Vol 10 No 6, June 2001
1009-1963/2001/10(06)/0480-06
Chinese Physics
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and IOP Publishing Ltd

THE MEAN-SQUARE EXPONENTIAL STABILITY AND INSTABILITY OF STOCHASTIC NONHOLONOMIC SYSTEMS*

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(Received 27 October 2000; revised manuscript received 15 February 2001)

We present a new methodology for studying the mean-square exponential stability and instability of nonlinear nonholonomic systems under disturbance of Gaussian white-noise by the first approximation. Firstly, we give the linearized equations of nonlinear nonholonomic stochastic systems; then we construct a proper stochastic Lyapunov function to investigate the mean-square exponential stability and instability of the linearized systems, and thus determine the stability and instability in probability of corresponding competing systems. An example is given to illustrate the application procedures.

Keywords: stochastic nonholonomic system, mean-square exponential stability and instability
PACC: 0320

I. INTRODUCTION

This paper concerns the question of mean-square exponential stability of stochastic nonholonomic systems described by stochastic differential equations of the form

$$\mathrm{d}\boldsymbol{Y} = \boldsymbol{F}(\boldsymbol{Y}(t))\mathrm{d}t + \boldsymbol{G}(\boldsymbol{Y}(t))\mathrm{d}\boldsymbol{B}(t), \qquad (1)$$

where $\boldsymbol{Y}(t)$ is an n-dimensional, $\boldsymbol{B}(t)$ an m-dimensional mutually independent Wiener process with $E\{B_i(t)\} = 0$, $E\{[B_i(t) - B_i(s)]^2\} = |t-s|$, and $\boldsymbol{F}(\boldsymbol{Y}(t))$ is an n-dimensional vector, $\boldsymbol{G}(\boldsymbol{Y}(t))$ an $n \times m$ matrix.

We state here the concepts of stability for stochastic systems that our paper involves. We shall always refer to the equilibrium or null solution, $\boldsymbol{Y}(t) \equiv 0$, as the solution whose stability properties are being tested. We will denote the solution with initial state $\boldsymbol{Y}_0$ at time t_0 by $\boldsymbol{Y}(t; \boldsymbol{Y}_0, t_0)$, $\boldsymbol{Y} \in R^n$.

Unless specified otherwise, $|\boldsymbol{Y}|$ will denote

$$|\boldsymbol{Y}| = (Y_1^2 + \cdots + Y_n^2)^{1/2}$$

Euclidean norm.

Definition 1.1. *The trivial solution* $\boldsymbol{Y}(t) \equiv 0$ *of stochastic differential equations* (1) *is said to possess exponential stability of pth mean if there exists* $\delta > 0$ *and constants* α, β, *such that* $|\boldsymbol{Y}_0| < \delta$, *implies for all* $t \geq t_0$,

$$E\{|\boldsymbol{Y}(t); \boldsymbol{Y}_0, t_0|^p\} \leq \beta |\boldsymbol{Y}_0|^p \exp\{-a(t-t_0)\}.$$

Definition 1.2. *The trivial solution* $\boldsymbol{Y}(t) \equiv 0$ *of stochastic differential equations* (1) *is said to possess exponential instability of qth mean if there exists* $\delta > 0$ *and constants* α, β, *such that* $|\boldsymbol{Y}_0| > \delta$, *implies for all* $t \geq t_0$,

$$E\{|\boldsymbol{Y}(t); \boldsymbol{Y}_0, t_0|^{-q}\} \leq \beta |\boldsymbol{Y}_0|^{-q} \exp\{-a(t-t_0)\}.$$

Specifically, when p=2 (or q=2) it is called mean-square exponential stability (or instability). The concept of mean-square stability was first introduced by Samuels,[1] and it means that the second moment of solution process is asymptotically bounded.

II. STABILITY AND INSTABILITY IN THE FIRST APPROXIMATION FOR STOCHASTIC SYSTEMS

The pth mean exponential stability (and instability) is usually studied via stochastic Lyapunov functions, that is, the criteria of stability is given by $E\dot{V}(\boldsymbol{Y}(t), t) < 0$, which requires calculating the total derivative with respect to time t for the stochastic Lyapunov function $V(\boldsymbol{Y}, t) \geq 0$ along the path of the system.

