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THERMODYNAMICS OF THE SLOWLY ROTATING KERR-NEWMAN BLACK HOLE IN THE GRAND CANONICAL ENSEMBLE*

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We investigate the thermodynamics of the slowly rotating Kerr-Newman (K-N) black hole in the grand canonical ensemble with York's formalism. Some thermodynamical properties, such as the thermodynamical action, entropy, thermodynamical energy and heat capacity are studied, and solutions of the slowly rotating K-N black hole with different boundary conditions are analysed. We find stable solutions and instantons under certain boundary conditions.

Keywords: thermodynamics, slowly rotating Kerr–Newman black hole

PACC: 0314

I. INTRODUCTION

Much effort has been concentrated on the study of the thermodynamics of black holes recently.[1−20] By using the path-integral approach,[9−11] Hawking has studied the thermodynamical properties of black holes with the micro-canonical ensemble.[11,12] York *et al.*[13−16] further developed the canonical ensemble by defining the boundary conditions. Braden *et al.*[17] used this method to study the charged black holes in the grand canonical ensemble and the black hole in asymptotically anti-de Sitter spacetimes.[18−20]

In the grand canonical ensemble, York *et al.*[13] assumed that the black hole is put in a finite-sized "box", just as Bose *et al.*[21] did in general relativity. The boundary conditions are chosen according to the thermodynamical ensemble under study, and the thermodynamical action for the black hole can be calculated in the given thermodynamical ensemble. In the canonical ensemble, the thermodynamical potential is the Helmholtz free energy, while in the grand canonical ensemble the thermodynamical potential is the grand canonical potential;[11,17] thus we can investigate all the thermodynamical quantities and relations if we know the thermodynamical action.[22]

Hawking and Page[23,24] have investigated the thermodynamics of the black hole by fixing the Hawking temperature of the black hole at infinity, and York *et al.*[13,17,18] have done this by fixing the local temperature at finite radius. Louko and Winters-Hilt[19] have studied the thermodynamics of the Reissner-Nordstrom anti-de Sitter black hole by fixing a renormalized temperature at infinity. The same procedure was used by Hawking and Page.[23,24] Peca and Lemos[20] have studied the thermodynamics of Reissner-Nordstrom following York's formalism.[13,17,18] All of the papers listed above only studied static black holes. The reason is that one can straightforwardly calculate the Euclidean reduced action for the static spherical metrics, but for the axial symmetric stationary black hole, we cannot easily calculate the Euclidean reduced action, and there is no real Euclidean metric that represents the stationary geometry of charged, rotating black holes. Although Brown *et al.*[25] introduced a complex geometry and a real thermodynamical action, the concrete calculation has not been carried out yet.

Katz *et al.*[26] used the turning point method only to study the thermodynamical stability of the K-N black hole, and found stable and unstable black holes. In this paper, we use Brown and York's formalism[13,17,18,25,27] to investigate the thermodynamics of the slowly rotating K-N black hole in the grand canonical ensemble, i.e. by fixing the local temperature and electric potential at finite radius. Firstly, we calculate the thermodynamical action of the K-N black hole and investigate some thermodynamical properties. Here, we mainly investigate the connection between the solutions of the black hole and the boundary conditions explicitly, and we present numerical solutions of the black hole under different boundary conditions in the figures. By investigating the stability of

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the solutions, we also find the stable and unstable solutions. Furthermore, we have pointed out that only the black hole with the higher event horizon is stable and the other case is unstable under the same boundary conditions. In section II, we calculate the thermodynamical action of the slowly rotating K-N black hole in the grand canonical ensemble and evaluate the thermoenergy, entropy, charges etc. In section III, we analyse the solutions of the slowly rotating K-N black hole in graphics for different boundary conditions. In section IV, we study the local stability of these solutions. The last section is the conclusion.

II. THERMODYNAMICS OF THE SLOWLY ROTATING K-N BLACK HOLE IN GR-AND CANONICAL ENSEMBLE

The K-N solution in general relativity depicting the spacetime of a rotating charged black hole is given as[28]

$$\begin{aligned}\mathrm{d}^2s =&\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 - \frac{2a\sin\theta(r^2+a^2-\Delta)}{\Sigma}\mathrm{d}t\mathrm{d}\varphi\\ &+\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\mathrm{d}\varphi^2\\ &+\frac{\Sigma}{\Delta}\mathrm{d}r^2+\Sigma\mathrm{d}\theta^2,\end{aligned}\tag{1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta,\tag{2}$$

$$\Delta = r^2 + a^2 + Q^2 - 2Mr,\tag{3}$$

M, Q and a are the mass, charge and specific angular momentum of the K-N black hole, respectively.

