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THE ELECTRON IN THE GRAVITATIONAL FIELD OF THE KERR BLACK HOLE*

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We investigate the electronic energy levels in the field of the micro-Kerr black hole under the approximation condition $a << GM/c^2$, and obtain some interesting results.

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I. INTRODUCTION

Some authors[1−4] have investigated the movement of scalar particles in the gravitational field of micro black holes, and have proved that the contact energy expression is complex, as follows:

$$\omega = \mu - \Delta\omega - \mathrm{i}\sigma. \tag{1}$$

The energy, corresponding to positive σ, will decrease when the particle drops into the black hole; and the negative σ corresponds to the excited energy of the particle (particle creation). In the quasi-stationary case, i.e. $\sigma << \Delta\omega$, when $2\mu M << l + 1/2, a << M, Q = 0$, the energy spectrum is hydrogen-like, and the energy decreases as the momentum and radical quantum numbers become larger. Some authors[5,6] have investigated thermodynamic effects of the black hole.

If the spin of the particle is a semi-integer, its state is different from the scalar particle state; the cross section for which the electron is captured by the micro black hole is one eighth of the cross section for which the scalar particle is captured by the micro black hole.[7] In the gravitational field of the Kerr-Newman black hole, there is no super-radiation of the electron and neutrino. Using a similar quantum electro-dynamics (QED) method, the process of creating the constraint state of the Reissner-Nordstrom field, by using the numerical solutions of the Dirac equation, has been studied.[8]

Following the above formalism, we investigate the electronic energy levels in the gravitational field of Schwarzschild and Kerr black holes. In the Schwarzschild case, relativistic effect and spin effect make the energy level split (on angular momentum and orbit momentum), but this is different from that in the Coulomb field. In flat space, the electronic energy depends on quantum numbers n and j. The contact energy of particles with $m_l < 0$ $(j = l + 1/2)$ is smaller than that of the particles with $m_l > 0$ $(j = l - 1/2)$, when they have the same value of l. Nonlinear spin effect is obvious in the lower energy level. In Kerr space, the electronic spatial energy spectra are split. We acquired its split energy level expressions.

II. DIRAC EQUATION

The Langrangian of the system is

$$L = \overline{\Psi}\mathrm{i}\gamma^\mu\nabla_\mu - \mu\overline{\Psi}\Psi, \tag{2}$$

where μ is the electronic mass, and the Dirac matrix satisfies $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $g^{\mu\nu}$ being the metric of Kerr space

$$\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \sigma^{\mu A\dot{B}} \\ \sigma^\mu_{AB} & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} P^A \\ Q_{\dot{A}} \end{pmatrix}. \tag{3}$$

So we can obtain the spin form of the Dirac equation

$$\sqrt{2}\sigma^\mu_{A\dot{B}}\nabla_\mu P^B = -\mathrm{i}\mu Q_{\dot{A}}, \quad \sqrt{2}\sigma^{\mu A\dot{B}}\nabla_\mu Q_{\dot{B}} = -\mathrm{i}\mu P^{\dot{A}}. \tag{4}$$

We take the spin field $\overline{\Psi}$ as in the form $\overline{\Psi} = (Q_A, P^{\dot{A}})$, choose frame $(l, n, m, \overline{m})$ and its basic spin $(\zeta^A\chi^A)$, so we have

$$P^A = P_0\zeta^A + P_1\chi^A, \quad Q_{\dot{A}} = Q_0\zeta_{\dot{A}} + Q_1\chi_{\dot{A}}. \tag{5}$$

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Choosing Kinnersley frame,[9,10] and taking a transformation

$$
\begin{aligned}
P_0 &= \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}m\varphi}\frac{\overset{(-)}{S}(\theta)\overset{(-)}{R}(r)}{\sqrt{2}(r-\mathrm{i}a\cos\theta)},\\
Q_0 &= -\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}m\varphi}\frac{\overset{(+)}{S}(\theta)\overset{(-)}{R}(r)}{\sqrt{2}(r+\mathrm{i}a\cos\theta)},\\
P_1 &= \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}m\varphi}\Delta^{-1/2}\overset{(+)}{S}(\theta)\overset{(+)}{R}(r),\\
Q_1 &= \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}m\varphi}\Delta^{-1/2}\overset{(-)}{S}(\theta)\overset{(+)}{R}(r),
\end{aligned}
\tag{6}
$$

