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# DIRECT APPROACH FOR SOLITON PERTURBATIONS BASED ON THE GREEN'S FUNCTION*

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A direct approach has been developed for soliton perturbations based on the Green's function. We first linearized the soliton equation, and then derived the Green's function corresponding to approximation equations of different orders. Finally, we obtained the effects of perturbation on a soliton, namely both the slow time dependence of the soliton parameters and the corrections up to the second-order approximation. The higher-order effects can also be obtained in the same way.

**Keywords:** soliton theory, soliton perturbation
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## I. INTRODUCTION

In recent years, we have developed a new direct approach for soliton perturbations based on the separation of variables.[1−5] Since this approach does not rely on the tools of the inverse scattering transformation, it can be employed to deal not only with integrable but also with unintegrable systems (for example, the $\phi - 4$ model[6] etc). Besides, no additional approximations have been employed, and it is very simple and easy to understand. The procedure is described roughly as follows. Firstly, we linearize the perturbed soliton equations in the usual way. Then we solve the linearized approximation equations of different orders and, at the same time, we obtain the effects of perturbation on a soliton, namely both the slow time dependence of the soliton parameters and the different order corrections.

By introducing a coordinate system moving with the soliton, i.e. by choosing the soliton phase as a new space coordinate system, the time and space variables are separated in different terms of the linearized approximation equations. These equations can be solved, in principle, by the separation of variables. The key to the problem is how to find the eigenfunctions of an appropriate operator $\hat{L}$ and its adjoint $\hat{L}^{\dagger}$, based on which we can construct two complete sets of states as the bases of a perturbation expansion. In our previous papers,[1−5] the eigenfunctions of $\hat{L}$ and $\hat{L}^{\dagger}$ are orthogonal to each other, and so the expansion coefficients can be easily obtained. However, in many cases (including the case in this paper), the proof of orthogonality is not an easy thing, and so we must improve the method. In this paper, we shall first find the Green's function of the linearized equations instead of solving the equations themselves, since to do this we do not require the orthogonalities. According to this point of view, the approach developed in this paper has more widespread application prospects. At least, the soliton perturbations of all the nonlinear equations we have studied by the separation of variables (for example, KdV, MKdV, KdV–MKdV, nonlinear Schrödinger, sine-Gordon, $\phi$-4, double-quadratic equations, etc) can also be treated by our approach. However, we choose a KdV-like equation as an example to demonstrate our approach, since it has application in the study of long wavelength theory in shallow water[7−9] and also because there will be some new characteristics appearing in its soliton perturbation. In the first place, the appropriate operator $\hat{L}$ is neither self-adjoint, nor anti-self-adjoint, and then its eigenvalues are neither real nor imaginary, but complex numbers. In the second place, there are two independent eigenfunctions corresponding to a continuous eigenvalue. In other words, the eigenvalue problem is degenerate of second degree. Obviously, the study of the above characteristics will enrich the soliton perturbation theory.

## II. LINEARIZATION

Consider the following nonlinear evolution equation

$$u_t - u_x - uu_x - u_{xx} = 0, \tag{1}$$

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where the subscripts represent partial differentiation with respect to time $t$ and space $x$. The difference between Eq.(1) and the KdV equation is merely that the last term on the left-hand side of the KdV equation is $u_{xxx}$. It is easy to check directly that Eq.(1) has the following single soliton solution

$$u(x,t) = 2a\ \tanh[a(x - x_0 + t)], \tag{2}$$

where $a$ and $x_0$ are two real parameters that determine the soliton height (and also width) and initial position, respectively. The appropriate perturbed equation is

$$u_t - u_x - uu_x - u_{xx} = \epsilon R[u], \tag{3}$$

where $\epsilon$ is a small positive constant measuring the weakness of the perturbation ($0 < \epsilon \ll 1$), and the perturbation term $R[u]$ is a known function of $u, u_x, u_{xx}, \cdots$. To linearize Eq.(1) in the usual way, we introduce the slow time variables of different scales $t_n = \epsilon^n t, n = 1, 2, \cdots$, thus the time derivative should be replaced by the expansion

$$\partial_t \longrightarrow \partial_t + \epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \cdots. \tag{4}$$