We first consider a stationary-time hyper-surface Σ embedded in the K-N spacetime with the metric (1). We define the two-surface B with some constant value for the Boy-Lindquist (B-L) radius $r = r_{\mathrm{B}}$ embedded in Σ. In the slowly rotating approximation, i.e. $a << r_{\mathrm{B}}$, the line-element on this surface can be written as[29]

$$\begin{aligned}\mathrm{d}^2s \approx& r_{\mathrm{B}}^2\left(1+\frac{a^2\cos^2\theta}{r_{\mathrm{B}}^2}\right)\mathrm{d}^2\theta + r_{\mathrm{B}}^2\left\{1+\frac{a^2}{r_{\mathrm{B}}^2}\right.\\ &\left.\cdot\left[1+\left(\frac{2M}{r_{\mathrm{B}}}-\frac{Q^2}{r_{\mathrm{B}}^2}\right)\sin^2\theta\right]\right\}\sin^2\theta\mathrm{d}^2\varphi,\end{aligned}\tag{4}$$

where $r = r_{\mathrm{B}} \geq r_+$ and r_+ is the outer horizon of the black hole.

The thermodynamical action of the K-N black hole is given by[30]

$$\begin{aligned}I =&\frac{1}{8\pi}\oint_B \mathrm{d}^2x\sqrt{\sigma}\beta k - \frac{1}{8\pi N}\oint_B \mathrm{d}^2x\sqrt{\sigma}n^iD_iN\\ &-\frac{1}{\sqrt{h}}\oint_B \mathrm{d}^2x\sqrt{\sigma}\beta[\omega 2n_i\phi_jp^{ij}\\ &+\Psi n_i\varepsilon^i+\omega(A_j\phi^j)n_i\varepsilon^i\big|_H^B + I_{\mathrm{subt}},\end{aligned}\tag{5}$$

where σ is the determinant of the metric induced on B, n_i is a radially outward-pointing normal defined on Σ, k is the trace of the extrinsic curvature of B as embedded in Σ, N is the lapse function, D_i the spatial covariant derivative operator, Ψ the electrical potential, p^{ij} the corresponding momentum, ω the angular velocity, and I_{subt} is the arbitrary term that can be used to define the zero of energy, as will be seen later.

The first term is just the B-Y quasi-local energy E[27], which is, up to $O(a^4/r_{\mathrm{B}}^4)$, explicitly given by[27]

$$\begin{aligned}E =&\frac{1}{8\pi}\oint_B \mathrm{d}^2x\sqrt{\sigma}\beta k\\ =&-\beta r_{\mathrm{B}}\sqrt{1-\frac{2M}{r_{\mathrm{B}}}+\frac{a^2+Q^2}{r_{\mathrm{B}}^2}}\\ &\cdot\left[1-\frac{a^2}{6r_{\mathrm{B}}^2}\left(1+\frac{2M}{r_{\mathrm{B}}}-\frac{Q^2}{r_{\mathrm{B}}^2}\right)\right].\end{aligned}\tag{6}$$

The second term is[30]

$$-\frac{1}{8\pi N}\oint_B \mathrm{d}^2x\sqrt{\sigma}n^iD_iN = -\pi r_+^2.\tag{7}$$

For the slowly rotating K-N black hole $A_j \approx 0$, so integrating the last term, we obtain

$$\begin{aligned}&-\frac{1}{\sqrt{h}}\oint_B \mathrm{d}^2x\sqrt{\sigma}\beta[\omega 2n_i\phi_jp^{ij}\\ &+\Psi n_i\varepsilon^i+\omega(A_j\phi^j)n_i\varepsilon^i\big|_H^B\\ =&-\beta\Psi Q - \frac{5\beta a^2(r_+^2+a^2+Q^2)}{8r_+^3},\end{aligned}\tag{8}$$

where Ψ is the potential between the boundary and the horizon. Therefore, the thermodynamical action of the slowly rotating K-N black hole in the grand canonical ensemble is

$$\begin{aligned}I =&-\beta r_{\mathrm{B}}\sqrt{1-\frac{2M}{r_{\mathrm{B}}}+\frac{a^2+Q^2}{r_{\mathrm{B}}^2}}\\ &\cdot\left[1-\frac{a^2}{6r_{\mathrm{B}}^2}\left(1+\frac{2M}{r_{\mathrm{B}}}-\frac{Q^2}{r_{\mathrm{B}}^2}\right)\right]\\ &-\pi r_+^2-\beta\Psi Q-\frac{5\beta a^2(r_+^2+a^2+Q^2)}{8r_+^3}+I_{\mathrm{subt}}.\end{aligned}\tag{9}$$

For the K-N black hole, the outer horizon radius is

$$r_+ = M + \sqrt{M^2-a^2-Q^2},\tag{10}$$

which can be rewritten as

$$2M = (r_+^2+a^2+Q^2)/r_+.\tag{11}$$

Substituting Eq.(11) into Eq.(9) yields

$$\begin{aligned} I =& -\beta r_{\rm B}\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}} \\ & \cdot\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right] \\ & -\beta\Psi Q-\pi r_+^2-\frac{5\beta a^2(r_+^2+a^2+Q^2)}{8r_+^3} \\ & +I_{\rm subt}, \end{aligned} \tag{12}$$