It is very difficult to construct a proper stochastic Lyapunov function which meets the requirement

*Project supported by the Natural Science Foundation of Liaoning Province of China (Grant No. 002083) and by the Science Research Foundation of Liaoning Education Commission of China (Grant Nos. 990111004, 20021004).

of the problem. People still hope to find some simple methods to solve such complicated problems in spite of these difficulties. For example, for a given nonlinear stochastic dynamical system (1), one can obtain its corresponding linearized system expressed by Eqs.(2), by first making a Taylor expansion for $\boldsymbol{F}(\boldsymbol{Y}(t))$ and $\boldsymbol{G}(\boldsymbol{Y}(t))$ at the neighbouring region of $\boldsymbol{Y}=0$, then omitting the higher terms and taking their linear terms

$$\mathrm{d}y_i(t) = \sum_{j=1}^{n} b_i^j y_j \mathrm{d}t + \sum_{r=1}^{m}\sum_{j=1}^{n} \sigma_{ir}^j y_j \mathrm{d}B_r(t)$$
$$(i = 1, ..., n). \tag{2}$$

Or in another form

$$\mathrm{d}\boldsymbol{Y}(t) = \boldsymbol{B}(t)\boldsymbol{Y}(t)\mathrm{d}t + \sum_{r=1}^{m} \sigma_r(t)\boldsymbol{Y}(t)\mathrm{d}B_r(t), \tag{3}$$

where b_i^j and σ_{ir}^j are constants; $\boldsymbol{B}(t) \equiv \boldsymbol{B}$ and $\boldsymbol{\sigma}_r(t) \equiv \boldsymbol{\sigma}_r$ are $n \times n$ constants matrices.

It was Kac and Krasovskii[2] who first proved the theorem on the stability of linearized stochastic systems. They proved that if Eq.(3) has mean-square exponential stability, then its corresponding complete system (1) is stable in probability. An analogous result of the equations was given by Gihman.[3]

We also know from Ref.[4] that if Eq.(3) has exponential instability of qth mean, then its corresponding complete system (1) is unstable in probability.

III. LINEARIZED STOCHASTIC NONHOLONOMIC SYSTEMS

We can establish the perturbation equations of a stochastic nonholonomic system by the method described in Refs.[5,6]. Suppose that the configuration of a mechanical system is determined by n generalized coordinates $q_1, \cdots, q_n$. The system is subject to g ideal, homogeneous and linear nonholonomic constraints

$$\dot{q}_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon} b_{\varepsilon+\beta,\sigma}(q_\nu)\dot{q}_\sigma$$
$$(\beta = 1, \cdots, g; \nu = 1, \cdots, \varepsilon; \varepsilon = n - g), \tag{4}$$

where the coefficients $b_{\varepsilon+\beta,\sigma}$ do not contain $q_{\varepsilon+\beta}$; the generalized forces imposed on the system are $\overline{Q}_s = Q_s + \eta_s (s = 1, \cdots, n)$, $Q_s = \dfrac{\partial U}{\partial q_s} - \dfrac{\partial F}{\partial \dot{q}_s}$, where $U = U(\boldsymbol{q})$ is the potential force function and does not contain $q_{\varepsilon+\beta}$; $F = \dfrac{1}{2}\sum\limits_{s,k=1}^{n} f_{sk}\dot{q}_s\dot{q}_k$ is a dissipative function; $\eta_s(t) = \sum\limits_{i=1}^{m} C_{si}(q_\nu)\xi_i(t)$ are stochastic forces, ξ_i are Gaussian white-noises and C_{si} do not contain $q_{\varepsilon+\beta}$. Suppose that the system is stationary, therefore the kinetic energy T is the quadratic form of $\dot{q}_s$, namely, $T = T_2 = \dfrac{1}{2}\sum\limits_{s,k=1}^{n} a_{sk}\dot{q}_s\dot{q}_k$, and also a_{sk} do not contain $q_{\varepsilon+\beta}$. Then the equations of motion of the system can be written in the explicit form of Chaplygin's equation[5,6]

