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ANGULAR VELOCITY AND CORIOLIS EFFECT IN TIME-DEPENDENT QUANTUM MECHANICAL SU_2 ROTATION*

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Starting from a time-dependent rotation $U(t)$ in SU_2 group element space, we derive its corresponding quantum mechanical dynamic Coriolis term and the relationship between $U(t)$ and rotational angular velocity. Throughout our discussion, the technique of integration within an ordered product of operators is fully used, which has the advantage that the correspondence between the classical rotation and the quantum rotation is in a transparent fashion. A new angular-velocity formula is also derived.

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Based on Dirac's symbolic method, representation theory and transformation theory,[1] the technique of integration within an ordered product of operators (IWOP) has been introduced in Ref.[2]. This technique can help us to find more new useful quantum mechanical representations, especially the entangled state representation with continuum variables.[3] In this paper, we consider the following problems: for a time-dependent $SU(2)$ group element $U(t)$, which represents a rotation in group parametric space, what is the corresponding angular velocity and what is its quantum mechanical correspondence so far as the Coriolis force is concerned? We shall employ the IWOP technique to solve these problems. For the given $U(t)$ let us introduce the following time-dependent quantum mechanical operator expressed in bosonic coherent state representation

$$D(U(t))=\pi^{-2}\int \mathrm{d}^2z_1\mathrm{d}^2z_2\left|U(t)\left(\begin{array}{c}z_1\\z_2\end{array}\right)\right\rangle\left\langle\left(\begin{array}{c}z_1\\z_2\end{array}\right)\right|, \tag{1}$$

where $\left|\left(\begin{array}{c}z_1\\z_2\end{array}\right)\right\rangle \equiv |z_1,z_2\rangle$ is the two-mode coherent state, defined as

$$|z_1,z_2\rangle=\exp\left\{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1a_1^\dagger+z_2a_2^\dagger\right\}|00\rangle, \tag{2}$$

a_i (a_i^+) $(i=1,2)$ is Bose annihilation (creation) operator,$|z_1,z_2\rangle$ satisfy the overcompleteness relation[4]

$$\pi^{-2}\int \mathrm{d}^2z_1\mathrm{d}^2z_2\left|\left(\begin{array}{c}z_1\\z_2\end{array}\right)\right\rangle\left\langle\left(\begin{array}{c}z_1\\z_2\end{array}\right)\right|=1.$$

In the following, as a first step, we want to identify Eq.(1) as a quantum mechanical rotation operator. Then we search for the time-dependent Hamiltonian which is responsible for such a quantum mechanical rotation. As a by-product we shall derive the corresponding angular velocity of time-dependent $SU(2)$ rotation. As the derivation continues, we can identify the term $\mathrm{i}\dot{U}U^\dagger$ as a Coriolis term, where the dot over U means the differentiation with respect to time t.

We start by using the IWOP technique[2,3,5] (the essence of the IWOP technique lies in that Bose operators can be permuted with normal ordering symbol) and

$$|00\rangle\langle 00|=:\exp\{-a_1^\dagger a_1-a_2^\dagger a_2\}:, \tag{3}$$

where : : denotes normal ordering. We can directly integrate Eq.(1). Here and henceforth we adopt the Einstein convention—if an index is repeated in a term, summation over it from 1 to 2 is implied. Substituting Eqs.(2) and (3) into Eq.(1) we have

$$D(U(t))=\int\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}:\exp\left\{-\frac{1}{2}|U_{ij}z_j|^2+a_i^\dagger U_{ij}z_j-\frac{1}{2}|z_i|^2+a_iz_i^*-a_i^\dagger a_i\right\}:. \tag{4}$$

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Noticing

$$U^{\dagger}U = UU^{\dagger} = 1,\ \sum_{i=1}^{2}|U_{ij}z_j|^2 = \sum_{i=1}^{2}|z_i|^2,$$

then Eq.(4) becomes

$$\begin{aligned} D(U(t)) = \int\int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} &: \exp\{-|z_i|^2 \\ &+ a_i^{\dagger}U_{ij}z_j + a_i z_i^{*} - a_i^{\dagger}a_i\} : \\ &= : \exp\{a_i^{\dagger}(U_{ij} - \delta_{ij})a_j\} : . \end{aligned} \tag{5}$$