$I_{\rm subt}$ is an arbitrary term that can be used to fix the zero of the energy, but does not affect other thermodynamical variables; it has the form $I_{\rm subt}=\beta E_{\rm subt}$.[13] For convenience, we use it for the zero of the energy

$$E(r_+=0,a=0,Q=0)=0. \tag{13}$$

To evaluate $I_{\rm subt}$, we calculate the thermal energy of the K-N black hole from Eq.(9) and use condition (13). The thermal energy is[22]

$$\begin{aligned} E =& F+\beta\left(\frac{\partial F}{\partial\beta}\right)_{\Psi,r_{\rm B}}-\Psi\left(\frac{\partial F}{\partial\Psi}\right)_{\beta,r_{\rm B}} \\ =& \left(\frac{\partial\overline{I}}{\partial\beta}\right)_{\Psi,r_{\rm B}}-\frac{\Psi}{\beta}\left(\frac{\partial\overline{I}}{\partial\Psi}\right)_{\beta,r_{\rm B}} \\ =& -r_{\rm B}\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}} \\ & \cdot\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right] \\ & -\frac{5\beta a^2(r_+^2+a^2+Q^2)}{8r_+^3}-E_{\rm subt}, \end{aligned} \tag{14}$$

where F is the grand canonical potential, the classical action $\overline{I}$ responds to the minimum of the thermodynamical action. Hence we have the equalities $\left(\frac{\partial\overline{I}}{\partial\beta}\right)_{\Psi,r_{\rm B}}=\left(\frac{\partial I}{\partial\beta}\right)_{\Psi,r_{\rm B},r_+,Q}$ and $\left(\frac{\partial\overline{I}}{\partial\Psi}\right)_{\beta,r_{\rm B}}=\left(\frac{\partial I}{\partial\Psi}\right)_{\beta,r_{\rm B},r_+,Q}$. Equations (13) and (14) then yield

$$E_{\rm subt}=-r_{\rm B}. \tag{15}$$

Substituting Eq.(15) into Eq.(12), we finally obtain the thermodynamical action of the slowly rotating K-N black hole

$$\begin{aligned} I =& \beta r_{\rm B}-\beta r_{\rm B}\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}} \\ & \cdot\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right] \\ & -\beta\Psi Q-\pi r_+^2-\frac{5\beta a^2(r_+^2+a^2+Q^2)}{8r_+^3}. \end{aligned} \tag{16}$$

From now on, we will mainly investigate the thermodynamical properties of this slowly rotating K-N black hole in the grand canonical ensemble. Substituting (15) into (14), we obtain the thermal energy

$$\begin{aligned} E =& r_{\rm B}-r_{\rm B}\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}} \\ & \cdot\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right] \\ & -\frac{5a^2(r_+^2+a^2+Q^2)}{8r_+^3}. \end{aligned} \tag{17}$$

The charge is given by

$$q=-\left(\frac{\partial F}{\partial\Psi}\right)_{\beta,r_{\rm B}}=-\frac{1}{\beta}\left(\frac{\partial F}{\partial\Psi}\right)_{\beta,r_{\rm B}}=Q. \tag{18}$$

The entropy is

$$\begin{aligned} S =& \beta^2\left(\frac{\partial F}{\partial\beta}\right)_{\Psi,r_{\rm B}}=\beta\left(\frac{\partial I}{\partial\beta}\right)_{\Psi,r_{\rm B}}-I \\ =& \pi r_+^2=\frac{A_+}{4}. \end{aligned} \tag{19}$$

This result agrees with the Bekenstein-Hawking entropy.[31,32] In the grand canonical ensemble we fix β, Ψ, a and $r_{\rm B}$ of the boundary, so the thermodynamical action is a function of r_+ and Q only. The equilibrium conditions of the black hole are[17]: 1) δI=0, and 2) the Hessian matrix is positive. From the first condition of equilibrium, we know

$$\begin{aligned} \frac{\partial I}{\partial r_+} =& -\beta r_{\rm B}\left[1-\frac{a^2}{6r_{\rm B}^2}\left(\frac{1}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}\right)\right]\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}} \\ & -\beta r_{\rm B}\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right]\left(\frac{a^2+Q^2}{r_+r_{\rm B}}-\frac{1}{r_{\rm B}}\right)\Big/ \\ & 2\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}}-2\pi r_++\frac{5\beta a^2}{8r_+^2}+\frac{15\beta a^2(a^2+Q^2)}{8r_+^4}=0, \end{aligned} \tag{20}$$

$$\frac{\partial I}{\partial Q} = \frac{\beta Q a^2}{3r_{\rm B}^2}\left(\frac{1}{r_+}-\frac{1}{r_{\rm B}}\right)\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+ r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}}$$
$$-\beta Q\left[1-\frac{a^2}{6r_{\rm B}^2}\left(1+\frac{r_+}{r_{\rm B}}+\frac{a^2+Q^2}{r_+ r_{\rm B}}-\frac{Q^2}{r_{\rm B}^2}\right)\right]\left(\frac{1}{r_{\rm B}}-\frac{1}{r_+}\right)\Big/$$
$$2\sqrt{1-\frac{r_+}{r_{\rm B}}-\frac{a^2+Q^2}{r_+ r_{\rm B}}+\frac{a^2+Q^2}{r_{\rm B}^2}}-\beta\Psi-\frac{15\beta a^2 Q}{4r_+^3}=0. \tag{21}$$

