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A GENERATION OF EXACTLY SOLVABLE ANHARMONIC SYMMETRIC OSCILLATORS*

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Using the ideas of supersymmetric quantum mechanics, we exactly solve a continuous family of anharmonic potentials, which are the supersymmetric partners of the linear harmonic oscillators. The family includes a series of potentials in which the excited-state energy is the same as that of the harmonic oscillators, but the ground-state energy can be any value lower than the excited states. The shape of the potential is variable, which includes the double-well and triple-well potentials. All the potentials obtained in this paper are free of singularities, and the supersymmetry of the solutions is unbroken.

Keywords: anharmonic potential, double-well potential, exact solution
PACC: 0361

I. INTRODUCTION

In quantum mechanics, there are only a few potentials for which the Schrödinger equation is exactly solvable.[1] However, the class of exactly solvable problems can be enlarged by using the method of generating isospectral Hamiltonians.[2−6] Recently, a number of anharmonic oscillators have been added to the class of exactly solvable ones.[6−7] These solutions can be regarded as a family in which the eigenvalues or the spectral are almost isospectral to the harmonic ones, but a few of the excited energy level(s) are absent; in other words, the family is a non-continuous one.

Many of the exact solutions for the anharmonic potential, obtained in resent years, have been obtained by using the ideas of supersymmetric quantum mechanics (SUSY QM).[8,9] In this paper, we study a general family of one-dimensional (1D) anharmonic potentials also by means of the methods of SUSY QM. It shall be shown that the family is a continuous one, and many of the exact solvable potentials obtained before are the particular cases of this family, including the Abraham–Moses potential[2], the conditional exact solvable potentials[6] and the family recently obtained by Fernandez *et al.*[7]

In the next section, we shall first discuss how to obtain the exactly solvable potential and the energy level. The general form of the potential includes the symmetric and asymmetric potentials, but only the former, which is simpler, will be discussed in detail in this paper. In section III, the shape of the anharmonic potentials and the energy levels, as well as the wavefunction, will be discussed.

II. GENERAL FORM OF THE EXACT SOLVABLE ANHARMONIC POTENTIAL

In supersymmetric quantum mechanics,[6−8] a pair of Hamiltonians, $H_+(x)$ and $H_-(x)$, called the partner of Hamiltonians, are defined by

$$H_+ = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x),$$
$$H_- = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x), \tag{1}$$

where V-(x) and $V_+(x)$ are two potentials satisfying the following equations,[7]

$$V_-(x) = V_+(x) + W'(x), \tag{2}$$

$$W'(x) + W^2(x) = 2(V_+(x) - \varepsilon), \tag{3}$$

where $W'(x) = (\mathrm{d}/\mathrm{d}x)W(x)$, $W(x)$ is the SUSY potential to be determined, and ε is a constant called the factorization energy. The Hamiltonians have a relationship with the operator A^+,[7]

$$A_-A^+ = A^+H_+$$

where the operator A^+ is defined by

$$A^+ = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right). \tag{4}$$

If a solution to Eq.(3) is found, then Eq.(2) can produce a new potential $V_-(x)$ with the same

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spectrum as $V_+(x)$ except for an additional level at ε (provided that the wavefunction $\Psi_\varepsilon^{(-)} \propto \exp\left(-\int_0^x w(y)\mathrm{d}y\right)$ has no singularities and is square integrable).

It was reported that some of the solutions for the cases that $V_-(x) = x^2/2 - \varepsilon$ with the factorization energies $\varepsilon = -2k - 1/2$ for $k = 0, 1, 2...$ have already been solved. However, the author has found that there is a family of potentials that have exact solutions for the non-integral k (i.e., $k \in \mathcal{R}$ with $-1/2 < k < \infty$). Therefore the solvable family is of the continuous type. In this paper, we shall discuss these potentials in detail in the following sections.

In order to cut down the length of this paper, we only give the solution of the symmetric potential, that is, the potential with an even parity (i.e., $V_-(-x) = V_-(x)$), although the solution is easy to extend to the non-parity cases.

We now consider the construction of a pair of potentials $V_\pm(x)$ with $V_+(x)$ being the harmonic potential,

$$V_+(x) = x^2/2,$$

and $V_-(x)$ the anharmonic potential to be determined, provided that $W(x)$ is known.

One can see that Eq.(3) is a first-order differential equation, mathematically, known as the Riccati equation. In order to obtain an anharmonic potential, one may first solve this equation, and some authors have studied the anharmonic potential in this way. In order to obtain some new solutions, we shall study Eq.(3) in a different way as follows.

