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QUANTUM FLUCTUATIONS IN MESOSCOPIC RESISTANCE-INDUCTANCE-CAPACITANCE ELECTRIC CIRCUITS AT FINITE TEMPERATURE*

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By using the charge and current in a quantization resistance–inductance–capacitance (RLC) electric circuit, we construct a pair of canonical variables. Using this pair of variables and the thermal field dynamics, we obtain the fluctuations of charge and current in the RLC electric circuit at finite temperatures. It is shown that the fluctuations increase with increasing temperature and decrease with prolonging of time.

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I. INTRODUCTION

Since Louisell's proposition of the quantization scheme for an inductance–capacitance (LC) electric circuit with a source,[1] much interest has been paid to mesoscopic LC and resistance–inductance–capacitance (RLC) circuits[2−4] especially in recent years, due to the progress of mesoscopic physics. However, most research is focused on fluctuations of charge and current at zero temperature.[5,6] In fact, any practical electric circuits work in a finite temperature environment, so the effect of temperature should be taken into account in calculating quantum fluctuations of charge and current. One of the methods for this aim is to correlate the effect of environment with the ensemble average, which can be calculated by thermal field dynamics (TFD).[7] In Refs.[8] and [9] we investigated the quantum fluctuations in a single LC lossless circuit and a coupled two LC lossless circuits at finite temperatures, respectively. It is shown that the fluctuations of the charge and current increase with the increase of temperature. A question, thus, naturally arises: for a single loss LC electric circuit, namely, an RLC electric circuit, is the influence of temperature on fluctuations of charge and current still important? We will solve this problem in the following.

The equation of motion for a single RLC electric circuit is

$$\ddot{q}(t)+\frac{R}{L}\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{L}\varepsilon(t), \tag{1}$$

where $q(t)$ is the charge on capacitance C, $\dot{q}(t)=j(t)$ is the current through inductance L and resistance R, $\omega_0=1/\sqrt{LC}$ is the resonance frequency of lossless ($R=0$) electric circuit. Set $\dot{q}(t)=p(t)/L$, and $q(t), p(t)\ j(t)$ are denoted by q, p, j for simplicity. So Eq.(1) can be written as

$$\begin{aligned}\dot{q}&=p/L,\\ \dot{p}&=-\frac{R}{L}p-\frac{1}{C}q+\varepsilon(t).\end{aligned} \tag{2}$$

From Eq.(2) we can obtain

$$\frac{\partial\dot{q}}{\partial q}+\frac{\partial\dot{p}}{\partial p}=-\frac{R}{L}. \tag{3}$$

If we introduce the canonicalization transformation[10] (not the common canonical transformation) as follows:

$$q=Q\mathrm{e}^{-Rt/2L},\qquad p=\left(P-\frac{R}{2}Q\right)\mathrm{e}^{-Rt/2L}, \tag{4}$$

then the classical equation of motion of Eq.(2) becomes

$$\ddot{Q}+\omega^2 Q=\frac{1}{L}\varepsilon(t)\,\mathrm{e}^{Rt/2L}, \tag{5}$$

where $\omega^2=\omega_0^2-(R/2L)^2$. When $\omega^2>0$, the Hamiltonian of Eq.(5) is

$$H=\frac{P^2}{2L}+\frac{1}{2}L\omega^2Q^2-Q\varepsilon(t)\,\mathrm{e}^{Rt/2L}. \tag{6}$$

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When we quantize Eq.(2) in the Heisenberg picture, the equations of motion for coordinate and momentum operators share the same form of Eq.(2). But the classical variables q and p become operators $\hat{q}$ and $\hat{p}$, which observe appropriate commutation relations, namely, the quantization condition. From the operator equation of motion corresponding to Eq.(2) we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{q}\hat{p}-\hat{p}\hat{q}\right)=-\frac{R}{L}\left(\hat{q}\hat{p}-\hat{p}\hat{q}\right). \tag{7}$$

In a dissipative system, namely $R\neq 0$, Eq.(3) implies that q and p cannot be constructed as common canonical variables, and Eq.(7) means that in this system the quantization condition should be modified as

$$\hat{q}\hat{p}-\hat{p}\hat{q}=\mathrm{i}\hbar\mathrm{e}^{-Rt/L}. \tag{8}$$

Equation (8) is coincided with Eq.(7) and then coincided with Eqs.(2) and (3). In the Heisenberg picture, due to Eq.(8) and the operator expression of Eq.(4), we see

$$\left[\hat{Q},\hat{P}\right]=\mathrm{i}\hbar,\quad \left[\hat{Q},\hat{Q}\right]=0,\quad \left[\hat{P},\hat{P}\right]=0.$$