$$\begin{aligned}
&\sum_{\nu=1}^{\varepsilon}\tau_{\sigma\nu}\ddot{q}_\nu + \sum_{\nu=1}^{\varepsilon}\dot{\tau}_{\sigma\nu}\dot{q}_\nu \\
&- \frac{1}{2}\sum_{\mu,\nu=1}^{\varepsilon}\frac{\partial\tau_{\mu\nu}}{\partial q_\sigma}\dot{q}_\mu\dot{q}_\nu + \sum_{\nu=1}^{\varepsilon}\varphi_{\sigma\nu}\dot{q}_\nu \\
&- \sum_{\beta=1}^{g}\sum_{\mu=1}^{\varepsilon}\left(a_{\varepsilon+\beta,\mu} + \sum_{\gamma=1}^{g} a_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\gamma,\mu}\right)\dot{q}_\mu \\
&\cdot \sum_{\nu=1}^{\varepsilon}\left(\frac{\partial b_{\varepsilon+\beta,\sigma}}{\partial q_\nu} - \frac{\partial b_{\varepsilon+\beta,\nu}}{\partial q_\sigma}\right)\dot{q}_\nu \\
=&\frac{\partial U}{\partial q_\sigma} + \eta_\sigma + \sum_{\beta=1}^{g}\eta_{\varepsilon+\beta}b_{\varepsilon+\beta,\sigma},
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\tau_{\sigma\nu} =& a_{\sigma\nu} + \sum_{\beta=1}^{g}(a_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + a_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) \\
&+ \sum_{\beta,\gamma=1}^{g} a_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\gamma,\nu}, \\
\varphi_{\sigma\nu} =& f_{\sigma\nu} + \sum_{\beta=1}^{g}(f_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + f_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) \\
&+ \sum_{\beta,\gamma=1}^{g} f_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\gamma,\nu}.
\end{aligned}$$

In order to discuss the stability of the stochastic nonholonomic systems (4) and (5), we first give their perturbation equations.

Let

$$\begin{cases} q_s = x_s, \\ \dot{q}_\nu = x_{n+\nu} \end{cases}$$
$$(s = 1, \cdots, n; \quad \nu = 1, \cdots, \varepsilon; \quad \varepsilon = n - g), \tag{6}$$

suppose $x_\sigma^0, x_{\varepsilon+\beta}^0$ is a point on the manifolds of equilibrium and let

$$\begin{cases} x_\sigma = x_\sigma^0 + y_\sigma, \\ x_{\varepsilon+\beta} = x_{\varepsilon+\beta}^0 + y_{\varepsilon+\beta} + \sum\limits_{\sigma=1}^{\varepsilon} b_{\varepsilon+\beta,\sigma}(x_\nu^0)y_\sigma, \\ x_{n+\nu} = y_{n+\nu}. \end{cases} \tag{7}$$

Considering the transformations (6) and (7), we obtain the perturbation equations of stochastic nonholonomic systems (4) and (5)

$$\begin{aligned}
&\dot{y}_\nu = y_{n+\nu},\\
&\dot{y}_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon}[b_{\varepsilon+\beta,\sigma}(x_\nu^0+y_\nu)-b_{\varepsilon+\beta,\sigma}(x_\nu^0)]y_{n+\sigma}\\
&\sum_{\nu=1}^{\varepsilon}\tau_{\sigma\nu}\dot{y}_{n+\nu}+\sum_{\nu=1}^{\varepsilon}\dot{\tau}_{\sigma\nu}y_{n+\nu}\\
&-\frac{1}{2}\sum_{\mu,\nu=1}^{\varepsilon}\frac{\partial\tau_{\mu\nu}}{\partial y_\sigma}y_{n+\mu}y_{n+\nu}+\sum_{\nu=1}^{\varepsilon}\varphi_{\sigma\nu}y_{n+\nu}\\
&-\sum_{\beta=1}^{g}\sum_{\mu=1}^{\varepsilon}\left(a_{\varepsilon+\beta,\mu}+\sum_{\gamma=1}^{g}a_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\gamma,\mu}\right)y_{n+\mu}\\
&\cdot\sum_{\nu=1}^{\varepsilon}\left(\frac{\partial b_{\varepsilon+\beta,\sigma}}{\partial q_\nu}-\frac{\partial b_{\varepsilon+\beta,\nu}}{\partial q_\sigma}\right)y_{n+\nu}\\
=&\frac{\partial U}{\partial y_\sigma}+\eta_\sigma+\sum_{\beta=1}^{g}\eta_{\varepsilon+\beta}b_{\varepsilon+\beta,\sigma} \qquad (8)\\
&(\mu,\nu=1,\cdots,\varepsilon;\quad \beta=1,\cdots,g;\\
&s=1,\cdots,n;\quad \varepsilon=n-g).
\end{aligned}$$