For convenience, we introduce the following variables:

$$x=r_+/r_{\rm B},\quad \overline{Q}=Q/r_{\rm B},\quad \overline{\beta}=\beta/4\pi r_{\rm B},\quad \overline{a}=a/r_{\rm B}, \tag{22}$$

Then eliminating $O(\overline{a}^{\,4})$, Eqs.(20) and (21) become:

$$6\overline{a}^2x^6+6\overline{Q^2}x^6-6x^7-6\overline{a}^2x^7-6\overline{Q^2}x^7+6x^8-45\overline{a}^2\overline{\beta Q^4}x+45\overline{a}^2\overline{\beta Q^2}x^2$$
$$+45\overline{a}^2\overline{\beta Q^4}x^2-60\overline{a}^2\overline{\beta Q^2}x^3+15\overline{a}^2\overline{\beta}x^4+15\overline{a}^2\overline{\beta Q^2}x^4-15\overline{a}^2\overline{\beta}x^5-6\overline{a}^2\overline{\beta^2Q^6}$$
$$+6\overline{\beta^2Q^4}x+2\overline{a}^2\overline{\beta^2Q^4}x+6\overline{a}^2\overline{\beta^2Q^6}x+6\overline{a}^2\overline{\beta^2Q^4}x^2-12\overline{a}^2\overline{\beta^2}x^3-12\overline{\beta^2Q^2}x^2$$
$$-4\overline{a}^2\overline{\beta^2Q^2}x^3-12\overline{a}^2\overline{\beta^2Q^4}x^3+6\overline{a}^2\overline{\beta^2Q^2}x^4+6\overline{\beta^2}x^5+2\overline{a}^2\overline{\beta^2}x^5+6\overline{a}^2\overline{\beta^2Q^2}x^5$$
$$-6\overline{a}^2\overline{\beta^2}x^6+12\overline{a}^2\overline{\beta^2Q^2}x=0, \tag{23}$$
$$6\overline{a}^2\overline{Q^4}x-6\overline{Q^4}x^2-2\overline{a}^2\overline{Q^2}x^2-12\overline{a}^2\overline{Q^4}x^2+6\overline{Q^2}x^3+8\overline{a}^2\overline{Q^2}x^3+6\overline{a}^2\overline{Q^4}x^3$$
$$+6\overline{a}^2\overline{Q^2}x^4-15\overline{a}^2\overline{Q^3}\,\Psi+15\overline{a}^2\overline{Q}x\,\Psi-6\overline{a}^2\,\Psi^2x^3-6\,\Psi^2\overline{Q^2}x^3+6\,\Psi^2x^4=0. \tag{24}$$

The second condition will be used in section IV.

Ⅲ. NUMERICAL SOLUTIONS OF THE SLOWLY ROTATING K-N BLACK HOLE

In this section we study the numerical solutions of Eqs.(23) and (24). After eliminating the parameter $\overline{Q}$ of Eqs.(23) and (24), we find the new equation is very complicated, so we present, in graphics, an analysis of the equation in three steps. 1) We show the relation between x and Ψ by fixing $\overline{a}$ and $\overline{\beta}$ in Figs.1–4; 2) Figs.5–7 are devoted to the case for fixed $\overline{a}$; 3) Figs.8–12 show the cases for fixed $\overline{\beta}$.

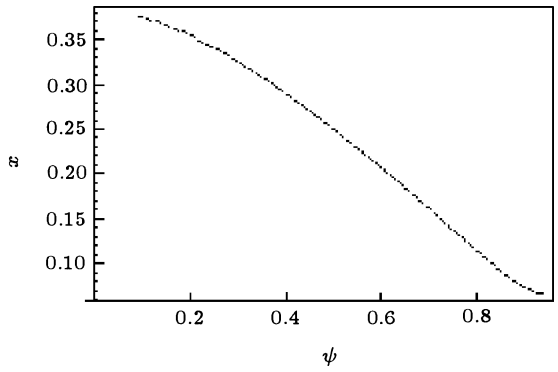


Fig.2. Solution of the slowly rotating K-N black hole for $\overline{a}$=0.01, $\overline{\beta}$=0.3.