where $\Delta = r^2 - 2Mr + a^2$, so we can gain the following equations:

$$
\begin{aligned}
&[\hat{D}_0 + \hat{D}_1(a)]\overset{(-)}{S}(\theta, a)\\
&= -(\lambda^2 + 2a\omega m - 1/4)\overset{(-)}{S}(\theta, a),\\
&\overset{(+)}{S}(\pi-\theta) = \overset{(-)}{S}(\theta),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\Delta^{1/2}\left[\frac{\mathrm{d}}{\mathrm{d}r} + \mathrm{i}\frac{k}{\Delta}\right]\overset{(+)}{R} &= (\lambda - \mathrm{i}\mu r)\overset{(-)}{R},\\
\Delta^{1/2}\left[\frac{\mathrm{d}}{\mathrm{d}r} - \mathrm{i}\frac{k}{\Delta}\right]\overset{(-)}{R} &= (\lambda + \mathrm{i}\mu r)\overset{(+)}{R},
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\hat{D}_0 =& \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) - \frac{m^2 - m\cos\theta + 1/4}{\sin^2\theta},\\
\hat{D}_1 =& \frac{a\mu}{\lambda + a\mu\cos\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{1}{2}\cos\theta - m + a\omega\sin^2\theta\right)\\
&+ a\omega\cos\theta - a^2\omega^2\sin^2\theta - \lambda a^2\mu^2\cos^2\theta\\
k =& (r^2 + a^2)\omega - m_j a.
\end{aligned}
\tag{9}
$$

The field resource with angular velocity $a \approx GM/c^2$ will generate complicated nonlinear effects. We now only consider the slowly rotating case $a << GM/c^2$, and the Kerr metric is rewritten in a linear form in the following region

$$
r \geq r_+ = GM/c^2 + \sqrt{G^2M^2/c^4 - a^4} \approx 2GM/c^2. \tag{10}
$$

Then the Kerr metric turns into

$$
\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \mathrm{e}^{\lambda}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 + \frac{4Ma}{r}\sin^2\theta\mathrm{d}\varphi\mathrm{d}t, \tag{11}
$$

where

$$
\mathrm{e}^{\nu} = \mathrm{e}^{-\lambda} = 1 - r_g/r, \quad r_g = 2M. \tag{12}
$$

For the metric (11) and the unit system ($c = G = \hbar = 1$), the Dirac equation becomes

$$
\gamma^{\mu}\nabla_{\mu}\Psi + \mathrm{i}\mu\Psi = 0, \tag{13}
$$

$$
\nabla_{\mu} = \partial_{\mu} + \gamma^{\mu}(\gamma_{\nu,\mu} - \gamma_{\rho}\Gamma^{\rho}_{\nu,\mu})/4. \tag{14}
$$

III. THE SCHWARZSCHILD BLACK HOLE CASE

In order to make the usual spherical spin decompose, we choose γ matrix in the following forms:

$$
\begin{aligned}
\gamma^t &= \tilde{\gamma}^0\mathrm{e}^{-\nu/2},\\
\gamma^{\varphi} &= r^{-1}\mathrm{arc}\sin\theta(\sin\varphi\tilde{\gamma}^1 - \cos\theta\tilde{\gamma}^2),\\
\gamma^r &= \mathrm{e}^{-\lambda/2}(\sin\theta\cos\varphi\tilde{\gamma}^1 + \sin\theta\sin\varphi\tilde{\gamma}^2 + \cos\theta\tilde{\gamma}^3),\\
\gamma^{\theta} &= r^{-1}(\cos\theta\cos\varphi\tilde{\gamma}^1 + \cos\theta\sin\varphi\tilde{\gamma}^2 - \sin\theta\tilde{\gamma}^3).
\end{aligned}
\tag{15}
$$