At the same time, $u$ and $R[u]$ are expanded in a power series of $\epsilon$ as follows

$$u(x,t) = u^{(0)}(x,t) + \epsilon u^{(1)}(x,t) + \epsilon^2 u^{(2)}(x,t) + \cdots. \tag{5}$$

$$R[u] = R[u^{(0)}] + \epsilon R^{(1)}[u^{(0)}, u^{(1)}] + \cdots. \tag{6}$$

Inserting Eqs.(4)–(6) into Eq.(3), and equating the coefficients of each power of $\epsilon$, one obtains the following approximation equations of different orders:

$$u_t^{(0)} - u_x^{(0)} - u^{(0)}u_x^{(0)} - u_{xx}^{(0)} = 0, \tag{7}$$

$$\begin{aligned} &u_t^{(1)} - u_x^{(1)} - u^{(0)}u_x^{(1)} - u_x^{(0)}u^{(1)} - u_{xx}^{(1)} \\ =&R[u^{(0)}] - u_{t_1}^{(0)}, \end{aligned} \tag{8}$$

$$\begin{aligned} &u_t^{(2)} - u_x^{(2)} - u^{(0)}u_x^{(2)} - u_x^{(0)}u^{(2)} - u_{xx}^{(2)} \\ =&R^{(1)}[u^{(0)}, u^{(1)}] - u_{t_2}^{(0)} - u_{t_1}^{(1)} + 2u^{(1)}u_x^{(1)}\ , \cdots \end{aligned} \tag{9}$$

The zeroth-order approximation equation (7) is formally the same as Eq.(1), so it has the single soliton solution expressed by Eq.(2), which can be rewritten as

$$\begin{aligned} &u^{(0)} = 2a\ \tanh z, \\ &z = a(x + \xi)\ , \qquad \xi_t = 1. \end{aligned} \tag{10}$$

We assume that $\xi$ depends not only on the fast variable $t$ (see the last equation in Eq.(10)), but also on the slow variables $t_1, t_2, \cdots$, due to perturbation. It follows from Eq.(10) that

$$u_{t_n}^{(0)} = \xi_{t_n}\ 2a^2\mathrm{sech}^2 z, \qquad n = 1, 2, \cdots. \tag{11}$$

Since, in this paper, we study the effects of perturbation on a single soliton, Eqs.(7), (8) and (9) are subject to the initial conditions $u^{(0)}(x,0) = 2a\ \tanh[a(x - x_0)], u^{(1)}(x,0) = 0$ and $u^{(2)}(x,0) = 0$, respectively. Substituting Eqs.(10) and (11) into Eqs.(8) and (9), and taking $z$ (proportional to the space coordinates in a system moving with the soliton) as a new independent variable instead of $x$, we can rewrite Eqs.(8) and (9) and the appropriate initial conditions as

$$\begin{aligned} &u_t^{(1)} - a^2\hat{L}u^{(1)} \\ =&R[u^{(0)}(z)] - \xi_{t_1}\ 2a^2\mathrm{sech}^2 z \equiv F^{(1)}(z), \\ &u^{(1)}(z,0) = 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &u_t^{(2)} - a^2\hat{L}u^{(2)} = R^{(1)}[u^{(0)}, u^{(1)}] \\ &- u_{t_2}^{(0)} - u_{t_1}^{(1)} + 2au^{(1)}u_z^{(1)} \equiv F^{(2)}(z), \\ &u^{(2)}(z,0) = 0, \end{aligned} \tag{13}$$

where $\hat{L}$ is a linear differential operator defined as

$$\hat{L} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\tanh z\frac{\mathrm{d}}{\mathrm{d}z} + 2\mathrm{sech}^2 z. \tag{14}$$