Using the operator identity[4]

$$\exp\{a_i^{\dagger}\Lambda_{ij}a_j\} =: \exp\{a_i^{\dagger}(\mathrm{e}^{\Lambda} - 1)_{ij}a_j\} :, \tag{6}$$

where Λ is a 2×2 matrix we can put Eq.(5) into the compact form

$$D(U(t)) = \exp\{a_i^{\dagger}(\ln U)_{ij}a_j\}. \tag{7}$$

Without loss of generality, we let the group parameters be

$$\begin{aligned} U &= \exp\left(\mathrm{i}\boldsymbol{\sigma}\cdot\frac{\psi}{2}\boldsymbol{n}\right) \\ &= \begin{bmatrix} \cos\dfrac{\psi}{2} + \mathrm{i}\sin\dfrac{\psi}{2}\cos\theta, & \mathrm{i}\sin\theta\sin\dfrac{\psi}{2}\exp[-\mathrm{i}\phi] \\ \mathrm{i}\sin\theta\sin\dfrac{\psi}{2}\exp[\mathrm{i}\phi], & \cos\dfrac{\psi}{2} - \mathrm{i}\sin\dfrac{\psi}{2}\cos\theta \end{bmatrix}, \end{aligned} \tag{8}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The logarithm of U is

$$\ln U = \mathrm{i}\psi \begin{pmatrix} \dfrac{1}{2}\cos\theta, & \dfrac{1}{2}\mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \dfrac{1}{2}\mathrm{e}^{\mathrm{i}\phi}\sin\theta, & -\dfrac{1}{2}\cos\theta \end{pmatrix}. \tag{9}$$

With the aid of Schwinger bosonic representation of angular momentum,[6] we know

$$\boldsymbol{J} = \left(a_1^{\dagger}, a_2^{\dagger}\right)\frac{\boldsymbol{\sigma}}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

$$\begin{aligned} J_x &= \frac{a_1^{\dagger}a_2 + a_2^{\dagger}a_1}{2}, \\ J_y &= \frac{a_1^{\dagger}a_2 - a_2^{\dagger}a_1}{2\mathrm{i}}, \\ J_z &= \frac{a_1^{\dagger}a_1 - a_2^{\dagger}a_2}{2}. \end{aligned} \tag{10}$$

Substituting Eq.(9) into Eq.(7) and using Eq.(10) gives the explicit form of quantum mechanical rotation:

$$D\left(\exp\left(\mathrm{i}\boldsymbol{\sigma}\cdot\frac{\psi}{2}\boldsymbol{n}\right)\right) = \exp\{\mathrm{i}\psi\boldsymbol{n}\cdot\boldsymbol{J}\}. \tag{11}$$

Therefore, we have naturally derived the quantum mechanical SU_2 rotation operator directly from the classical SU_2 transformation from $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ to $U(t)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ in the coherent state basis via the technique of IWOP, and Eq.(11) clearly shows that $D(U(t))$ is a rotation operator.

Another important question is: what is the Hamiltonian that dominates this time evolution of rotation? Because a_i and $a_i^{\dagger}$ commute with each other within : :, we can differentiate Eq.(5) with respect to time t and obtain

$$\begin{aligned} \partial_t D(U(t)) &= : \partial_t \exp\{a_i^{\dagger}(U_{ij} - \delta_{ij})a_j\} : \\ &= : a_i^{\dagger}\dot{U}_{ij}a_j \exp\{a_i^{\dagger}(U_{ij} - \delta_{ij})a_j\} : \\ &= a_i^{\dagger}\dot{U}_{ij} : \exp\{a_i^{\dagger}(U_{ij} - \delta_{ij})a_j\} : a_j \\ &= a_i^{\dagger}\dot{U}_{ij}Da_jD^{-1}D. \end{aligned} \tag{12}$$

Using the Baker–Hausdoff formula and Eq.(7), we have

$$\begin{aligned} Da_lD^{-1} &= \exp\{a_i^{\dagger}(\ln U)_{ij}a_j\}a_l\exp\{a_i^{\dagger}(-\ln U)_{ij}a_j\} \\ &= \exp\{-\ln(U)_{lk}\}a_k = U_{lk}^{\dagger}a_k. \end{aligned} \tag{13}$$