The quantization of Eq.(6) also only means that the classical variables Q and P are replaced by operators $\hat{Q}$ and $\hat{P}$. Furthermore, if we construct the following non-Hermitian operators:

$$\begin{aligned}A&=\left(2L\omega\hbar\right)^{-1/2}\left(\omega L\hat{Q}+\mathrm{i}\hat{P}\right),\\ A^{+}&=\left(2L\omega\hbar\right)^{-1/2}\left(\omega L\hat{Q}-\mathrm{i}\hat{P}\right),\end{aligned} \tag{9}$$

then Eq.(8) can be written as

$$H=\left(A^{+}A+\frac{1}{2}\right)\omega\hbar-\sqrt{\frac{\hbar}{2L\omega}}\left(A^{+}+A\right)\varepsilon\left(t\right)\mathrm{e}^{Rt/2L}, \tag{10}$$

where $\omega=\sqrt{1/LK}, K=1/L\omega^{2}$. The inverse transformation of Eq.(9) is

$$\begin{aligned}\hat{Q}&=\sqrt{\frac{\hbar}{2\omega L}}\left(A+A^{+}\right),\\ \hat{P}&=-\mathrm{i}\sqrt{\frac{L\omega\hbar}{2}}\left(A-A^{+}\right).\end{aligned} \tag{11}$$

So we have

$$\begin{aligned}&\left[A,A^{+}\right]=1,\quad [A,A]=0,\\ &\left[A^{+},A^{+}\right]=0,\quad A\left|\left|0\right\rangle\right\rangle=0.\end{aligned} \tag{12}$$

From Eq.(8) we can construct another pair of non-Hermitian operators

$$\begin{aligned}a&=\left(2L\omega\hbar\right)^{-1/2}\left(\omega L\hat{q}+\mathrm{i}\hat{p}\right)\mathrm{e}^{Rt/2L},\\ a^{+}&=\left(2L\omega\hbar\right)^{-1/2}\left(\omega L\hat{q}-\mathrm{i}\hat{p}\right)\mathrm{e}^{Rt/2L},\end{aligned} \tag{13}$$

which observe the commutation relation of Eq.(8), and

$$\begin{aligned}&\left[a,a^{+}\right]=1,\quad [a,a]=0,\\ &\left[a^{+},a^{+}\right]=0,\quad a\left|0\right\rangle=0.\end{aligned} \tag{14}$$

When the electric source has not been introduced, namely $t=0$ and the temperature $T\to 0$, the states of this mesoscopic RLC electric circuit are denoted by $|0\rangle$and $||0\rangle\rangle$ in q and Q representations, respectively. According to Eqs.(9) and (13) it can be easily shown that $|0\rangle=||0\rangle\rangle$. From Eqs.(4) and (11), we can obtain the quantum fluctuations of $\hat{q}$ and $\hat{p}$ in vacuum state $|0\rangle$ as

$$\begin{aligned}\left(\Delta\hat{q}\right)^{2}&=\left\langle 0\right|\hat{q}^{2}\left|0\right\rangle-\left\langle 0\right|\hat{q}\left|0\right\rangle^{2}\\ &=\left[\left\langle\left\langle 0\right|\right|\hat{Q}^{2}\left|\left|0\right\rangle\right\rangle-\left\langle\left\langle 0\right|\right|\hat{Q}\left|\left|0\right\rangle\right\rangle^{2}\right]\mathrm{e}^{-Rt/L}\Big|_{t=0}\\ &=\frac{\hbar}{2\omega L}=\frac{\hbar}{2}\sqrt{\frac{C}{4L-R^{2}C}},\end{aligned} \tag{15}$$

$$\begin{aligned}\left(\Delta\hat{p}\right)^{2}=&\left\langle 0\right|\hat{p}^{2}\left|0\right\rangle-\left\langle 0\right|\hat{p}\left|0\right\rangle^{2}\\ =&\left[\left\langle 0\right|\left|\left(\hat{P}-\frac{R}{2}\hat{Q}\right)^{2}\right|\left|0\right\rangle\right.\\ &\left.-\left\langle 0\right|\left|\left(\hat{P}-\frac{R}{2}\hat{Q}\right)\right|\left|0\right\rangle^{2}\right]\mathrm{e}^{-Rt/L}\Big|_{t=0}\\ =&\frac{\hbar}{2}\left(L\omega+\frac{R^{2}}{2\omega L}\right)=\frac{\hbar}{2}\frac{L}{\sqrt{C\left(4L-R^{2}C\right)}}.\end{aligned} \tag{16}$$