## III. EIGENVALUE PROBLEM

Our main task, in this paper, is to derive the effects of perturbation on a soliton through finding the solutions of Eqs.(12) and (13) by separation of variables. The key is to solve the eigenvalue problem of operator $\hat{L}$

$$\hat{L}\phi = \lambda\phi. \tag{15}$$

In appendix A, the eigenfunctions for continuous eigenvalues are derived in a new way, different from that given in Refs.[1] and [2].

$$\begin{aligned} &\phi_\pm(z,k) \\ =&\frac{1}{\sqrt{4\pi\lambda_\pm(k)}}\mathrm{e}^{\mathrm{i}kz}(1 \pm \mathrm{i}k + \mathrm{i}k\tanh z - \tanh^2 z) \\ &\text{for } \lambda_\pm(k) = -k^2 \pm 2\mathrm{i}k, \quad -\infty < k < \infty. \end{aligned} \tag{16}$$

It is easy to check directly that $\hat{L}$ also has an eigenfunction for discrete eigenvalue $\lambda = 0$,

$$\phi_0(z) = \frac{1}{2}\mathrm{sech}^2 z\ , \qquad \lambda = \lambda_0 = 0\ . \tag{17}$$

It should be pointed out that though the two sets of complex numbers $\lambda_+(k) = -k^2 + 2ik$ and $\lambda_-(k) = -k^2 - 2ik$ coincide with each other, the corresponding eigenfunctions $\phi_+(z,k)$ and $\phi_-(z,k)$ satisfies different boundary conditions. The former tends to zero as $z \to -\infty$, while the latter tends to zero as $z \to +\infty$, and so they are linearly independent. Therefore, the eigenvalue problem (15) is degenerate of second degree.

In appendix B, we shall prove the following completeness relation

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}k \left[\phi_-^*(z',k)\phi_+(z,k) + \phi_+^*(z',k)\phi_-(z,k)\right] \\ &+ \phi_0^*(z')\phi_0(z) \\ =&\mathrm{sech}^2 z\ \delta(z-z') = \mathrm{sech}^2 z'\ \delta(z-z') \\ =&\mathrm{sech}z\ \mathrm{sech}z'\ \delta(z-z'), \end{aligned} \tag{18}$$

where, as usual, '*' stands for complex conjugate, and $\delta(z-z')$ is the Dirac function. Note that there is an additional factor $\mathrm{sech}z\ \mathrm{sech}z'$ on the right-hand side of Eq.(18). If we define $\psi_\pm(z,k) = \cosh z\ \phi_\pm(z,k), \psi_0(z) = \cosh z\ \phi_0(z)$ ($\cosh z$ can be understood as a certain weight factor), then we obtain the usual completeness relation: $\int_{-\infty}^{\infty} \mathrm{d}k[\psi_-^*(z',k)\psi_+(z,k) + \psi_+^*(z',k)\psi_-(z,k)] + \psi_0^*(z')\psi_0(z) = \delta(z-z')$.

## IV. GREEN'S FUNCTION

Now, we solve Eqs.(12) and (13). According to Refs.[1]–[5], we should expand $u^{(1)}$ and $u^{(2)}$ on the complete set of eigenfunctions, and determine the coefficients with the aid of orthogonalities. However, the proof of orthogonality is not an easy thing. To avoid this difficulty, we first seek the Green's function corresponding to Eqs.(12) and (13). The Green's function $G(z,t;z')$ is determined by the following equation and initial condition

$$\begin{aligned}&G_t - a^2\hat{L}G = \delta(z-z')\ , \\ &G(z,0;z') = 0. \end{aligned} \tag{19}$$

Once the Green's function is derived, the first-order and second-order corrections can be expressed in terms of it,

$$u^{(1)}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}z' G(z,t;z')F^{(1)}(z'). \tag{20}$$

$$u^{(2)}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}z' G(z,t;z')F^{(2)}(z'). \tag{21}$$