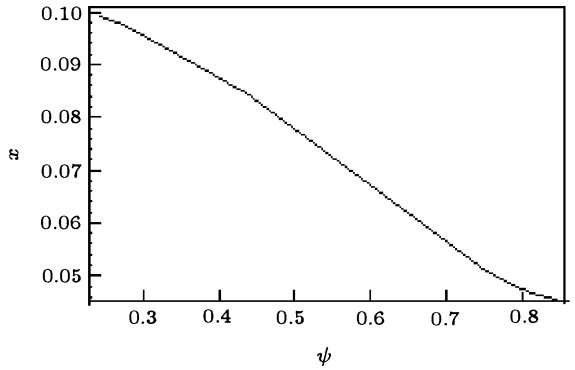


Fig.1. Solution of the slowly rotating K-N black hole for $\overline{a}$=0.01, $\overline{\beta}$=0.1.

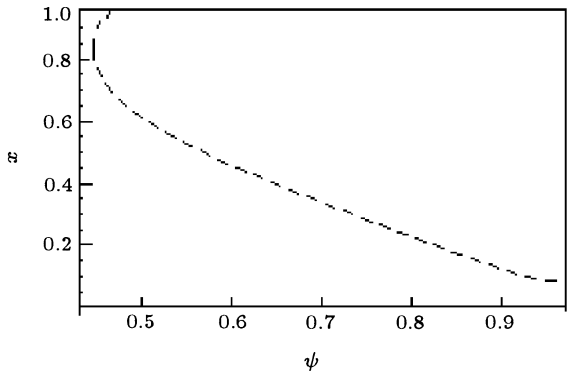


Fig.3. Solution of the slowly rotating K-N black hole for $\overline{a}$=0.01, $\overline{\beta}$=0.6. Only the upper branch is stable, the others are instantons.

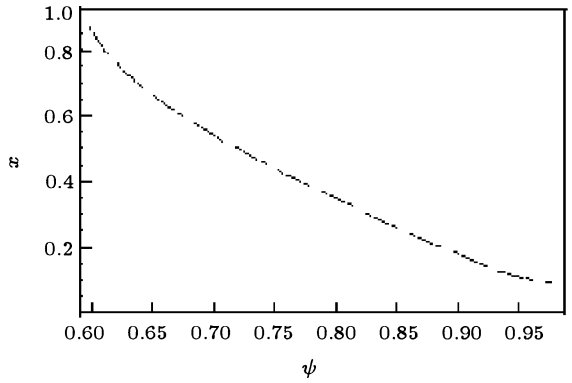


Fig.4. Solution of the slowly rotating K-N black hole for $\overline{a}$=0.01, $\overline{\beta}$=0.9.

From Figs.1, 2 and 4, we find that there is no or one solution of different values of Ψ for fixed $\overline{a}$ and $\overline{\beta}$. In Fig.3, we can see that x has zero, one, or two solutions for different values of Ψ. When there are two solutions, we regard the branch of the curve with the higher value of x as the higher branch and the branch with the lower value of x as the lower branch. The stability of these solutions will be investigated in section IV.

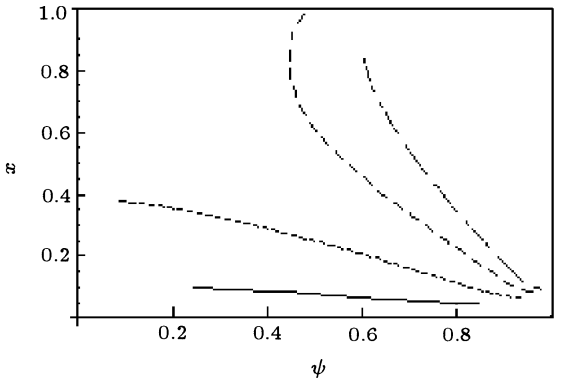


Fig.5. Solutions of the slowly rotating K-N black hole for $\overline{a}$=0.01, $\overline{\beta}$=0.1(solid), 0.3, 0.6, 0.9. Only the upper branch is stable, the others are instantons.

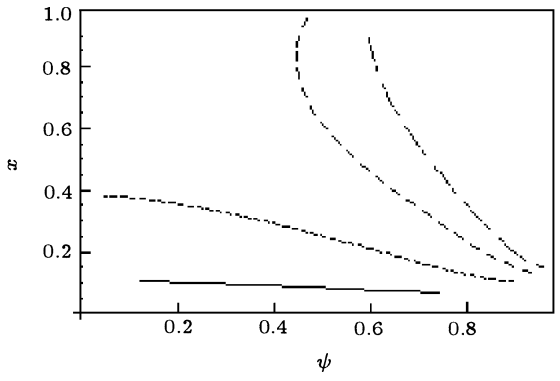


Fig.6. Solution of the slowly rotating K-N black hole for $\overline{a}$=0.02, $\overline{\beta}$=0.1(solid), 0.3, 0.6, 0.9. Only the upper branch is stable, the others are instantons.

