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STEADY SOLUTIONS OF THE OPTICAL SOLITON EQUATION WITH A NONLINEAR RESPONSE DELAY TERM*

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The exact solution of the optical soliton equation with a nonlinear response delay term has been obtained by using the method of separating variables. The new type of optical solitary wave solution, which is quite different from the bright and dark soliton solutions, has been found for a special case.

Keywords: optical soliton equation, solitary wave solution, method of separating variables
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Study of the solitary wave solution of the nonlinear Schrödinger (NLS) equation is one of the important works in nonlinear science. There have been many studies[1−4] attempting to find out the analytical solution of the NLS equation and to determine the theoretical method for the investigation of the NLS equation. These results have important scientific values and application prospects. The nonlinear evolution of short pulses in an optical fibre is usually described by the NLS equation. It is well known that the NLS equation properly describes the nonlinear dynamics of pulses on a picosecond timescale. As a consequence of the investigations of experiment and theory in high-power and ultrashort pulse-propagation regimes, the NLS equation must be modified to include higher-order dispersion and nonlinear effects. The optical soliton equation with a nonlinear response delay term is a basic evolutional equation, which involves the nonlinear single-mode optical field in silica fibre,[5] and it mainly describes the phenomenon of the change of optical field envelope in the scale of subpicosecond.[6] This equation also is used to discuss the decay problem of the ps degenerate soliton and the effect of the frequency downshift.[7−9] In this paper, we give a new analytical expression of the optical solitary wave, which involves the optical soliton equation with a nonlinear response delay term, at the assumption of separating variables of waveforms.

In the anomalous dispersion regime, the optical soliton equation, including higher-order terms, is called the first-order modified NLS equation[7] and takes the following form: (we assume β_3 to be zero in the optical soliton equation)

$$i\frac{\partial q}{\partial z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+|q|^2q+i\beta\frac{\partial^3 q}{\partial T^3} +i\alpha\frac{\partial}{\partial T}(|q|^2q)-\gamma\frac{\partial}{\partial T}(|q|^2)q=0, \tag{1}$$

where q is the complex envelope of the electric field, z and T are the propagation distance and the retarded time, the terms with the real constants β,α and γ describe third-order dispersion of the fibre, the self-gradient effect and the nonlinear response delay effect, respectively.

By the background of physics, one is only interested in the steady solution of Eq.(1), which has the following form, considering the special case for separating variables

$$q(T,z)=y(T)\exp[i,\theta(z)], \tag{2}$$

where $y(T)$ is a real function. Substituting Eq.(2) into Eq.(1), separating imaginary and real parts, we obtain simple coupled algebra equations:

$$\begin{cases} -y\theta_z+y_{TT}/2+y^3-2\gamma y^2y_T=0, & (3)\\ \beta y_{TTT}+3\alpha y^2y_T=0. & (4)\end{cases}$$

Equation (4) is a third-order differential equation, after integrating, we obtain

$$y_{TT}=-\frac{\alpha}{\beta}y^3+C_1, \tag{5}$$

where $C_1=(y_{TT})_0+\frac{\alpha}{\beta}y_0^3$.

Substituting Eq.(5) into Eq.(3) the latter can be transformed as

$$-4\gamma y^2y_T=-\left(2-\frac{\alpha}{\gamma}\right)y^3+2\eta y-C_1. \tag{6}$$

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Considering $\theta_z = \eta(\text{const})$, then we have $\theta(z) = \eta z + \theta_0$, where η and θ_0 are arbitrary constants determined from the initial condition. This is the inevitable result from Eq.(3)

$$\frac{y_{TT}/2}{y} + y^2 - 2\gamma y y_T = \theta_z = \eta(\text{const}). \tag{7}$$

Equation (6) is a first-order nonlinear differential equation, which can be integrated,

$$\int \left(\frac{2-\alpha/\beta}{4\gamma}y - \frac{2\eta}{4\gamma}\frac{1}{y} + \frac{C_1}{4\gamma}\frac{1}{y^2}\right)^{-1} \mathrm{d}y = T - T_0, \tag{8}$$

where

$$T_0 = \left(\int \left(\frac{2-\alpha/\beta}{4\gamma}y - \frac{2\eta}{4\gamma}\frac{1}{y} + \frac{C_1}{4\gamma}\frac{1}{y^2}\right)^{-1} \mathrm{d}y\right)_{y=y_0}.$$

Equation (8) implies that the envelope function of the solitary wave depends on the effects of the higher-order terms in the NLS equation. The analytical equation expressed by Eq.(8) includes three parameters and three types of function in the integrated function. They are β (the third-order dispersion of the fibre), α (the self-gradient effect), and γ (the nonlinear response delay effect of the silica fibre). The whole effect of the solution completely describes the mathematical characteristics of the envelope of the optical solitary wave.