So we can obtain the wavefunction

$$
\Psi = r^{-1}\mathrm{e}^{\nu/4}\mathrm{e}^{-\mathrm{i}\omega t}\begin{pmatrix} P(r)\mathrm{Y}_l^m j^{-1/2}(\theta,\varphi)\\ -\mathrm{i}Q(r)(n,\sigma)\mathrm{Y}_l^m j^{-1/2}(\theta,\varphi)\end{pmatrix}, \tag{16}
$$

where

$$
\mathrm{Y}_l^m j^{(j=l-1/2)}(\theta,\varphi) = \begin{pmatrix} \left(\frac{j+m_j+1}{2j+2}\right)^{1/2}\mathrm{Y}_l^m j^{-1/2}(\theta,\varphi)\\ \left(\frac{j+m_j+1}{2j+2}\right)^{1/2}\mathrm{Y}_l^m j^{+1/2}(\theta,\varphi)\end{pmatrix}, \tag{17}
$$

$$
\mathrm{Y}_l^m j^{(j=l+1/2)}(\theta,\varphi) = \begin{pmatrix} \left(\frac{j+m_j}{2j}\right)^{1/2}\mathrm{Y}_l^m j^{-1/2}(\theta,\varphi)\\ \left(\frac{j+m_j}{2j}\right)^{1/2}\mathrm{Y}_l^m j^{+1/2}(\theta,\varphi)\end{pmatrix}. \tag{18}
$$

Here σ is the Pauli matrix: $(n,\sigma) = \sin\theta\cos\varphi\sigma_1\sin\theta\sin\varphi + \sigma_2\cos\theta\sigma_3$,

$$
\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{19}
$$

Solving Eq.(16), the radius functions satisfy the following equations:

$$e^{-\lambda/2}P' + \frac{m_l}{r}P = (e^{-\nu/2}\omega + \mu)Q, \tag{20}$$

$$e^{-\lambda/2}Q' + \frac{m_l}{r}Q = (e^{-\nu/2}\omega + \mu)P, \tag{21}$$

where m_l is a positive or negative integration number

$$l = |m_l + 1/2| - 1/2 = \begin{cases} |m_l| & m_l > 0; \\ |m_l| - 1 & m_l < 0. \end{cases} \tag{22}$$

A four-dimensional fluid density vector is

$$J^\mu = \tilde{\Psi}\gamma^\mu \Psi, \tag{23}$$

and its normalization integration for a stationary case is

$$\int\limits_{r \geq r_g} \tilde{\Psi}\,\Psi\sqrt{-g}e^{\nu/2}d^3x = \int_{r_g}^{\infty}(|P|^2 + |Q|^2)dr. \tag{24}$$

From Eqs.(20) and (21), we have

$$F^n + \left\{\frac{\omega^2}{r^4} - \frac{1}{r^2}\left[\mu^2 + \frac{1}{r^2}m_l(m_l+1)\right]\right\}F = E_{m_l}F, \tag{25}$$

where

$$E_{m_l} = \frac{\omega+\mu r}{r^2}\left(\frac{r^2}{\omega+\mu r}\right)'\frac{d}{dr} + \frac{m_l}{r}\,\frac{\omega+\mu r}{r^2}\left(\frac{r^2}{\omega+\mu r}\right)'. \tag{26}$$

Now we will prove that the operator E_{m_l} contains gravitational spin-orbit interaction and other nonlinear effects.

In non-relativistic (Pauli) approximation, function G equals zero, and function F can be expressed as Laguerre polynomials for the constrain states. Introducing an infinite-minimum ε:

$$\omega = \mu + \varepsilon, \qquad |\varepsilon| << \mu,$$

we have

$$\frac{\omega}{\mu} \equiv 1 + \frac{\varepsilon}{\mu} = 1 - \frac{\mu^2 M^2}{2n^2}$$
$$n = 1 + l + n_r = 1, 2, 3, \cdots. \tag{27}$$

In the weak relativity approximation Eqs.(26) and (27) become

$$\left\{\frac{1}{2\mu}\frac{d^2}{d\rho} + \varepsilon - \frac{\mu M}{\rho} - \frac{l(l+1)}{\rho^2}\right\}F = (U^{(1)} + U^{(2)} + U^{(3)} + U^{(4)})F, \tag{28}$$

where $\rho = r - r_g$, and the right-side terms of the above equation are perturbation potentials. For these approximations, Eq.(20) becomes

$$Q = \frac{1}{2\mu}\left(P' + \frac{m_l}{\rho}P\right), \tag{29}$$

and Eq.(24) turns into

$$\int_{\rho=0}^{\infty}(|P|^2 + |Q|^2)d\rho = \int_{\rho=0}^{\infty}|F|^2\left(1 + \frac{\varepsilon - U_M}{2\rho}\right)d\rho = 1, \tag{30}$$

where $U_M = \dfrac{\mu M}{\rho}$.