Substituting Eq.(13) into Eq.(12), we obtain

$$\partial_t D(U(t)) = a_i^{\dagger}\dot{U}_{ij}U_{jl}^{\dagger}a_l D(U(t)). \tag{14}$$

Comparing Eq.(14) with the standard form of Schrödinger equation

$$\mathrm{i}\partial_t D(U(t)) = HD(U(t)) \quad (\hbar = 1), \tag{15}$$

we should recognize the Hamiltonian is

$$H = \mathrm{i}a_i^{\dagger}\dot{U}_{ij}U_{jl}^{\dagger}a_l. \tag{16}$$

Let

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},\ \tilde{a}^{\dagger} = (a_1, a_2),$$

then H can be written as

$$H = \tilde{a}^{\dagger}[\mathrm{i}\dot{U}U^{\dagger}]a. \tag{17}$$

For the $SU(2)$ group element expressed by Eq.(8), we calculate

$$\mathrm{i}\dot{U}U^{\dagger}=\begin{bmatrix}\frac{\omega_z}{2}, & \mathrm{i}\frac{\omega_0}{2}\\ -\mathrm{i}\frac{\omega_0^*}{2}, & -\frac{\omega_z}{2}\end{bmatrix}, \tag{18}$$

where

$$\omega_z=\dot{\theta}\sin\theta\sin\psi+2\dot{\phi}\sin^2\theta\sin^2\frac{\psi}{2}-\dot{\psi}\cos\theta, \tag{19}$$

$$\begin{aligned}\omega_0=&\mathrm{e}^{-\mathrm{i}\phi}\Big\{-2\sin\frac{\psi}{2}\left(\sin\frac{\psi}{2}-\mathrm{i}\cos\theta\cos\frac{\psi}{2}\right)\dot{\theta}\\&+2\sin\theta\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2}+\mathrm{i}\cos\theta\sin\frac{\psi}{2}\right)\dot{\phi}+\mathrm{i}\dot{\psi}\sin\theta\Big\}.\end{aligned}$$

Defining

$$\begin{aligned}\omega_x=&\frac{\omega_0^*-\omega_0}{2\mathrm{i}}\\=&-2\sin\frac{\psi}{2}\left(\sin\frac{\psi}{2}\sin\phi+\cos\frac{\psi}{2}\cos\theta\cos\phi\right)\dot{\theta}\\&+2\sin\theta\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2}\sin\phi-\cos\theta\cos\phi\sin\frac{\psi}{2}\right)\dot{\phi}\\&-\dot{\psi}\sin\theta\cos\phi,\\\omega_y=&-\frac{\omega_0^*+\omega_0}{2}\\=&-2\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2}\cos\theta\sin\phi-\sin\frac{\psi}{2}\cos\phi\right)\dot{\theta}\\&-2\sin\theta\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2}\cos\phi+\cos\theta\sin\phi\sin\frac{\psi}{2}\right)\dot{\phi}\\&-\dot{\psi}\sin\theta\sin\phi,\end{aligned} \tag{20}$$

consequently Eq.(18) becomes

$$\mathrm{i}\dot{U}U^{\dagger}=\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}. \tag{21}$$

Thus we can identify $\boldsymbol{\omega}=(\omega_x,\omega_y,\omega_z)$ as the angular velocity for the rotation expressed by Eq.(8). So far as we know, this angular velocity expression has not been reported in the literature before. Furthermore, substituting Eqs.(19) and (10) into Eq.(17) yields

$$H=\boldsymbol{\omega}\cdot\boldsymbol{J}. \tag{22}$$

It is clear that $\boldsymbol{\omega}$ is the angular velocity of the SU_2 rotation (8). Remarkably, our method points out the physical source of the angular velocity of SU_2 rotation and naturally derives the relationship between $\boldsymbol{\omega}$ and the SU_2 transformation: $\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}=\mathrm{i}\dot{U}U^{\dagger}$. Alternatively, one may choose three Euler angles (α,β,γ) as three parameters describing the $SU(2)$ group element U_{E}, i.e.,

$$U_{\mathrm{E}}=\exp\left(-\frac{\mathrm{i}}{2}\sigma_z\beta\right)\exp\left(-\frac{\mathrm{i}}{2}\sigma_y\alpha\right)\exp\left(-\frac{\mathrm{i}}{2}\sigma_z\gamma\right). \tag{23}$$

