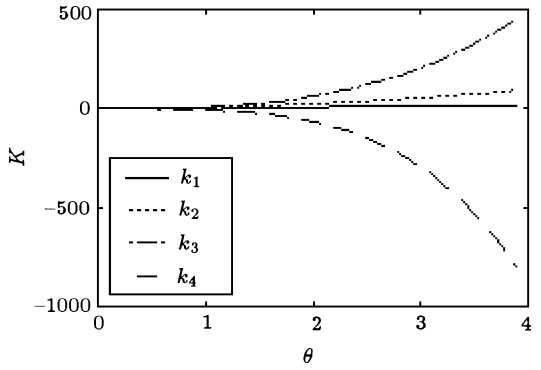


Fig.2. Feedback gain K varying with θ in the WGY system.

(2) Chua's oscillator

Consider Chua's oscillator[7]

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1 - f(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = \beta x_2 - \gamma x_3, \end{cases} \tag{16}$$

$$y = x_1, \tag{17}$$

where $f(x_1) = bx_1 = \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|)$. The system parameters are α=10.0, β=15.0, γ=0.0385, a=−1.27, and b=−0.68.

The initial states of the drive and the response systems are $\boldsymbol{x}(0)=(-1.0,-0.1,1.5)^T$ and $\hat{\boldsymbol{x}}(0)=(1.5,0.3,-1.0)^T$. We have T from Eq.(5), and the feedback gain is $K = T^{-1}P^{-1}(\theta)C^T$.

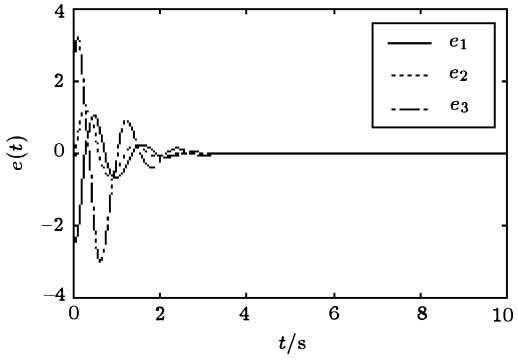


Fig.3. Synchronization error in two Chua's oscillators performed with θ=1.0.

We find that the synchronization occurs when $\theta \geq 1.0$. When θ=1.0, we have $K=(3.0, 3.3, 3.4)^T$, and the error variables varying with time are shown in Fig.3. The feedback gain K varying with θ is shown in Fig.4.

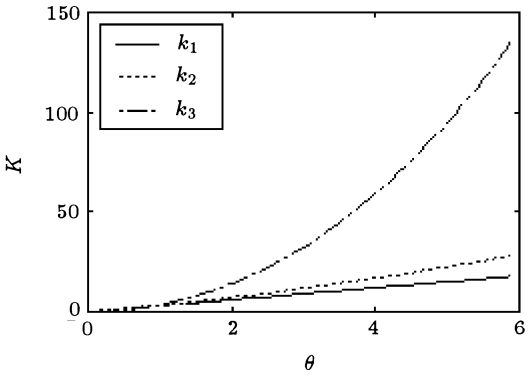


Fig.4. Feedback gain K varying with θ in Chua's oscillator.

In computer simulations, the response system can synchronize the drive system for arbitrary initial states, because the presented synchronization rule is globally asymptotic stable.

The synchronization method in Wang's work is a special case of our work, so its example TNC hyperchaotic system can be done with our methods. We do not use it as an example here. Wang's method cannot fit the WGY system, but ours can. Compared with Ref.[8], our method is also more simple than that of Ref.[8], which takes two variables x_1 and x_2 as the synchronization driving signals, and its controller separates the linear and nonlinear parts.

V. CONCLUSION

We have presented a linear output error feedback approach to globally synchronize a class of chaotic systems with the first state variable as the driving signal. The feedback gain is chosen as a function of a free parameter. We prove that globally asymptotic synchronization can be attained when the parameter is large enough. This enlarges the application range of Wang's method, and is a more simple method. By using this method, we have shown the simulation results, taking the WGY hyperchaotic system and Chua's oscillator as examples. These simulation results show that the proposed scheme might be useful for developing practical synchronized chaotic systems.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821

[2] Peng J H and Ding E J 1996 *Phys. Rev. Lett* **76** 904

[3] Pecora L M and Carroll T L 1991 *Phys. Rev.* A **44** 2374

[4] Curran P F, Suykens J A K and Chua L O 1997 *Int. J. Bifurcation Chaos* **7** 2891

[5] Nijmeijer H and Mareels I M Y 1997 *IEEE Trans. Circuits Syst. I* **44** 882

[6] Gauthier J P, Hammouri H and Othman S 1992 *IEEE Trans. Autom. Control* **37** 875

[7] Wang X F and Wang Z Q 1998 *Int. J. Bifurcation Chaos* **8** 1363

[8] Gao J F, Luo X J and Ma X K 1999 *Acta Phys. Sin.* **48** 1618 (in Chinese)