In order to solve Eq.(19), we expand $G$ as follows

$$\begin{aligned}G(z,t;z') = \int_{-\infty}^{\infty} \mathrm{d}k \big[&C_+(t,z';k)\phi_+(z,k) \\ &+ C_-(t,z';k)\phi_-(z,k)\big] \\ &+ C_0(t,z')\phi_0(z). \end{aligned} \tag{22}$$

Substituting Eq.(22) into Eq.(19), and making use of Eqs.(15)–(18), we obtain the first-order ordinary differential equations and the appropriate initial conditions governing the coefficients:

$$\begin{aligned}&\dot{C}_+ - \lambda_+(k)a^2 C_+ = \phi_-^*(z',k)\cosh^2 z', \\ &C_+(0,z';k) = 0, \end{aligned} \tag{23}$$

$$\begin{aligned}&\dot{C}_- - \lambda_-(k)a^2 C_- = \phi_+^*(z',k)\cosh^2 z', \\ &C_-(0,z';k) = 0, \end{aligned} \tag{24}$$

$$\dot{C}_0 = \phi_0^*(z')\cosh^2 z', \quad C_0(0,z') = 0, \tag{25}$$

where the overdot signifies the derivative with respect to $t$. It is easy to obtain the solutions of Eqs.(23)–(25) in a standard way that

$$C_+ = -\frac{1}{\lambda_+(k)a^2}\left[1 - \mathrm{e}^{\lambda_+(k)a^2 t}\right]\phi_-^*(z',k)\cosh^2 z', \tag{26}$$

$$C_- = -\frac{1}{\lambda_-(k)a^2}\left[1 - \mathrm{e}^{\lambda_-(k)a^2 t}\right]\phi_+(z',k)\cosh^2 z', \tag{27}$$

$$C_0 = t\ \phi_0^*(z')\cosh^2 z'. \tag{28}$$

Inserting Eqs.(26)–(28) into Eq.(22), we immediately obtain the Green's function

$$\begin{aligned}G =& \cosh^2 z' \Big\{ -\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_+(k)a^2}[1 - \mathrm{e}^{\lambda_+(k)a^2 t}] \\ &\cdot \phi_-^*(z',k)\phi_+(z,k) \\ &- \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_-(k)a^2}[1 - \mathrm{e}^{\lambda_-(k)a^2 t}] \\ &\cdot \phi_+^*(z',k)\phi_-(z,k) + t\ \phi_0^*(z')\phi_0(z)\Big\}. \end{aligned} \tag{29}$$

## V. EFFECTS OF PERTURBATION ON A SOLITON

Substituting Eq.(29) into Eq.(20), we obtain the expression of the first-order correction $u^{(1)}(z,t)$. In this expression, the term proportional to $t$ (which comes from the last term on the right-hand side of Eq.(29)) will grow in time infinitely. It is just the so-called secular term. In order to eliminate it, we must put

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}z' \cosh^2 z'\ \phi_0^*(z')F^{(1)}(z') \\ =&\frac{1}{2}\int_{-\infty}^{'} \mathrm{d}z' \left\{R[u^{(0)}(z')] - \xi_{t_1} 2a^2 \mathrm{sech}^2 z'\right\} = 0. \end{aligned} \tag{30}$$

From Eq.(30) one easily obtains the $t_1$-dependence of $\xi$,

$$\xi_{t_1} = \frac{1}{4a^2}\int_{-\infty}^{\infty} \mathrm{d}z\ R[u^{(0)}(z)]. \tag{31}$$

Now, the first-order correction reduces to

$$\begin{aligned} u^{(1)} = \int_{-\infty}^{\infty} \mathrm{d}z' \cosh^2 z' F^{(1)}(z') \Big\{ & -\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_+(k)a^2} \\ & \cdot \left[1 - \mathrm{e}^{\lambda_+(k)a^2 t}\right] \phi_-^*(z',k)\phi_+(z,k) \\ & - \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_-(k)a^2} \left[1 - \mathrm{e}^{\lambda_-(k)a^2 t}\right] \\ & \cdot \phi_+^*(z',k)\phi_-(z,k) \Big\}. \end{aligned} \tag{32}$$

Both Eqs.(31) and (32) represent the first-order effects of perturbation. The former determines how the soliton velocity is affected, while the latter signifies the first-order correction on the wave profile of the soliton.