With the help of the Taylor formula, we can make the Taylor expansion for each term of system (8), then omit higher terms and keep only the linear terms, such that we obtain the linearized stochastic nonholonomic system

$$\begin{aligned}
&\dot{y}_\nu = y_{n+\nu},\qquad \dot{y}_{\varepsilon+\beta}=0,\\
&\sum_{\nu=1}^{\varepsilon}\tau_{\sigma\nu}(q_\mu^0)\dot{y}_{n+\nu}=-\varphi_{\sigma\nu}(q_s^0)y_{n+\nu}+\sum_{\mu=1}^{\varepsilon}\left.\frac{\partial^2U}{\partial y_\sigma\partial y_\mu}\right|_0 y_\mu\\
&+\sum_{i=1}^{m}\left\{\sum_{\mu=1}^{\varepsilon}\left[\left.\frac{\partial C_{\sigma i}}{\partial y_\mu}\right|_0\right.\right.\\
&\left.\left.+\sum_{\beta=1}^{g}\left.\frac{\partial(b_{\varepsilon+\beta,\sigma}C_{\varepsilon+\beta,i})}{\partial y_\mu}\right|_0\right]y_\mu\right\}\xi_i(t),
\end{aligned}$$

or in another form

$$\left\{\begin{aligned}
&\dot{y}_\nu = y_{n+\nu},\qquad \dot{y}_{\varepsilon+\beta}=0,\\
&\dot{y}_{n+\nu}=\sum_{\sigma=1}^{\varepsilon}\frac{\Delta_{\nu\sigma}}{\Delta}\left\{-\varphi_{\sigma\nu}(q_s^0)y_{n+\nu}\right.\\
&\qquad\left.+\sum_{\mu=1}^{\varepsilon}\left.\frac{\partial^2U}{\partial y_\sigma\partial y_\mu}\right|_0 y_\mu\right\}\\
&\qquad+\sum_{i=1}^{m}\left[\sum_{\mu=1}^{\varepsilon}\left(\left.\frac{\partial C_{\sigma i}}{\partial y_\mu}\right|_0\right.\right.\\
&\qquad\left.\left.+\sum_{\beta=1}^{g}\left.\frac{\partial b_{\varepsilon+\beta,\sigma}C_{\varepsilon+\beta,i}}{\partial y_\mu}\right|_0\right)y_\mu\right]\xi_i(t)\Bigg\},
\end{aligned}\right. \qquad (9)$$

where $\Delta=\det(\tau_{\sigma\nu}(q_\mu^0))$; $\Delta_{\nu\sigma}$ is the algebraic complement of element (ν,σ) in the Δ. Let

$$\left\{\begin{aligned}
F_\nu=&\sum_{\sigma=1}^{\varepsilon}\frac{\Delta_{\nu\sigma}}{\Delta}\left[-\varphi_{\sigma\nu}(q_s^0)y_{n+\nu}+\sum_{\mu=1}^{\varepsilon}\left.\frac{\partial^2U}{\partial y_\sigma\partial y_\mu}\right|_0 y_\mu\right],\\
G_{\nu i}=&\sum_{\sigma=1}^{\varepsilon}\frac{\Delta_{\nu\sigma}}{\Delta}\left\{\sum_{\mu=1}^{\varepsilon}\left[\left.\frac{\partial C_{\sigma i}}{\partial y_\mu}\right|_0\right.\right.\\
&\left.\left.+\sum_{\beta=1}^{g}+\left.\frac{\partial(b_{\varepsilon+\beta,\sigma}C_{\varepsilon+\beta,i})}{\partial y_\mu}\right|_0\right]y_\mu\right\},
\end{aligned}\right.$$

then Eqs.(9) can be expressed as

$$\left\{\begin{aligned}
&\dot{y}_\nu = y_{n+\nu},\\
&\dot{y}_{\varepsilon+\beta}=0,\\
&\dot{y}_{n+\nu}=F_\nu+\sum_{i=1}^{m}G_{\nu i}\xi_i(t).
\end{aligned}\right. \qquad (10)$$