Let us now assume that the function $W(x)$ can be written in two terms

$$W(x) = x + g(x)/f(x), \tag{5}$$

where $g(x)$ and $f(x)$ are two new functions to be determined. Substituting Eq.(5) to Eq.(3), for $f(x) \neq 0$ $(-\infty < x < \infty)$, one obtains two differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) + 2xg(x) = (-\lambda_1 - 2\varepsilon - 1)f(x), \tag{6}$$

where λ_1 is an arbitrary constant satisfying the following equation,

$$g^2(x) - g(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lambda_1 f^2(x). \tag{7}$$

There will be a number of solutions of $g(x)$ and $f(x)$ for Eqs.(6) and (7); however, one is able to obtain some new solutions by simplifying the solutions of the equations. One of the simplest cases that Eqs.(7) and (6) are solvable is that λ_1=0, and we shall only discuss this case in this paper. In this case, one obtains the relation between the two functions $g(x)$ and $f(x)$

$$g(x) = \mathrm{d}f/\mathrm{d}x. \tag{8}$$

Substituting this relation into Eq.(6) and putting λ_1=0, one finally obtains a second-order differential equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) + 2x\frac{\mathrm{d}}{\mathrm{d}x}f(x) = -(2\varepsilon+1)f(x). \tag{9}$$

Therefore, we have transformed the first-order differential equation (4) into a second-order equation. Equation (9) can be solved exactly, and the general solution is

$$\begin{aligned} f(x) =& C_1\cdot F\left(\frac{2\varepsilon+1}{4}, \frac{1}{2}, -x^2\right) \\ &+ C_2 x\cdot F\left(\frac{1}{2}+\frac{2\varepsilon+1}{4}, \frac{3}{2}, -x^2\right), \end{aligned} \tag{10}$$

where C_1 and C_2 are two arbitrary constants, and $F(a,b,t)$ is the Kummer confluent hypergeometric function defined by

$$\begin{aligned} F(a,b,t) =& 1 + \frac{1}{1!}\frac{a}{b}t^1 + \frac{1}{2!}\frac{a}{b}\frac{1+a}{1+b}t^2 + ... \\ &+ \frac{1}{n!}\frac{a\cdots n-1+a}{b\cdots n-1+b}t^n ... \\ =& \sum_{n=0}^{\infty}\frac{1}{n!}\frac{(a)_n}{(b)_n}t^n, \end{aligned} \tag{11}$$

where $(a)_n$ is the Pochhammer symbol related to the Gamma function $\Gamma(a)$ by the relationship $(a)_n = \Gamma(a+1)/\Gamma(a)$. One can verify that the eigenvalue, $-(2\varepsilon+1)$, can take any continuous real value, rather than the discontinuous one.

In order to obtain the singular free potential $V_-(x)$ and SUSY potential $W(x)$ by Eq.(5), for $C_1 \neq 0$, the function $f(x)$ must satisfy $f(x) \neq 0$ for x in the domain $(-\infty, \infty)$.

By solving the inequality $f(x) > 0$, one obtains that ε must be in the domain $(-\infty, 1/2)$. It is easy to know, from Eq.(11), that the first term on the right-hand side of Eq.(10) is an even function; therefore, if one puts C_2=0, the potential $V_-(x)$ must be an even parity, or it is a symmetric potential. To simplify our discussion, we mainly discuss this case in this paper.

Substituting Eq.(10) into Eqs.(8) and (5), one finally obtains the exact (analytical) solutions of the SUSY potential, for C_2=0 and $2\varepsilon + 1 \geq 2$,

$$W(x) = x + \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln F\left(\frac{2\varepsilon+1}{4}, \frac{1}{2}, -x^2\right)\right]. \tag{12}$$

Substituting Eq.(12) into Eq.(2), one finally obtains a family of "exactly solvable potentials" with even parity,

$$V_-(x) = \frac{1}{2}(x^2 - 2\varepsilon) + \frac{2x(F')}{F} - \left(\frac{F'}{F}\right)^2, \qquad (13)$$

where F is a short symbol for $F((2\varepsilon+1)/4, 1/2, -x^2)$.