Similarly, inserting Eq.(29) into Eq.(21), one obtains the second-order correction $u^{(2)}(z,t)$, and the non-secular condition $\int_{-\infty}^{\infty} \mathrm{d}z \cosh^2 z\ \phi_0^*(z)F^{(2)}(z) = 0$, which leads to

$$\begin{aligned} \xi_{t_2} = & \frac{1}{4a^2}\int_{-\infty}^{\infty} \mathrm{d}z \{R^{(1)}[u^{(0)}, u^{(1)}] \\ & - u_{t_1}^{(1)} + 2u^{(1)}u_z^{(1)}\}. \end{aligned} \tag{33}$$

Now, the second-order correction reduces to

$$\begin{aligned} u^{(2)} = \int_{-\infty}^{\infty} \mathrm{d}z' \cosh^2 z' F^{(2)}(z') \Big\{ & -\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_+(k)a^2} \\ & \cdot \left[1 - \mathrm{e}^{\lambda_+(k)a^2 t}\right] \phi_-^*(z',k)\phi_+(z,k) \\ & - \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\lambda_-(k)a^2} \left[1 - \mathrm{e}^{\lambda_-(k)a^2 t}\right] \\ & \cdot \phi_+^*(z',k)\phi_-(z,k) \Big\}. \end{aligned} \tag{34}$$

The second-order effects of perturbation on a soliton are given by Eqs.(33) and (34). Obviously, the higher-order effects can also be derived in the same way.

Finally, as an example, we consider the case

$$R[u] = -u_x = -u_x^{(0)} - \epsilon u_x^{(1)} - \epsilon^2 u_x^{(2)} - \cdots, \tag{35}$$

which corresponds to the refractive index of a slightly varying medium. Since $R[u^{(0)}] = -u_x^{(0)} = -2a^2\mathrm{sech}^2 z$, it follows from Eqs.(31) and (12) that

$$\begin{aligned} \xi_{t_1} = & \frac{1}{4a^2}\int_{-\infty}^{\infty} \mathrm{d}z R[u^{(0)}(z)] \\ = & -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}z\ \mathrm{sech}^2 z = -1, \end{aligned} \tag{36}$$

and

$$F^{(1)}(z) = R[u^{(0)}(z)] - \xi_{t_1} 2a^2 \mathrm{sech}^2 z = 0, \tag{37}$$

respectively. Therefore, we have from Eq.(32) that $u^{(1)}(z) = 0$, and then $R^{(1)} = -u_x^{(1)} = 0$. Similarly, it follows from Eqs.(33) and (13) that

$$\begin{aligned} & \xi_{t_2} = \frac{1}{4a^2}\int_{-\infty}^{\infty} \mathrm{d}z (R^{(1)} - u_{t_1}^{(1)} + 2u^{(1)}u_x^{(1)}) = 0\ , \\ & F^{(2)} = R^{(1)} - u_{t_2}^{(0)} - u_{t_1}^{(1)} + 2u^{(1)}u_x^{(1)} = 0, \end{aligned} \tag{38}$$

which leads to $u^{(2)} = 0$. Repeating the above steps in the higher-order approximations, we can easily obtain that $\xi_{t_n} = 0$ and $u^{(n)} = 0$, for $n > 2$. Consequently, we finally obtain

$$\xi_t = 1 + \epsilon\xi_{t_1} + \epsilon^2\xi_{t_2} + \cdots = 1 - \epsilon, \tag{39}$$

and

$$\begin{aligned} u &= u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots \\ &= u^{(0)} = 2a\tanh\{a[x - x_0 + (1-\epsilon)t]\}. \end{aligned} \tag{40}$$

Equation (40) shows that in the presence of perturbation $R[u] = -u_x$, the soliton merely changes its velocity a little, while its shape remains unchanged.