Furthermore, we can write Eqs.(10) in Itô's form:

$$\mathrm{d}\boldsymbol{Y}(t)=\boldsymbol{f}(\boldsymbol{Y}(t))\mathrm{d}t+\boldsymbol{H}(\boldsymbol{Y}(t))\mathrm{d}\boldsymbol{B}(t), \qquad (11)$$

where

$$\begin{aligned}
&\boldsymbol{Y}=(Y_1,Y_2,\cdots,Y_{n+\varepsilon})^T,\\
&\boldsymbol{f}=(y_{n+1},\cdots,y_{n+\varepsilon},0,\cdots,0,F_1,\cdots,F_\varepsilon)^T,\\
&\boldsymbol{H}=\begin{bmatrix}
0 & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & 0\\
G_{11} & G_{12} & \cdots & G_{1m}\\
\vdots & \vdots & \cdots & \vdots\\
G_{\varepsilon 1} & G_{\varepsilon 2} & \cdots & G_{\varepsilon m}
\end{bmatrix}_{(n+\varepsilon)\times m}.
\end{aligned}$$

IV. THE MEAN-SQUARE EXPONENTIAL STABILITY AND INSTABILITY OF STOCHASTIC NONHOLONOMIC SYSTEMS

Considering stochastic nonholonomic systems (11), we introduce a differential operator of solution process $\boldsymbol{Y}(t)$.

$$L=\sum_{i=1}^{n+\varepsilon}f_i(\boldsymbol{Y})\frac{\partial}{\partial y_i}+\frac{1}{2}\sum_{i,j=1}^{n+\varepsilon}a_{ij}(\boldsymbol{Y})\frac{\partial^2}{\partial y_i\partial y_j}$$

where

$$a_{ij}(\boldsymbol{Y})=\sum_{r=1}^{m}H_{ir}(\boldsymbol{Y})H_{jr}(\boldsymbol{Y})\quad i,j=1,\cdots,n+\varepsilon.$$

Definition 4.1. *Function $V(\boldsymbol{Y},t)$ is said to be positive definite in the neighbouring region of $\boldsymbol{Y}=0$ under the meaning of Lyapunov if*

(i) $V(0,t) \equiv 0$;
(ii) $V(\boldsymbol{Y},t) > W(\boldsymbol{Y})$ *hold in this neighbouring region (where* $W(\boldsymbol{Y}) > 0$, *for all* $\boldsymbol{Y} \neq 0$*).*

Theorem 1. [4] *The solution process* $\boldsymbol{Y} = 0$ *of systems* (11) *is mean-square exponentially stable, if and only if there exists a quadratic homogeneous function* $V(\boldsymbol{Y},t)$ *and for constants* $K_i > 0$, *there are*

$$\begin{aligned} &K_1|\boldsymbol{Y}|^2 \leq V(\boldsymbol{Y},t) \leq K_2|\boldsymbol{Y}|^2, \\ &LV(\boldsymbol{Y},t) \leq -K_3|\boldsymbol{Y}|^2, \\ &\left|\frac{\partial V}{\partial y_i}\right| \leq K_4|\boldsymbol{Y}|, \quad \left|\frac{\partial^2 V}{\partial y_i \partial y_j}\right| \leq K_4 \\ &(i,j = 1,\cdots,n+\varepsilon; \quad \varepsilon = n-g). \end{aligned} \tag{12}$$

Analogously, we have the following theorem of mean-square exponential instability.

Theorem 2. [4] *The solution process* $\boldsymbol{Y} = 0$ *of systems* (11) *is mean-square exponentially unstable, if and only if there exists a quadratic homogeneous function* $V(\boldsymbol{Y},t)$ *and for constants* $K_i > 0$, *there are*

$$\begin{aligned} &K_1|\boldsymbol{Y}|^{-2} \leq V(\boldsymbol{Y},t) \leq K_2|\boldsymbol{Y}|^{-2}, \\ &LV(\boldsymbol{Y},t) \leq -K_3|\boldsymbol{Y}|^{-2}, \\ &\left|\frac{\partial V}{\partial y_i}\right| \leq K_4|\boldsymbol{Y}|^{-2}, \quad \left|\frac{\partial^2 V}{\partial y_i \partial y_j}\right| \leq K_4|\boldsymbol{Y}|^{-4} \\ &(i,j = 1,\cdots,n+\varepsilon; \quad \varepsilon = n-g), \end{aligned} \tag{13}$$

