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NOETHER'S THEOREM OF NONHOLONOMIC SYSTEMS OF NON-CHETAEV'S TYPE WITH UNILATERAL CONSTRAINTS*

Li Yuan-cheng(李元成)[a], Zhang Yi(张 毅)[b], Liang Jing-hui(梁景辉)[c], and Mei Feng-xiang(梅凤翔)[d]

[a] *Department of Applied Physics, University of Petroleum, Dongying 257062, China*
[b] *Department of Basic Courses, Suzhou Institute of Urban Construction & Environmental Protection, Suzhou 215011, China*
[c] *Department of Physics, Shanxi Normal University, Linfen 041004, China*
[d] *Department of Applied Mechanics, Beijing Institute of Technology, Beijing 100081, China*

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In this paper, we present Noether's theorem and its inverse theorem for nonholonomic systems of non-Chetaev's type with unilateral constraints. We present first the principle of Jourdain for the system and, on the basis of the invariance of the differential variational principle under the infinitesimal transformations of groups, we have established Noether's theory for the above systems. An example is given to illustrate the application of the result.

Keywords: analytical mechanics, unilateral constraint, nonholonomic system, Noether's theorem
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I. INTRODUCTION

In 1918, using the invariance of the Hamilton action under the infinitesimal transformations, Noether[1] studied the symmetries of a mechanical system and presented the famous Noether's theorem. The conservation laws and symmetries can also be studied by using the differential variational principles. In 1978[2] and 1986,[3] Vujanovic studied the conservation laws of dynamical systems on the basis of the D'Alembert, Jourdain and Gauss principles. Great progress[4−12] has been achieved in the research of symmetries and conservation laws of dynamical systems in the last ten or more years, but all the studies are primarily limited to ideal bilateral systems. In fact, the unilateral constraints are more general than the bilateral ones, and they are more difficult to study. The equations of motion and Noether's theorem for unilateral constraint systems in the configuration space have been studied in Ref.[9]. In this paper, Noether's theorem and its inverse theorem for the nonholonomic systems of non-Chetaev's type with unilateral constraints are further studied and an example is given to illustrate the application of the result.

II. JOURDAIN PRINCIPLE AND THE NONSIMULTANEOUS VARIATION OF THE SYSTEM

Suppose that the configuration of a system is determined by n generalized coordinates $q_s(s = 1, \cdots, n)$, the motion of the system is subjected to the ideal unilateral constraints as follows:

$$\varphi_\alpha(q_s, t) \geq 0 \qquad (\alpha = 1, \cdots, b), \tag{1}$$

$$\varphi_\beta(q_s, \dot{q}_s, t) \geq 0 \qquad (\beta = 1, \cdots, g). \tag{2}$$

Also, we suppose that the constraint hypersurface is absolutely smooth and the collision is perfectly elastic. The constraints (1) are unilateral holonomic and constraints (2) are unilateral nonholonomic.

If the sign of inequality in constraints (1) or (2) holds strictly, it is said that the system is divorced from the constraint; and if constraints (1) or (2) take the sign of equality, it is said that the system is in the constraint.

If constraints (2) are the nonholonomic constraints of non-Chetaev's type and if the restriction which constraints (2) impose on virtual displacements is

$$\sum_{s=1}^{n} f_{\beta s}(t, q_s, \dot{q}_s)\delta_1 \dot{q}_s = 0 \qquad (\beta = 1, \cdots, g), \tag{3}$$

where $f_{\beta s}$ is generally independent of $\partial\varphi_{\beta s}/\partial\dot{q}_s$; especially if $f_{\beta s} = \partial\varphi_{\beta s}/\partial\dot{q}_s$, then constraints (2) will be nonholonomic of Chetaev's type.

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The principle of Jourdain of the system can be expressed in the Euler–Lagrange form

$$\sum_{s=1}^{n}\Big[-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}+\frac{\partial L}{\partial q_s}+Q_s+\sum_{\alpha=1}^{b}\lambda_\alpha\frac{\partial\varphi_\alpha}{\partial q_s}+\sum_{\beta=1}^{g}\mu_\beta f_{\beta s}\Big]\delta_1\dot{q}_s=0,$$
$$\lambda_\alpha\geq 0,\quad \varphi_\alpha\geq 0,\quad \lambda_\alpha\varphi_\alpha=0\quad(\alpha=1,\cdots,b),$$
$$\mu_\beta\geq 0,\quad \varphi_\beta\geq 0,\quad \mu_\beta\varphi_\beta=0\quad(\beta=1,\cdots,g),\tag{4}$$

where $L=T-V$ is the Lagrangian of the system, Q_s is the non-potential generalized force and λ_α,μ_β are constraint multipliers.