## APPENDIX A: DERIVATION OF EIGENFUNCTION $\phi(z,k)$ IN Eq.(16)

Consider the eigenvalue problem

$$\hat{L}\phi = \lambda\phi \quad \hat{L} = \frac{d^2}{\mathrm{d}z^2} + 2\tanh z\frac{d}{\mathrm{d}z} + 2\mathrm{sech}^2 z. \tag{A1}$$

Letting

$$\phi(z,k) = \mathrm{e}^{\mathrm{i}kz}\theta(z,k), \tag{A2}$$

we obtain the ordinary differential equation governing $\theta$ as follows

$$\begin{aligned} & \theta_{zz} + (2\mathrm{i}k + 2\tanh z)\theta_z \\ & + (2 - k^2 + 2\mathrm{i}k\tanh z - 2\tanh^2 z)\theta = \lambda\theta\ . \end{aligned} \tag{A3}$$

Introducing a new independent variable $y = \tanh z$, we can rewrite Eq.(A3) as

$$\begin{aligned} & (1-y^2)\theta_{yy} + 2\mathrm{i}k(1-y^2)\theta_y \\ & + (2 - k^2 + 2\mathrm{i}ky - 2y^2)\theta = \lambda\theta. \end{aligned} \tag{A4}$$

We assume that Eq.(A4) has a polynomial solution:

$$\theta(z,k) = a_0(k) + a_1(k)y + a_2(k)y^2 + \cdots + a_n(k)y^n. \tag{A5}$$

Substituting Eq.(A5) into Eq.(A4), and equating the coefficients of each power of $y$, we easily obtain $n = 2$ and the following relations successively,

$$\begin{aligned} & a_1 = -\mathrm{i}ka_2\ , \quad a_0 = -\left[\frac{\lambda}{2} + (1 + \frac{1}{2}k^2)\right] a_2 \\ & \lambda_\pm(k) = -k^2 \pm 2\mathrm{i}k. \end{aligned} \tag{A6}$$

Thus, we obtain immediately

$$\phi_{\pm}(z,k) = ce^{ikz}(1 \pm ik + ik\tanh z - \tanh^2 z), \tag{A7}$$

where $c = -a_2$ is a constant, determined by the completeness relation (18), to be $c = [4\pi\lambda_{\pm}(k)]^{-1/2}$.

## APPENDIX B: PROOF OF COMPLETENESS RELATION (18)

We start from the integral

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi_{\pm}^*(z',k)\phi_{\mp}(z,k) \\
=&\frac{1}{4\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')}(1 \mp \tanh z)(1 \pm \tanh z') \\
&- \int_{-\infty}^{\infty} \frac{dk}{\lambda_{\pm}(k)} e^{ik(z-z')} \Big[ \pm 2ik(1 \mp \tanh z) \\
&\cdot (1 \pm \tanh z') \mp ik\,\mathrm{sech}^2 z(1 \pm \tanh z') \\
&\mp ik(1 \mp \tanh z)\mathrm{sech}^2 z' + \mathrm{sech}^2 z\,\mathrm{sech}^2 z' \Big].
\end{aligned} \tag{B1}$$

Since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} = \delta(z-z'), \tag{B2}$$

$$\begin{aligned}
&\pm 2ik(1 \mp \tanh z)(1 \pm \tanh z') \mp ik\,\mathrm{sech}^2 z \\
&\cdot (1 \pm \tanh z') \mp ik(1 \mp \tanh z)\mathrm{sech}^2 z' \\
=&-ik(1 \mp \tanh z)(1 \pm \tanh z')(\tanh z - \tanh z'),
\end{aligned} \tag{B3}$$

Eq.(B1) can be rewritten as

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk \phi_{\pm}^*(z',k)\phi_{\pm}(z,k) \\
=&\frac{1}{2}(1 \mp \tanh z)(1 \pm \tanh z')\delta(z-z') \\
&+ \frac{1}{4\pi}(1 \mp \tanh z)(1 \pm \tanh z')(\tanh z - \tanh z') \\
&\cdot \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k \pm 2i} e^{ik(z-z')} \\
&- \frac{1}{4\pi}\mathrm{sech}^2 z\,\mathrm{sech}^2 z' \int_{-\infty}^{\infty} \frac{dk}{k(k \pm 2i)} e^{ik(z-z')}.
\end{aligned} \tag{B4}$$