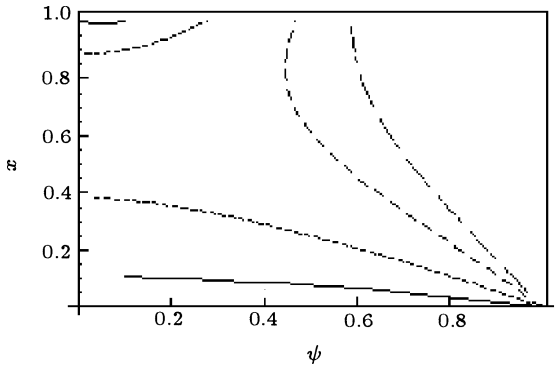


Fig.7. Solution of the slowly rotating K-N black hole for $\overline{a}$=0, $\overline{\beta}$=0.1(solid), 0.3, 0.6, 0.9. Only the upper branch is stable, the others are instantons.

In Figs.5–7, we plot different values of $\overline{\beta}$ (fixed $\overline{a}$) in the same picture. From these pictures, we can see that, for fixed Ψ, there exists a maximum of $\overline{\beta}$, $(\overline{\beta}_{\max}(\Psi))$, so for $\overline{\beta} > \overline{\beta}_{\max}(\Psi)$, there are no solutions; for $\overline{\beta} < \overline{\beta}_{\max}(\Psi)$, there are one or two solutions depending on $\overline{\beta}$. Comparing Figs.5–7, we find higher $\overline{\beta}_{\max}(\Psi)$ for higher $\overline{a}$. Especially, for $\overline{a}$=0, Fig.7 presents the solutions of the Reissner-Nordstrom black hole.[17,20] For Ψ=0, it presents the Schwarzschild black hole; there are two solutions for $\overline{\beta} < \overline{\beta}_{\max}(0)=2/\sqrt{27}$. In the limiting case, $\overline{\beta} \to 0, (T \to \infty)$, we find that there is a solution $x = r_+/r_{\rm B} = 1$. (See Ref.[13]) In the other limiting case $\overline{\beta} \to \infty$ $(T \to 0)$, we find that Ψ is nearly a constant 1 as x has different values. This corresponds to the extreme Reissner-Nordstrom black hole.

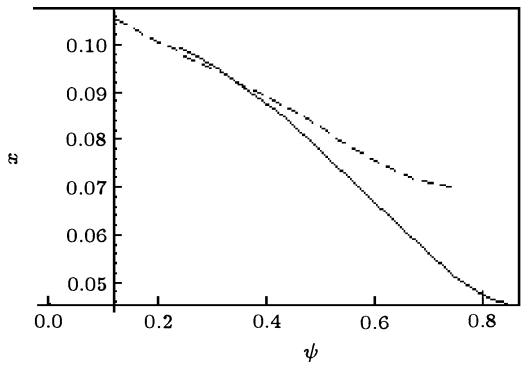


Fig.8. Solution of the slowly rotating K-N black hole for $\overline{\beta}$=0.1, $\overline{a}$=0.01(solid), 0.02.

Investigating the solutions for fixed $\overline{\beta}$ in Figs.8–11, we can also find that for fixed Ψ there is a maximum $\overline{\beta}_{\max}(\overline{a})$, so for $\overline{\beta} > \overline{\beta}_{\max}(\overline{a})$ there is no solution. For $\overline{\beta} < \overline{\beta}_{\max}(\overline{a})$, we can have one or two solutions depending on the precise value of $\overline{\beta}$. In the extreme cases $\overline{\beta} \to \infty$ $(T \to 0)$ (Fig.12), when the black hole

rotates slowly, we find that Ψ increases sharply to 1 as x increases. This result corresponds well to the extreme Reissner-Nordstrom black hole, identical to the case discussed above.

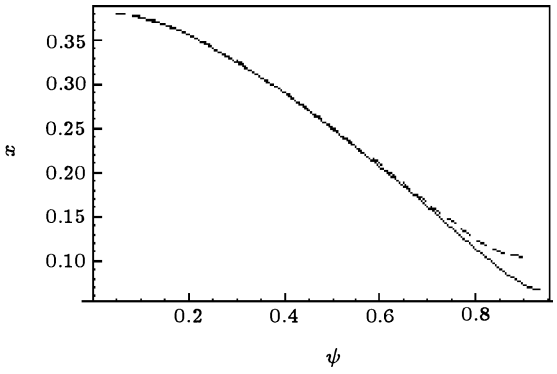


Fig.9. Solution of the slowly rotating K-N black hole for $\overline{\beta}$=0.3, $\overline{a}$=0.01 (solid), 0.02.

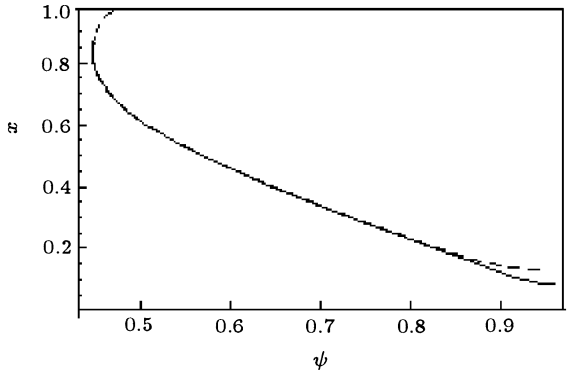


Fig.10. Solution of the slowly rotating K-N black hole for $\overline{\beta}$=0.6, $\overline{a}$=0.01(solid), 0.02. Only the upper branch is stable, the others are instantons.