Defining the Jourdain nonsimultaneous variation $\Delta_1\dot{q}_s$, $(\Delta_1 q_s)^{\cdot}$ and $(\Delta_1 t)^{\cdot}$, we have

$$\Delta_1 q_s=0,\quad \Delta_1 t=0.\tag{5}$$

By using the relation between Δ_1 and δ_1, it is easy to verify that[3]

$$\Delta_1\dot{q}_s=\delta_1\dot{q}_s,\tag{6}$$

$$(\Delta_1 q_s)^{\cdot}=\delta_1\dot{q}_s+\dot{q}_s(\Delta_1 t)^{\cdot},\tag{7}$$

$$\Delta_1\dot{q}_s=(\Delta_1 q_s)^{\cdot}-\dot{q}_s(\Delta_1 t)^{\cdot}.\tag{8}$$

We make the infinitesimal transformations of generalized coordinates, time and velocities[3] as follows:

$$q_s^*=q_s,\quad t^*=t,$$
$$\frac{\mathrm{d}q_s^*}{\mathrm{d}t}-\frac{\mathrm{d}q_s}{\mathrm{d}t}=\delta_1\dot{q}_s=\Delta_1\dot{q}_s,$$
$$\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*}-\frac{\mathrm{d}q_s}{\mathrm{d}t}=(\Delta_1 q_s)^{\cdot}.\tag{9}$$

Considering the infinitesimals $(\Delta_1 q_s)^{\cdot}$ and $(\Delta_1 t)^{\cdot}$ as the composition unit for Jourdain infinitesimal transformations and introducing the Jourdain generators of space and time, we have

$$(\Delta_1 q_s)^{\cdot}=\varepsilon F_s(t,q_s,\dot{q}_s),\quad (\Delta_1 t)^{\cdot}=\varepsilon f(t,q_s,\dot{q}_s).\tag{10}$$

From Eqs.(6)–(8), the Jourdain simultaneous variation can be expressed, by those generators, as

$$\delta_1\dot{q}_s=\varepsilon[F_s(t,q_s,\dot{q}_s)-\dot{q}_s f(t,q_s,\dot{q}_s)].\tag{11}$$

Ⅲ. NOETHER'S THEOREM

Substituting formula (11) into Eqs.(4) , we obtain

$$\varepsilon\Bigg\{\sum_{s=1}^{n}Q_s(F_s-\dot{q}_s f)+\sum_{\alpha=1}^{b}\lambda_\alpha\sum_{s=1}^{n}\frac{\partial\varphi_\alpha}{\partial q_s}(F_s-\dot{q}_s f)+\sum_{\beta=1}^{g}\mu_\beta\sum_{s=1}^{n}f_{\beta s}(F_s-\dot{q}_s f)+\sum_{s=1}^{n}\frac{\partial L}{\partial q_s}F_s+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{F}_s-\left(\frac{\partial L}{\partial t}+\sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\ddot{q}_s\right)f+\frac{\partial L}{\partial t}f-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\dot{f}-\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(F_s-\dot{q}_s f)\right]\Bigg\}=0.\tag{12}$$

Adding and subtracting a function $\varepsilon\dot{G}(t,q_s,\dot{q}_s)$ from Eq.(12) and noticing that

$$\dot{L}=\frac{\partial L}{\partial t}+\sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\ddot{q}_s,$$

we have

$$\varepsilon\Bigg\{\sum_{s=1}^{n}Q_s(F_s-\dot{q}_s f)+\sum_{\alpha=1}^{b}\lambda_\alpha\sum_{s=1}^{n}\frac{\partial\varphi_\alpha}{\partial q_s}(F_s-\dot{q}_s f)+\sum_{\beta=1}^{g}\mu_\beta\sum_{s=1}^{n}f_{\beta s}(F_s-\dot{q}_s f)+\sum_{s=1}^{n}\frac{\partial L}{\partial q_s}F_s+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{F}_s-\left(L-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)\dot{f}+\dot{G}+\frac{\partial L}{\partial t}f+\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}F_s+\left(L-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)f+G\right]\Bigg\}=0,\tag{13}$$

where $G(t,q_s,\dot{q}_s)$ is a gauge function. Equation (13) is the condition of invariance for the Jourdain principle of nonholonomic systems of non-Chetaev's type with unilateral constraints.

Hence we have

Theorem 1 If the generators F_s, f of the infinitesimal transformations and gauge function G satisfy

$$\sum_{s=1}^{n}\frac{\partial\varphi_\alpha}{\partial q_s}(F_s-\dot{q}_s f)=0,\tag{14}$$

$$\sum_{s=1}^{n}f_{\beta s}(F_s-\dot{q}_s f)=0,\tag{15}$$

$$\sum_{s=1}^{n}\frac{\partial L}{\partial q_s}F_s+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{F}_s+\frac{\partial L}{\partial t}f+\left(L-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)\dot{f}+\sum_{s=1}^{n}Q_s(F_s-\dot{q}_s f)+\dot{G}=0,\tag{16}$$

then the system has the following conserved quantity

$$\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}F_s+\left(L-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)f+G=\text{const.}\quad(17)$$

Theorem 1 can be called the generalized Noether's theorem for nonholonomic systems of non-Chetaev's type with unilateral constraints. Formulae (14)–(16) are the generalized Noether's identities for nonholonomic systems of non-Chetaev's type with unilateral constraints. Theorem 1 presents the conditions of the existence and form of conserved quantities for the nonholonomic systems of non-Chetaev's type with unilateral constraints.