Now, we obtain the exact analytical equations for the envelope function of the solitary wave, which involves the effects of the higher-order terms in the NLS equation. Thus, Eq.(8) can easily be integrated,

$$\begin{aligned}
&(ay^2+by+c)^{X/2a}(y+d)^Y \\
&\left(\frac{2ay+b+\sqrt{b^2-4ac}}{2ay+b-\sqrt{b^2-4ac}}\right)^{\frac{Xb-2aZ}{2a\sqrt{b^2-4ac}}} \\
&= \mathrm{sh}(T-T_0) + \mathrm{ch}(T-T_0), \\
&(b^2 > 4ac)
\end{aligned} \tag{9a}$$

$$\begin{aligned}
&(ay^2+by+c)^{X/2a}(y+d)^Y \\
&= \mathrm{sh}\left(T - T_0 + \frac{Xb-2aZ}{a\sqrt{4ac-b^2}}\mathrm{arctg}\frac{2ay+b}{\sqrt{4ac-b^2}}\right) \\
&\quad + \mathrm{ch}\left(T - T_0 + \frac{Xb-2aZ}{a\sqrt{4ac-b^2}}\mathrm{arctg}\frac{2ay+b}{\sqrt{4ac-b^2}}\right), \\
&(b^2 < 4ac)
\end{aligned} \tag{9b}$$

$$\begin{aligned}
&(ay^2+by+c)^{X/2a}(y+d)^Y \\
&= \mathrm{sh}\left(T - T_0 - \frac{Xb-2aZ}{a(2ay+b)}\right) \\
&\quad + \mathrm{ch}\left(T - T_0 - \frac{Xb-2aZ}{a(2ay+b)}\right), \\
&(b^2 = 4ac)
\end{aligned} \tag{9c}$$

where $A = \dfrac{2-\alpha/\beta}{4\gamma}, B = \dfrac{2\eta}{4\gamma}, C = \dfrac{C_1}{4\gamma}$;

$$a = A, b = -Ad, c = -B + Ad^2, C + Bd - Ad^3 = 0;$$

$$X = \frac{-B+2Ad^2}{-B+3Ad^2}, Y = \frac{d^2}{-B+3Ad^2}, Z = \frac{(B-Ad^2)d}{-B+3Ad^2}.$$

The nonlinear structures of the envelope function are described by the analytical solutions, Eqs.(9a), (9b) and (9c), being valid for separating variables in the special case. Thereby, under the condition of Eq.(2), the envelope function $y(T)$ is determined completely by Eqs.(9a), (9b) and (9c).

Thus, in this case, Eqs.(9a), (9b) and (9c) indicate that the amplitude $y(T)$ is a function of time and also gives the mathematical description of this envelope function. Therefore, the optical solitary wave $q(z,T)$ can be written as

$$q(T,z) = y(T)\exp[i(\eta z + \theta_0)], \tag{10}$$

where T_0 and θ_0 are constants. They are governed by the initial physical conditions. The envelope function $y(T)$ is described by the analytical solutions, Eqs.(9a), (9b) and (9c). This is a new type of optical solitary wave solution, and it is quite different from the bright and dark soliton solution, which was obtained using the perturbation theory of the NLS equation. As is shown in Eq.(10), the steady solution depends not only on the dispersion character of silica material, but also on the waveguide property of the fibre.

For the nonlinear function $y(T)$ describing the wave envelope, Fourier transform is performed using the Fourier space variables $\Delta\omega(=\omega-\omega_0)$ which presents a small frequency shift of the side band from the carrier frequency

$$\overline{y}(\Delta\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(T)\mathrm{e}^{i\Delta\omega T}\mathrm{d}T. \tag{11}$$

The inverse transform is given by

$$y(T) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{y}(\Delta\omega) \mathrm{e}^{-i\Delta\omega T} \mathrm{d}(\Delta\omega). \qquad (12)$$

The Fourier transform of Eq.(2) gives the frequency spectrum of the solitary wave solution. The Fourier spectrum of the envelope in the fibre is a possible modal expansion, but it is hard to show that any other complete orthogonal set of modes has the same property.

The solitary wave power is given by

$$P(T) = |q|^2 = y^2(T) = -\frac{\beta}{3\alpha} \frac{y_{TTT}}{y_T}, \qquad (13)$$

and its energy spectrum by

$$W(\Delta\omega) = 2\pi|\overline{y}|^2 = \left| \int_{-\infty}^{\infty} y(T) \mathrm{e}^{i\Delta\omega T} \mathrm{d}T \right|^2. \qquad (14)$$

In conclusion, we have exactly solved the optical soliton equation of the nonlinear problem in special forms using the method of separating variables. An analytical result has been found by means of the optical soliton equation for the solitary wave. This shows the mathematical characteristics of the steady solution. We regard it as a particular kind of the nonlinear optical field. The results may be important in the investigation of the propagation of the solitary electromagnetic wave envelope.

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