Now we investigate, one by one, the perturbation terms $U^{(i)}$. Term $U^{(1)}$ determines the electronic energy in the gravitational field, which does not split the levels due to the track quantum number

$$\frac{1}{\mu} < U^{(1)} > = \frac{\Delta\varepsilon}{\mu} = \frac{15}{8}\frac{\mu^4 M^4}{n^4}. \tag{31}$$

Term $U^{(2)}$ gives the perihelion oscillation effects in classical trajectory, which splits the levels due to the orbit quantum numbers

$$\frac{\Delta\varepsilon_{nl}}{\mu} = \frac{1}{\mu} < U^{(2)} > = -\frac{3(\mu M)^4}{n^3(l+1/2)}\left(1 - \frac{1}{3}\delta_{l0}\right). \tag{32}$$

Term $U^{(3)}$ denotes the interaction between gravity spin and orbit

$$U^{(3)} = \frac{L_\sigma}{4\mu^2\rho}\frac{dU_M}{d\rho}, \tag{33}$$

where $L_\sigma = -(1 + m_l)$.

The total of the above interactions which contribute to the energy, is

$$\frac{\Delta\varepsilon_{nm_l l}}{\mu} = \pm\frac{\mu^4 M^4(1 - \delta_{l0})}{4n^3 m_l(l+1/2)}. \tag{34}$$

When $m_l < 0$, i.e. $j = l + 1/2$ (the spin angular momentum parallel to the track momentum), Eq.(34) is positive; when $m_l > 0$, it is negative.

The term $U^{(4)}$ is related to the particle spin

$$\frac{1}{\mu} < U^{(4)} > + < -\frac{M}{2\mu\rho(\rho + r_g)}\frac{dU_M}{d\rho} > = -\frac{\mu^4 M^4(1 - \delta_{l0})}{2n^3 l(l+1/2)(l+1)}. \tag{35}$$

In the Schwarzschild case, the total energy for

$2\mu M << l+1/2$ is

$$\frac{\omega_{nm_l l}}{\mu} = 1 - \frac{\mu^2 M^2}{2n^2} - \frac{3\mu^4 M^4}{n^4} \cdot \left\{ \frac{n}{l+1/2}\left[\frac{n}{l+1/2}\left(1-\frac{\delta_{l0}}{3}\right) + \frac{1}{12}(1-\delta_{l0})\left(\frac{1}{m_l}+\frac{2}{l(l+1)}\right)\right] - \frac{5}{8}\right\}. \quad (36)$$

Thus we find that the relativistic effect and spin effects split the electronic energy levels in the gravitational field, and it is different from that of Coulomb field in flat space. The electronic energy levels, in flat space, only depend on quantum numbers n, j, and the contact energy of $m_l < 0$ ($j = l + 1/2$) is lower than the system energy of $m_l > 0$ when l is the same. We can find that the spin effect is obvious in low energy levels.

IV. THE KERR BLACK HOLE CASE

Corresponding to the Kerr metric, the γ matrix is[3]

$$\gamma_t = \overline{\gamma}_t, \quad \gamma_i = \overline{\gamma}_i + g_{i\varphi}\overline{\gamma}^{\varphi}, \quad \gamma^{\varphi} = \overline{\gamma}^{\varphi} + g^{i\varphi}\overline{\gamma}_i, \quad \gamma_{\varphi} = \overline{\gamma}_{\varphi}, \quad (37)$$

where $g_{i\varphi}$ is the Kerr metric. For Eq.(15) and in the region $e^{-\lambda/2} \approx 1$, the system Hamiltonian is

$$H = H_S + H_1 + H_2,$$

where H_S is the Hamiltonian of the electron in the Schwarzschild space and

$$H_1 = \frac{2(J\cdot L)}{r^3}, \quad H_2 = \frac{1}{2}\Omega\sigma. \quad (38)$$