So

$$\left(\Delta\hat{j}\right)^{2}=\frac{1}{L^{2}}\left(\Delta\hat{p}\right)^{2}=\frac{\hbar}{2}\frac{1}{L\sqrt{C\left(4L-R^{2}C\right)}} \tag{17}$$

and

$$\Delta\hat{q}\cdot\Delta\hat{j}=\frac{\hbar}{2}\frac{1}{\sqrt{L\left(4L-R^{2}C\right)}}. \tag{18}$$

After introducing the electric source and the temperature $T\neq 0$, the fluctuations are not only those in vacuum state. In this situation, the average of operators $\hat{Q},\hat{Q}^{2},\hat{P}$ and $\hat{P}^{2}$ should be calculated by ensemble average. In the following, we calculate the quantum fluctuations of mesoscopic RLC electric circuits at finite temperatures in TFD theory. According to TFD, the ensemble average of an operator $\hat{F}$ equals the average of operator $\hat{F}$ in the thermal vacuum state, namely,

$$Z^{-1}(\beta)\mathrm{Tr}(\hat{F}\mathrm{e}^{-\beta H})=\langle 0(\beta)||\hat{F}||0(\beta)\rangle. \tag{19}$$

The thermal vacuum state of system (10) is

$$||0\beta)\rangle=(1-\mathrm{e}^{-\beta\omega})^{1/2}\exp(\mathrm{e}^{-\beta\omega\hbar/2}A^{+}\tilde{A}^{+})||0\tilde{0}\rangle, \tag{20}$$

where $\beta = (kT)^{-1}, k$ is the Boltzmann's constant, $\tilde{A}^+$ is the creation operator which is defined in fictitious Hilbert space H and it corresponds to the A^+ defined in original Hilbert space H. $\left|\left|\tilde{0}\right\rangle\right.$ and $||0\rangle$ are vacuum states in $\tilde{H}$ and H. By using the Weyl correspondence[11] and Wigner theorem,[12] we have

$$\begin{aligned}&\langle 0(\beta)||F||0(\beta)\rangle\\ =&\int\int \mathrm{d}Q\mathrm{d}P F(Q,\pi)\langle 0(\beta)||\Delta(Q,\pi)||0(\beta)\rangle\\ =&2\int \mathrm{d}^2\alpha F(\alpha,\alpha^*)\left\langle 0\left(\beta\right)\right|\left|\Delta(\alpha,\alpha^*)\right|\left|0\left(\beta\right)\right\rangle, \quad (21)\end{aligned}$$

where $F(Q,P)$ is the classical correspondence of operator $\hat{F}$, and

$$\Delta(Q,P)=\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{2\pi}\mathrm{e}^{iPu}\left|Q+\frac{u}{2}\right\rangle\left\langle Q-\frac{u}{2}\right|$$

is called the Wigner operator. In the coherent state representation it takes the form

$$\Delta(\alpha,\alpha^*)=\int\frac{\mathrm{d}^2z}{\pi}\left|\alpha+z\right\rangle\left\langle\alpha-z\right|\exp(\alpha z^*-z\alpha^*), \quad (22)$$

where $\alpha=\dfrac{\hat{Q}+\mathrm{i}\hat{P}}{\sqrt{2}}, \alpha^*=\dfrac{\hat{Q}-\mathrm{i}\hat{P}}{\sqrt{2}}$. By introducing the coherent state of fictitious space

$$\begin{aligned}&\left|\tilde{z}\right\rangle=\exp\left[-\frac{1}{2}\left|\tilde{z}\right|^2+\tilde{z}^*\tilde{a}^+\right]\left|\tilde{0}\right\rangle,\\ &\tilde{a}\left|\tilde{z}\right\rangle=\tilde{z}\left|\tilde{z}\right\rangle, \quad (23)\end{aligned}$$

and its completeness relation

$$\int\frac{\mathrm{d}^2\tilde{z}}{\pi}\left|\tilde{z}\right\rangle\left\langle\tilde{z}\right|=1, \quad (24)$$