By SUSY QM,[6] the energy levels for the potential $V_+(x)$ are, $(n = 0, 1, 3..., -\infty < \varepsilon < 1/2)$

$$E_0^{(-)} = \varepsilon, E_{n+1}^{(-)} = n + \frac{1}{2}, \qquad (14)$$

From this expression one reads that all the excited energy levels are the same as those of the harmonic oscillator; however, the ground-state energy, ε, is an exception, which in the domain $(-\infty, 1/2)$ and the energy splitting $E_1^{(-)} - E_0^{(-)} = \varepsilon - 1/2$, can be any continuous value greater than zero. Each different value corresponds to a different potential.

So far we have obtained a general form of exactly solvable anharmonic potentials, and their energy levels.

III. SHAPE OF THE ANHARMONIC POTENTIALS AND THE WAVEFUNCTION

Some particular cases of the above solutions of the anharmonic potential deserve to be mentioned.

(i) If $\varepsilon = -(2n+1)/2, (n = 1, 2...)$, and $C_2 = 0$, then function (11) degenerates to a polynomial. The two simple cases are n=1 and n=2, the results are $F(-1, 1/2, -x^2) = 1 + 2x^2$, and $F(-2, 1/2, -x^2) = 1 + 4x^2 + 4x^4/3$, respectively. For n=1, one has

$$W(x) = x + 2x/(1 + 2x^2);$$

and for n=2, one has

$$W(x) = x + (8x + 16x^3/3)/(1 + 2x^2 + 4x^4/3).$$

These two particular cases have been discussed by Junker *et al.*[6]

(ii) If $\varepsilon = -1/2, C_2 \neq 0$, the functions on the right-hand side of Eq.(11) can be expressed in simpler forms, e.g., $F(0, /2, -x^2) = 1$, and $F(1/2, 3/2, -x^2) = (\sqrt{\pi})erf(x)/(2x)$,[11] where $erf(x)$ is the error function. In this case, the potential $V_-(x)$ is just the Abraham–Moses potential.[2]

(iii) If $\varepsilon = -2n - 3/2, (n = 0, 1, 2, 3...)$, the Kummer function of Eq.(11) can be reduced to the combination of the error function $erf(x)$ and the Gaussian function $\exp(-x^2)$, e.g., $F(2, 1/2, -x^2) = (\sqrt{\pi})x\, erf(x) + \exp(-x^2)$, etc. These cases were recently solved by Fernandez *et al.*[7] However, our method here might be more simple.

The particular cases discussed above can be included in the family $\varepsilon = -n - 1/2 (n = 0, 1, 2...)$. Besides these particular cases, however, the exact solution for a more general case of $-\infty < \varepsilon < 1/2$, (for $\varepsilon \in \mathcal{R}$) is also given by Eqs.(13) and (14). Therefore, we have obtained a continuous family of solutions, each solution with a different eigenvalue ε corresponding to a different shape of potential. These continuous families of solutions have not been obtained before, as far as we know.

A typical of the potential with $\varepsilon = -7/3 = -2.3333...$ is plotted in Fig.1. One can see that the shape of the potential is similar to that of the case with $\varepsilon = -5/2$, which was discussed and plotted in Ref.[6]; the harmonic potential $V_+(x) = x^2/2$ is also plotted in this figure but as a dashed line.

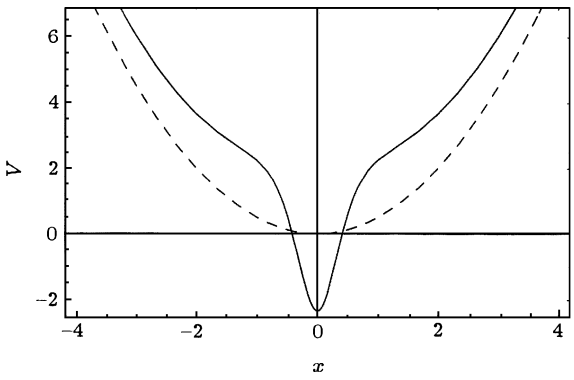


Fig.1. An anharmonic potential (solid line) for $\varepsilon = -7/3$ and its SUSY partner (dashed line).