Here Ω is the Lense-Tirring oscillation angular velocity,

$$\Omega = [3(j\cdot n)n - J]/r^3. \quad (39)$$

Hamiltonian H_1 is similar to the interaction between dipole moments of the centre body and orbit angular momentum of scalar particles. But the electronic state (n, l, j) of operator $\hat{L}_z$ is not determined, and this makes the energy levels split. Considering the perturbation $(H_1 + H_2)$, the Dirac equation turns into

$$\left[\varepsilon - U_M + \frac{1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{l(l+1)}{\rho^2}\right]\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} = [U^{(1)} + U^{(2)} + U^{(3)} + U^{(4)} + H_1 + H_2]\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}, \quad (40)$$

where

$$\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} = P_{nl}(\rho)\mathrm{Y}_l^m j^{(j)}(\theta,\varphi). \quad (41)$$

After a straight calculation, we obtain

$$\frac{1}{\mu} < H_1 > = \frac{4m_j}{2l+1}\frac{\mu a(\mu M)^4 |m_j|}{n^3 l(l+1/2)(l+1)}, \quad (42)$$

$$\frac{1}{\mu} < nlj|H_2|nlj > = < \frac{Ma}{2r^3} >_j 4m_j \cdot \frac{\pm l(l+1) + 3l(l+1/2) \mp 3(m_j^2+1/4)}{(2l-1)(2l+1)(2l+3)}. \quad (43)$$

Therefore, the energy levels of electrons, which are not in basic states, will split into $(2j+1)$ sublevels,

$$\frac{1}{\mu}\Delta\omega_{n_j^l}^{m_j}(a) = \frac{2m_j}{2l+1}\frac{\mu a(\mu M)^4}{n^3 l(l+1/2)(l+1)} \cdot \left[2|m_j| \mp \frac{3(m_j^2+1/4) - l(l+1) - 3(l+1/2)}{(2l-1)(2l+3)}\right]. \quad (44)$$

When $j + 1/2, m_l < 0$, we choose the upper sign and when $j - 1/2, m_l > 0$, we choose the lower sign in Eq.(43). For the gravitational orbit-spin interaction, there are two possible transitions

$$\frac{1}{\mu c^2}(E_{2\mathrm{p}_{1/2}} - E_{2\mathrm{s}_{1/2}}) = \frac{28}{96}\left(\frac{G\mu M}{\hbar c}\right)^4, \quad \frac{1}{\mu c^2}(E_{2\mathrm{p}_{3/2}} - E_{2\mathrm{s}_{1/2}}) = \frac{28}{96}\left(\frac{G\mu M}{\hbar c}\right)^4. \quad (45)$$

The dipole moment-orbit and dipole moment-spin interactions split the $2\mathrm{p}_{1/2}$ level into two levels and $2\mathrm{p}_{3/2}$ into four levels

$$\Delta E_{2\mathrm{p}_{1/2}}^{m_j=\pm 1/2} = \pm\frac{1}{15}\mu c^2\left(\frac{\mu ac}{\hbar}\right)\left(\frac{G\mu M}{\hbar c}\right)^4, \quad (46a)$$

$$\Delta E_{2\mathrm{p}_{3/2}}^{m_j=\pm 1/2} = \pm\frac{1}{24}\mu c^2\left(\frac{\mu ac}{\hbar}\right)\left(\frac{G\mu M}{\hbar c}\right)^4, \quad (46b)$$

$$\Delta E_{2\mathrm{p}_{3/2}}^{m_j=\pm 3/2} = \pm\frac{3}{40}\mu c^2\left(\frac{\mu ac}{\hbar}\right)\left(\frac{G\mu M}{\hbar c}\right)^4. \quad (46c)$$

When $\mu ac \approx \hbar$, the interaction between black hole dipole moment and electronic spin becomes obvious. And when $a \to GM/c^2$ and $\mu M \approx m_j M_{pl}$, the energy spectrum will be determined by the nonlinear effects.[13,14]

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