we can calculate the Wigner function of the thermal vacuum state

$$\begin{aligned}&\langle 0(\beta)||\Delta||0(\beta)\rangle\\ =&(1-\mathrm{e}^{-\beta\omega\hbar})\langle\tilde{00}||\exp(\mathrm{e}^{-\beta\omega\hbar/2}A\tilde{A})\\ &\cdot\int\frac{\mathrm{d}^2z}{\pi}|\alpha+z\rangle\langle\alpha-z|\\ &\cdot\exp(\alpha z^*-z\alpha^*)\int\frac{\mathrm{d}^2z}{\pi}|\tilde{z}\rangle\langle\tilde{z}|\\ &\cdot\exp(\mathrm{e}^{-\beta\omega\hbar/2}A^+\tilde{A}^+)||\tilde{00}\rangle. \quad (25)\end{aligned}$$

Using

$$||\tilde{00}\rangle\langle\tilde{00}||=:\exp(-A^+A-\tilde{A}^+\tilde{A}):, \quad (26)$$

where :: denotes normal ordering, we have

$$\begin{aligned}&\langle 0(\beta||\Delta||0(\beta)\rangle\\ =&\frac{1-\mathrm{e}^{-\beta\omega\hbar}}{1+\mathrm{e}^{-\beta\omega\hbar}}\exp\left[\frac{-2\left(1-\mathrm{e}^{-\beta\omega\hbar}\right)}{1+\mathrm{e}^{-\beta\omega\hbar}}\left|\alpha\right|^2\right]. \quad (27)\end{aligned}$$

The classical Weyl correspondence of $\widehat{Q}$ and $\widehat{P}$, are

$$\begin{aligned}Q&=(\alpha+\alpha^*)\sqrt{\frac{\hbar}{2\omega L}},\\ P&=(\alpha^*-\alpha)\mathrm{i}\sqrt{\frac{L\omega\hbar}{2}}. \quad (28)\end{aligned}$$

Substituting Eqs.(28) and (26) into Eq.(21), and using the mathematical formula

$$\int\frac{\mathrm{d}^2\alpha}{\pi}\alpha^{*n}\alpha^m\mathrm{e}^{\lambda|\alpha|^2}=\delta_{n,m}\left(-1\right)^{m+1}\lambda^{-(m+1)}m!, \quad (29)$$

we have

$$(\Delta\widehat{Q})^2=\langle 0(\beta)||\,\widehat{Q}^2\,||0(\beta)\rangle=\frac{\hbar}{2}\frac{1}{\omega L}\coth\frac{\beta\omega\hbar}{2}, \quad (30)$$

$$(\Delta\widehat{P})^2=\langle 0(\beta)||\,\widehat{P}^2\,||0(\beta)\rangle=\frac{\hbar}{2}\omega L\coth\frac{\beta\omega\hbar}{2}. \quad (31)$$

From Eq.(4) and $|0\rangle=||0\rangle$ we have

$$(\Delta\widehat{q})^2=\frac{\hbar}{2}\sqrt{\frac{C}{4L-R^2C}}\exp(-\frac{Rt}{L})\coth\frac{\beta\omega\hbar}{2}, \quad (32)$$

$$(\Delta\widehat{j})^2=\frac{\hbar}{2}\frac{1}{L\sqrt{C\left(4L-R^2C\right)}}\exp(-\frac{Rt}{L})\coth\frac{\beta\omega\hbar}{2}, \quad (33)$$

$$\Delta\widehat{q}\cdot\Delta\widehat{j}=\frac{\hbar}{2}\sqrt{\frac{1}{L\left(4L-R^2C\right)}}\exp(-\frac{Rt}{L})\coth\frac{\beta\omega\hbar}{2}. \quad (34)$$

Clearly, the fluctuation and uncertainty of $\widehat{q}$, $\widehat{j}$ increase with increasing temperature. As $T\to 0$ and $t\to 0$, Eqs.(32)–(34) turn back to Eqs.(16)–(18). The fluctuations and uncertainty of $\hat{q}$, $\hat{j}$ decrease with prolonged time, as shown from Eqs.(32)–(34) clearly.

To summarize, first, we have seen that at finite temperatures an RLC circuit exhibits more quantum noise than at $T\to 0$ case. In RLC electric circuit, the quantum fluctuations and uncertainty of $\widehat{q}$, $\widehat{j}$ grow with rising temperature. Secondly, it is shown that the fluctuations decrease with prolonged time. As the noise will affect the stability and precision of the signal, it is very important to consider the size and the match of R, L and C in devices of mesoscopic electric circuits.

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