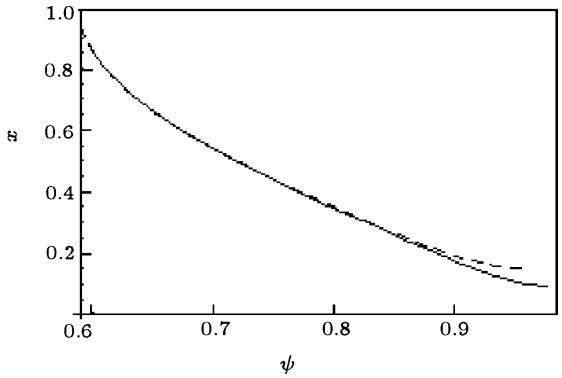


Fig.11. Solution of the slowly rotating K-N black hole for $\overline{\beta}$=0.9, $\overline{a}$=0.01(solid), 0.02.

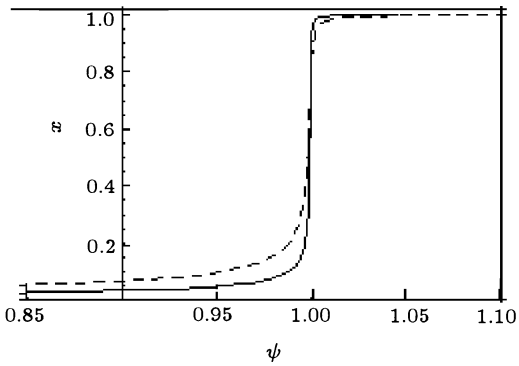


Fig.12. Solution of the slowly rotating K-N black hole for $\overline{\beta}=\infty$, $\overline{a}$=0.01(solid), 0.02.

IV. LOCAL STABILITY

We now study the stability of the solutions using the second condition of equilibrium that the Hessian matrix $I_{,ij}(i,j=Q,S$, where $S=\pi r_+^2)$ of the thermodynamical action is positive defined. The matrix is positive defined if its pivots are all positive. The pivots of the above matrix are

$$I_{,QQ}, \tag{25}$$

$$\frac{\det(I_{,ij})}{I_{,QQ}}. \tag{26}$$

It is easy to show that

$$\left(\frac{\partial I}{\partial Q}\right)_S=\beta\left(\frac{\partial E}{\partial Q}\right)_S-\beta\Psi=0\Rightarrow\Psi=\left(\frac{\partial E}{\partial Q}\right)_S, \tag{27}$$

$$\left(\frac{\partial I}{\partial S}\right)_Q=\beta\left(\frac{\partial E}{\partial S}\right)_Q-1=0\Rightarrow\beta^{-1}=T=\left(\frac{\partial E}{\partial S}\right)_Q. \tag{28}$$

From Eqs.(27) and (28), we get

$$\frac{\partial^2 I}{\partial Q^2}=\beta\frac{\partial^2 E}{\partial Q^2}=\beta\left(\frac{\partial\Psi}{\partial Q}\right)_S, \tag{29}$$

$$\frac{\partial^2 I}{\partial Q\partial S}=\beta\frac{\partial^2 E}{\partial Q\partial S}=\beta\left(\frac{\partial\Psi}{\partial S}\right)_Q=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial Q}\right)_S, \tag{30}$$

$$\frac{\partial^2 I}{\partial S^2}=\beta\frac{\partial^2 E}{\partial S^2}=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial S}\right)_Q. \tag{31}$$

Equation (29) is the pivot $I_{,QQ}$, The second pivot (26) can be easily obtained from Eqs.(29), (30) and (31)

$$\frac{\det I_{,ij}}{I_{,QQ}}=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial S}\right)_\Psi=\frac{1}{C_{\Psi,r_{\rm B}}}, \tag{32}$$

where $C_{\Psi,r_{\rm B}}$ is the heat capacity at constant Ψ and $r_{\rm B}$. Imposing positive pivots for local stability, we arrive at

$$\left(\frac{\partial\Psi}{\partial Q}\right)_{S,r_{\rm B}}\geq 0, \tag{33}$$

$$C_{\Psi,r_{\rm B}}\geq 0. \tag{34}$$

These conditions are identical to the classical thermodynamical stability conditions.[22] For the slowly rotating K-N black hole, the pivots obtained