where

$$\begin{aligned} L = &\sum_{\nu=1}^{\varepsilon} y_{n+1}\frac{\partial}{\partial y_\nu} + \sum_{\nu=1}^{\varepsilon} F_{\nu 1}\frac{\partial}{\partial y_{n+\nu}} \\ &+ \frac{1}{2}\sum_{\mu,\nu=1}^{\varepsilon}\sum_{r=1}^{m} G_{\nu r}G_{\mu r}\frac{\partial^2}{\partial y_{n+\nu}\partial y_{n+\mu}} \\ &(\nu,\mu = 1,\cdots,\varepsilon). \end{aligned} \tag{14}$$

Corollary *If linearized systems* (10) *are mean-square exponentially stable (unstable), then its complete systems* (8) *are stable (unstable) in probability.*

V. EXAMPLES

Assume that the configuration of a dynamical system is determined by generalized coordinates $q_i (i = 1,2,3)$. The kinetic energy of the system is

$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2);$$

the nonholonomic constraint imposed on the system is

$$\dot{q}_3 = cq_2\dot{q}_1; \tag{15}$$

the potential force function is

$$U = -\frac{1}{2}(Aq_1^2 + Bq_2^2);$$

the dissipative function is

$$F = -\frac{1}{2}(\mu_1\dot{q}_1^2 + \mu_2\dot{q}_2^2 + \dot{q}_3^2);$$

stochastic forces are

$$\eta_s = \sum_{i=1}^{2} C_{si}(q_\sigma)\xi_i \quad (s = 1,2,3),$$

where ξ_i are Gaussian white-noises with

$$\begin{aligned} &E[\xi_i] = 0, \\ &E[\xi_i(t)\xi_j(t+\tau)] = 2D_{ij}\delta(\tau) \\ &\qquad (i,j = 1,2,3), \end{aligned}$$

and

$$\begin{aligned} \eta_1 &= a_1q_2\xi_1 + a_2q_1\xi_2, \\ \eta_2 &= b_2q_2\xi_2, \\ \eta_3 &= q_1\xi_1 + a_1q_2\xi_2. \end{aligned}$$

Let us study the equilibrium position of the system and its stability in probability.

Firstly, we give the equilibrium position of the system.

Similar to the deterministic circumstance, we can obtain the equilibrium position of the system from the equation $\delta U = 0$, and investigate if this equilibrium position is stable in probability under excitation of white-noise.

$$\delta U = 0 \Rightarrow \begin{cases} \dfrac{\partial U}{\partial q_1} = 0 \\ \dfrac{\partial U}{\partial q_2} = 0 \end{cases} \Rightarrow \begin{cases} q_1 = 0 \\ q_2 = 0 \end{cases}.$$

Considering the limitation of nonholonomic constraint (15), we have the following equilibrium position:

$$q_1^0 = 0, \quad q_2^0 = 0, \quad q_3^0 = \text{constant},$$

i.e. the equilibrium state is a one-dimensional manifold.

Secondly, we can obtain the motion equations from Eqs.(5). We have

$$a_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad i,j = 1,2,3;$$

$$f_{ij} = 0 (i \neq j), f_{11} = \mu_1, f_{22} = \mu_2, f_{33} = 1;$$

$$b_{11} = cq_2, \quad b_{32} = 0;$$

$$\tau_{\sigma\nu} = a_{\sigma\nu} + \sum_{\beta=1}^{g}(a_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + a_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) + \sum_{\beta,\gamma=1}^{g} a_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\gamma,\nu},$$

$$\varphi_{\sigma\nu} = f_{\sigma\nu} + \sum_{\beta=1}^{g}(f_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + f_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) + \sum_{\beta,\gamma=1}^{g} f_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\gamma,\nu};$$

$$\tau_{ij} = 1 + c^2q_2^2, \quad \tau_{12} = \tau_{21} = 0, \quad \tau_{22} = 1;$$

$$\Delta_0 = \left[\begin{array}{cc} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{array}\right]\Bigg|_0 = 1,$$

$$\Delta_{11} = \Delta_{22} = 1, \quad \Delta_{12} = \Delta_{21} = 0;$$

$$\varphi_{11} = f_{11} = \mu_1, \quad \varphi_{22} = f_{22} = \mu_2,$$

$$\varphi_{12} = \varphi_{21} = 0,$$

$$\left\{\begin{array}{l} (1 + c^2q_2^2)\ddot{q}_1 + c^2q_2\dot{q}_1\dot{q}_2 + (\mu_1 + c^2q_2^2)\dot{q}_1 \\ = -Aq_1 + \eta_1 + b_{31}\eta_3, \\ \dot{q}_2 + 3c2q_2\dot{q}_1^2 + \mu_2\dot{q}_2 = -Bq_2 + \eta_2. \end{array}\right. \tag{16}$$