IV. NOETHER'S INVERSE THEOREM

Suppose that a first integral of the nonholonomic systems of non-Chetaev's type with unilateral constraints (1), (2), (4) is given

$$I=I(t,q_s,\dot{q}_s)=C,\quad(18)$$

then we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\sum_{s=1}^{n}\frac{\partial I}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial I}{\partial\dot{q}_s}\ddot{q}=0,\quad(19)$$

and from the principle (4) we can obtain the equation of motion as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q_s-\sum_{\alpha=1}^{b}\lambda_\alpha\frac{\partial\varphi_\alpha}{\partial q_s}-\sum_{\beta=1}^{g}\mu_\beta f_{\beta s}=0\quad(s=1,\cdots,n).\quad(20)$$

Multiplying $\overline{F}_s=(F_s-\dot{q}_s f)$ and Eq.(20), taking summation over s and adding the result to formula (19), and setting the coefficients of $\ddot{q}_k$ equal to zero, we obtain

$$\frac{\partial I}{\partial\dot{q}_k}-\sum_{s=1}^{n}\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\overline{F}_s=0\quad(k=1,\cdots,n).\quad(21)$$

Suppose

$$\det(h_{sk})=\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0,$$

from Eqs.(21) we obtain

$$\overline{F}_s=\tilde{h}_{sk}\frac{\partial I}{\partial\dot{q}_k}\quad(s=1,\cdots,n),\quad(22)$$

in which

$$\tilde{h}_{sk}h_{kl}=\delta_{sl}.$$

Now let

$$I=Lf+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\overline{F}_s+G,\quad(23)$$

from Eq.(23), we can obtain the solution

$$f=L^{-1}\left(I-\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}\overline{F}_s-G\right).\quad(24)$$

Equations (22) and (24) demonstrate that the generators F_s and f of the infinitesimal transformations can be obtained if the first integral of the system is given and the gauge function G is chosen.

V. AN ILLUSTRATIVE EXAMPLE

The Lagrangian of a system is

$$L=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-mgq_3,\quad(25)$$

and the unilateral constraints are

$$\varphi=q_1\geq 0,\quad(26)$$

$$\varphi=\dot{q}_1+\dot{q}_2-t\dot{q}_3\geq 0.\quad(27)$$

Suppose that the constraint (27) is of non-Chetaev's type and the restriction imposed on virtual displacements is

$$\delta\dot{q}_1+\delta\dot{q}_2-\delta\dot{q}_3=0.\quad(28)$$

Let us study the Noether's symmetries and the conserved quantities of the system.

Firstly, let us seek the conserved quantity from the given symmetry. By virtue of Eqs.(14), (15) and (16), the generators of the infinitesimal transformations satisfy

$$F_1-\dot{q}_1 f=0,\quad(29)$$

$$(F_1-\dot{q}_1 f)+(F_2-\dot{q}_2 f)-(F_3-\dot{q}_3 f)=0,\quad(30)$$

$$mgF_3+m\dot{q}_1\dot{F}_1+m\dot{q}_2\dot{F}_2+m\dot{q}_3\dot{F}_3-\left[\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)+mgq_3\right]\dot{f}+\dot{G}=0.\quad(31)$$

We can find a solution of Eqs.(29)–(31) as follows:

$$F_1=0,\quad f=0,\quad F_2=F_3=1,\quad G=-mgt.\quad(32)$$

Substituting Eqs.(25), (32) into Eq.(17), we obtain a conserved quantity of the system as

$$m\dot{q}_2+m\dot{q}_3-mgt=\text{const.}\quad(33)$$

Secondly, let us find the corresponding infinitesimal transformations from a known integral. Suppose

there is an integral in the form of Eq.(33), according to Eqs.(22)and (24), we can obtain

$$\overline{F}_1 = 0, \quad \overline{F}_2 = 1, \quad \overline{F}_3 = 1, \tag{34}$$

$$Lf + m\dot{q}_1\overline{F}_1 + m\dot{q}_2\overline{F}_2 + m\dot{q}_3\overline{F}_3 + G = m\dot{q}_2 + m\dot{q}_3 - mgt. \tag{35}$$

Letting

$$G = -mgt, \tag{36}$$

we obtain

$$f = 0. \tag{37}$$

So

$$f = 0, \quad F_1 = 0, \quad F_2 = 1, \quad F_3 = 1. \tag{38}$$

Equations (36) and (38) correspond to the strongly generalized quasi-symmetrical transformation of the nonholonomic system of non-Chetaev's type with unilateral constraints.

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