$$\left(\frac{\partial\Psi}{\partial\overline{Q}}\right)_{,\beta=0.1} = (12\overline{Q}(\overline{Q^2}-x)(x-1)x^6 + \overline{a}^2(-12\overline{Q^7}(x-1)^2x^4 + 30\overline{Q^5}(x-1)^2x^5$$
$$- 15\overline{Q^4}\sqrt{\overline{Q^2}(\overline{Q^2}-x)(x-1)x^5} + 30\overline{Q}^2x\sqrt{\overline{Q^2}(\overline{Q^2}-x)(x-1)x^5}$$
$$- 15x^2\sqrt{\overline{Q^2}(\overline{Q^2}-x)(x-1)x^5} + 2\overline{Q}x^6(3-2x-4x^2+3x^3)$$
$$- 4\overline{Q^3}x^6(-3+8x-11x^2+6x^3)/(12(\overline{Q^2}-x)^2x^3\sqrt{\overline{Q^2}(\overline{Q^2}-x)(x-1)x^5}. \quad (35)$$

In order to see that Eq.(33) is positive, we give $\overline{a}$=0.01 and plot it in a three-dimensional picture (Fig.13.), from which we find $\left(\frac{\partial\Psi}{\partial\overline{Q}}\right)_{\overline{a}=0.01,\overline{\beta}=0.1} \geq 0$. The first condition of the local stability (33) is satisfied. Then we investigate the second condition (34).

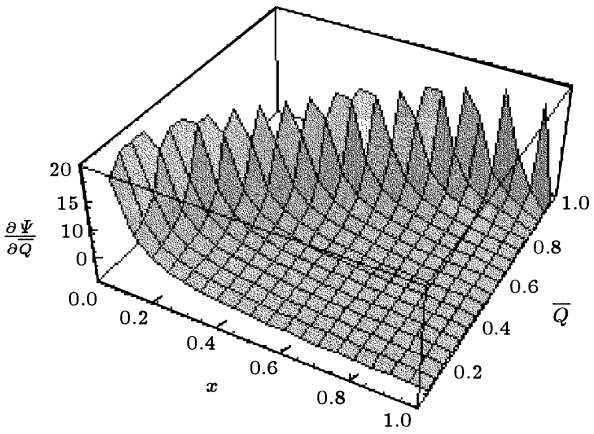


Fig.13. x-$\overline{Q}$-$\partial\Psi/\partial\overline{Q}$ three-dimensional picture for $\overline{a}$=0.01, $\overline{\beta}$=0.1.

We use Eq.(32) to calculate the heat capacity, but we cannot explicitly write down its expression because of its complexity. However, if we take $\overline{a}$=0.1, $\overline{\beta}$=0.1, we can plot x-$\overline{Q}$-$C_{\Psi,r_{\rm B}}$ in a three-dimensional picture (Fig.14). From Fig.14, we find that the heat capacity is positive when $x \to 1$, and it corresponds to the higher branch of Figs.3, 5, 6 and 7. Hence we say that only the solutions with a higher value of x, i.e. with a higher event horizon radius, are stable. The heat capacity is negative when x has a lower value, that is to say, the solutions with a lower value of x, i.e. with a lower event horizon radius, are unstable (instantons).

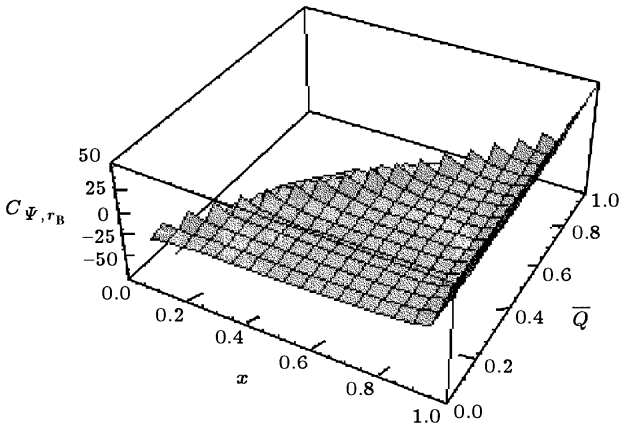


Fig.14. x-$\overline{Q}$-$C_{\Psi,r_{\rm B}}$ three-dimensional picture for $\overline{a}$=0.01, $\overline{\beta}$=0.1.

V. CONCLUSIONS

We have investigated the thermodynamics of the slowly rotating K-N black hole in the grand canonical ensemble by York's formalism, the BY quasi-local energy and the thermodynamical action of Brown *et al.* To simplify the calculation we have studied the slowly rotating case and omitted the higher infinitesimal; we find that for different boundary conditions there may be zero, one or two solutions for the black hole. Studying the stability, we show that when there are two solutions, only the one with a larger event horizon radius is stable, and the other is an instanton. We have also obtained the expressions for the thermal energy, entropy and charge. Owing to the complexity of K-N spacetime, we have only investigated certain thermodynamical properties, giving numerical solutions for the slowly rotating K-N black hole. In order to study the thermodynamical properties of the K-N black hole completely, we must introduce a more convenient thermodynamical action formalism. We will devote ourselves to this problem in the future.

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