The linearized equations corresponding to system (16) are

$$\left\{\begin{array}{l} \ddot{q}_1 + \mu_1\dot{q}_1 = -Aq_1 + a_1q_2\xi_1 + a_2q_1\xi_2, \\ \dot{q}_2 + \mu_2\dot{q}_2 = -Bq_2 + b_2q_2\xi_2. \end{array}\right. \tag{17}$$

Let

$$\left\{\begin{array}{l} q_s = x_s \\ \dot{q}_\nu = x_{3+\nu} \end{array}\right. \quad (s = 1, 2, 3; \nu = 1, 2);$$

$$\left\{\begin{array}{l} x_\sigma = y_\sigma \\ x_3 = x_3^0 + y_3 \\ x_{3+\nu} = y_{3+\nu} \end{array}\right. \quad (\sigma, \nu = 1, 2),$$

we obtain the perturbation equations of system (17)

$$\left\{\begin{array}{l} \dot{y}_1 = y_4, \\ \dot{y}_2 = y_5, \\ \dot{y}_0 = 0, \\ \dot{y}_4 = (-\mu_1y_4 - Ay_1) + a_1y_2\xi_1 + a_2y_1\xi_2, \\ \dot{y}_5 = (-\mu_2y_5 - By_2) + b_2q_2\xi \end{array}\right.$$

$$(s = 1, 2, 3; \quad \nu = 1, 2). \tag{18}$$

Finally, we investigate the stability in probability of system (16) by theorem 1.

We choose the stochastic Lyapunov function as

$$V(\boldsymbol{Y}) = |\boldsymbol{Y}|^2 = \sum_{i=1}^{5} Y_i^2.$$

It is obvious that, as long as $0 < k_1 < 1$, $k_2 > 1, k_4 > 2$, there are

$$k_1|Y|^2 \le V(Y, t) \le k_2|Y|^2,$$

$$\left|\frac{\partial V}{\partial y_i}\right| \le k_4|Y|, \quad \left|\frac{\partial^2 V}{\partial y_i \partial y_j}\right| \le k_4.$$

We have the result of LV from Eqs.(14),

$$LV = 2(1 - A)y_1y_4 + 2(1 - B)y_2y_5 - 2\mu_1y_4^2 - 2\mu_2y_5^2 + a_2^2y_1^2 + (a_1^2 + b_2^2)y_2^2.$$

To guarantee that the following expression,

$$LV \le -k_3|Y|^2 = -k_3\sum_{i=1}^{5} y_i^2$$

holds, we must choose parameters in such a way

$$A \le 1, \ \ B \le 1, \ \ \mu_1 > 0, \ \ \mu_2 > 0, \ \ \mu = \min(\mu_1, \mu_2),$$

$$M = \max\{a_2^2, a_1^2, b_2^2, k_3\},$$

$$\mu > 10M.$$

Hence, we know that system (18) is of mean-square stability by theorem 1, and furthermore the complete system (16) is stable in probability.

We must note that the differential operator L is defined in local region, so the investigation of stability of stochastic dynamical systems is within the neighbouring region of null solution of perturbation equations.

VI. CONCLUSIONS

A new methodology for studying the stability and instability in probability of nonlinear nonholonomic systems under Gaussian white-noise excitation has been presented. The procedure consists of applying a first-approximation method for nonlinear nonholonomic systems and constructing a proper stochastic Lyapunov function to examine the mean-square exponential stability and instability of the linearized systems, and thus we can determine the stability and instability in probability of the corresponding complete systems. The procedure has been shown to be effective by applying to an example of a nonlinear stochastic nonholonomic systems.

ACKNOWLEDGMENTS

We are indebted to our advisor, Mr F X Mei, who has given us much help.

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