Note that

$$\begin{aligned}
&(1 \mp \tanh z)(1 \pm \tanh z')\delta(z-z') \\
=&\mathrm{sech}^2 z\delta(z-z'),
\end{aligned} \tag{B5}$$

$$\begin{aligned}
&\frac{i}{k-2i}(1+\tanh z)(1-\tanh z')(\tanh z - \tanh z') \\
&+ \frac{i}{k+2i}(1-\tanh z)(1+\tanh z')(\tanh z - \tanh z') \\
=&\frac{2ik}{k^2+4}(1 - \tanh z\ \tanh z')(\tanh z - \tanh z') \\
&- \frac{4}{k^2+4}(\tanh z - \tanh z').
\end{aligned} \tag{B6}$$

We have immediately from Eq.(B4)

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk[\phi_-^*(z',k)\phi_+(z,k) + \phi_+^*(z',k)\phi_-(z,k)] \\
=&\mathrm{sech}^2 z\ \delta(z-z') \\
&+ \frac{1}{4\pi}(1 - \tanh z\ \tanh z')(\tanh z - \tanh z') \\
&\cdot \int_{-\infty}^{\infty} dk \frac{2ik}{k^2+4} e^{ik(z-z')} \\
&- \frac{1}{4\pi}(\tanh z - \tanh z')^2 \int_{-\infty}^{\infty} dk \frac{4}{k^2+4} e^{ik(z-z')} \\
&- \frac{4}{4\pi}\mathrm{sech}^2 z\ \mathrm{sech}^2 z' \int_{-\infty}^{\infty} dk \frac{2}{k^2+4} e^{ik(z-z')}.
\end{aligned} \tag{B7}$$

Making use of the residue theorem and Jordan's lemma, we can easily calculate

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk \frac{2ik}{k^2+4} e^{ik(z-z')} \\
=&-\theta(z-z')2\pi e^{-2(z-z')} + \theta(z'-z)2\pi e^{2(z-z')},
\end{aligned} \tag{B8}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk \frac{4}{k^2+4} e^{ik(z-z')} \\
=&\theta(z-z')2\pi e^{-2(z-z')} + \theta(z'-z)2\pi e^{2(z-z')},
\end{aligned} \tag{B9}$$

where $\theta(x)$ is the step function defined, $\theta(x) = \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,. \end{cases}$ Inserting Eqs.(B8) and (B9) into Eq.(B7), after a few steps of simple calculation, we finally obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk[\phi_-^*(z',k)\phi_+(z,k) + \phi_+^*(z',k)\phi_-(z,k)] \\
=&\mathrm{sech}^2 z\delta(z-z') - \frac{1}{4}\mathrm{sech}^2 z\ \mathrm{sech}^2 z',
\end{aligned} \tag{B10}$$

which is just the completeness relation (18).

# References


[1] an J R and Tang Y 1996 *Phys. Rev.* E **54** 6916

[2] an J R, Tang Y and Zhou G H 1998 *Phys. Rev.* E **58** 1064

[3] an J R, Tang Y and Chen Z H 1997 *Chin. Phys. Lett.* **14** 671

[4] an J R, Tang Y and Zhou G H 1999 *Commun. Theor. Phys.* **32** 375

[5] hao Y M and Yan J R 1999 *Acta Phys. Sin.* **48** 1976 (in Chinese)

[6] ang Y, Yan J R and Chen Z H 1997 *Chin. Phys. Lett.* **14** 845

[7] ang M L, Yu B and Li Z B 1997 *Phys. Lett.* A **216** 67

[8] hitham G B 1967 *Proc. R. Soc.* A **299** 6

[9] roef L T F 1975 *Appl. Sci. Res.* **31** 377