Using similar procedures to the derivation of Eq.(18), for U_{E} we calculate $\mathrm{i}\dot{U}_{\mathrm{E}}U_{\mathrm{E}}^{+}=\begin{bmatrix}\frac{\omega_{\mathrm{E}z}}{2} & \mathrm{i}\frac{\omega_{0\mathrm{E}}}{2}\\ -\mathrm{i}\frac{\omega_{0\mathrm{E}}^*}{2} & -\frac{\omega_{\mathrm{E}z}}{2}\end{bmatrix}$,

$$\omega_{0\mathrm{E}}=-\dot{\alpha}\mathrm{e}^{-\mathrm{i}\beta}-\dot{\gamma}(\sin\alpha\sin\beta+\mathrm{i}\sin\alpha\cos\beta),$$

$$\omega_{x\mathrm{E}}=\frac{\omega_{0\mathrm{E}}^*-\omega_{0\mathrm{E}}}{2\mathrm{i}}=-\dot{\alpha}\sin\beta+\dot{\gamma}\sin\alpha\cos\beta,$$

$$\omega_{y\mathrm{E}}=-\frac{\omega_{0\mathrm{E}}^*+\omega_{0\mathrm{E}}}{2}=\dot{\alpha}\cos\beta+\dot{\gamma}\sin\alpha\sin\beta,$$

$$\begin{aligned}\boldsymbol{\omega}_{\mathrm{E}}=&(-\dot{\alpha}\sin\beta+\dot{\gamma}\sin\alpha\cos\beta,\dot{\alpha}\cos\beta\\&+\dot{\gamma}\sin\alpha\sin\beta,\dot{\beta}+\dot{\gamma}\cos\alpha),\end{aligned} \tag{24}$$

this coincides with the well-known angular velocity formula in the classical mechanics book.[7] We can also obtain the angular velocity of the reverse rotation, $\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}^{'}=\mathrm{i}\dot{U}^{\dagger}U$. Because $U^{\dagger}U=UU^{\dagger}=1$, and $U^{\dagger}\dot{U}+\dot{U}^{\dagger}U=0$, we thus have

$$\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}^{'}=-\mathrm{i}U^{\dagger}\dot{U}, \tag{25}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\omega}'=-U^{\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}U. \tag{26}$$

It should be noticed that $\boldsymbol{\omega}$ is the angular velocity of the 'lab' frame; while $\boldsymbol{\omega}^{'}$ is the angular velocity of the 'body' frame ($D(U^{\dagger}(t))$ describes the reverse rotation). Equation (26) shows the relation between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^{'}$. Furthermore, if we compare Eq.(20) with the Hamiltonian which dominates a spin-system in a magnetic field $H=\boldsymbol{\mu}\cdot\boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic induction intensity, and $\boldsymbol{\mu}$ is the spin operator, we can formally regard $\boldsymbol{\omega}$ as $\boldsymbol{B}$.

We now give Eq.(21) another physical explanation. Recall that in the Heisenberg picture all operators are time-dependent, and their equations of motion are $-\mathrm{i}\dot{F}(t)=[H_{\mathrm{H}}(t),F(t)]$. When operating a time-dependent unitary transformation $U(t,t_0)$ on H_{H}, in order to keep the form of evolution equation unchanged, the Hamiltonian should be

$$H_{\mathrm{H}}^{'}(t)=U(t,t_0)H_{\mathrm{H}}(t)U^{\dagger}(t,t_0)-\mathrm{i}\dot{U}(t,t_0)U^{\dagger}(t,t_0), \tag{27}$$

where the last term has a similar form to that in Eq.(21). Thus we can refer to $\mathrm{i}\dot{U}U^{\dagger}=\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}$ as a 'Coriolis' effect, or a 'Coriolis' term demonstrated in a rotating frame.

In summary, we have naturally derived the dynamic Hamiltonian (Coriolis term) and the relationship between $U(t)$ and angular velocity $\boldsymbol{\omega}$ (which can

also be time-dependent) of quantum mechanical SU_2 rotation from the pre-assigned classical group transformation $U(t)$ in $SU(2)$ group element space. The new angular velocity formula for rotation (8) is shown in Eqs.(19) and (20). Throughout our discussions, the IWOP technique and the coherent state are fully used.

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