The shapes of the potentials for $0 < \varepsilon < 1/2$ are worthy of discussion. The even parity potential of Eq.(13) (i.e., C_2=0) can be expanded into a Taylor series,

$$V_-(x) = c_0^+ c_2 x^2 + c_4 x^4 + O(x^6), \qquad (15)$$

where c_0, c_2 and c_4 are constants, which can be calculated by using Eqs.(11) and (13),

$$c_0 = \varepsilon, c_2 = 1/2 + 4\varepsilon^2, c_4 = -4(1 + 2\varepsilon)^2, ... \qquad (16)$$

On the other hand, for $x \to \infty$, the asymptote of the potential $V_-(x)$ is the harmonic potential, i.e.,

$$V_-(x \to \infty) = x^2/2 - 2 - 3\varepsilon; \qquad (17)$$

therefore, if $c_2 < 0$, then the curve of the potential represented by Eq.(17) must have at least two minima, i.e., it must be a double-well potential. By solving $c_2 < 0$, one has the condition under which the

potential has symmetric double wells,

$$-\sqrt{2}/4 < \varepsilon < \sqrt{2}/4. \tag{18}$$

One of the double-well potentials, with ε=0, is plotted in Fig.2. It ought to be mentioned that one kind of double-well potential, which might be different from the one discussed here, has previously been obtained by Barclay *et al.*[9] However, our treatment here might be more simple.

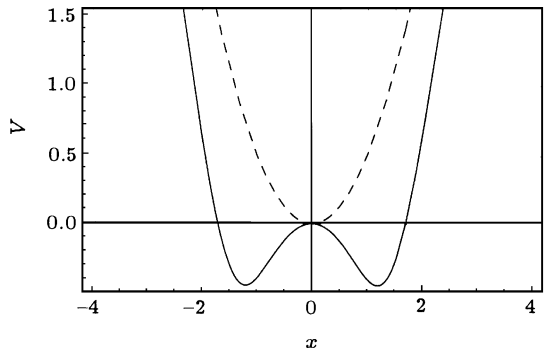


Fig.2. An anharmonic potential (solid line) for $\varepsilon = 0$, and its SUSY partner (dashed line). The potential has two wells.

For the parameter ε in the domain (-1/2, 1/2), the potential might be regarded as the superposition of a harmonic and a double-well potential, approximately. However, the double wells will not appear until the wells are deeper than the bottom of the harmonic potential; the condition of Eq.(18) expresses this case.

The separation of the wells becomes larger as ε approaches 1/2 from below; for $(\sqrt{2})/4 < \varepsilon < 1/2$, we have $c_2 < 0$ but $c_4 > 0$. In this case, the potential $V_-(x)$ must have three wells. The additional shallow well between the double wells is just the bottom of the harmonic potential. It appears again as the separation is large enough. One of these cases, with $\varepsilon = 0.45$, is plotted in Fig.3. Note that as ε becomes larger, the separation between the wells is also larger.

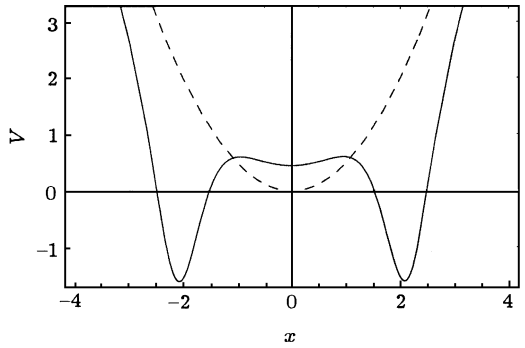


Fig.3. An anharmonic potential (solid line) for $\varepsilon = 0.45$ and its SUSY partner (dashed line). The potential has three wells, a shallow well is at the centre.

The ground-state wavefunction can be written as[6]

$$\begin{aligned}\Psi_0^{(-)}(x) &= C_0 \exp\left[-\int^{\pi} \mathrm{d}x' W(x')\right] \\ &= C_0 \frac{\exp(-x^2/2)}{F((2\varepsilon+1)/4, 1/2, -x^2)},\end{aligned} \tag{19}$$

where C_0 is the normalization constant. Other wavefunctions can be calculated by the following expression[8] $(n = 0, 1, 2...)$,

$$\Psi_{n+1}^{(-)}(x) = (E_n^{(+)} - \varepsilon)^{-1/2} A^+ \Psi_n^{(+)}(x), \tag{20}$$

where $E_n^{(+)}$ and $\Psi_n^{(+)}(x)$ are the energies and wavefunctions of the harmonic potential $V_+(x)$, respectively. The wavefunctions $\Psi_n^{(+)}(x)$ are related to the Hermit polynomial, by Eqs.(12) and (14), and the normalized eigenfunctions are $(n = 0, 1, 2...)$,

$$\begin{aligned}\Psi_{n+1}^{(-)}(x) =& \frac{\exp(-x^2/2)}{(2^{n+1} n!(n+1+\lambda/2)\sqrt{\pi})^{1/2}} \\ &\cdot \left\{\left(2x + \frac{\mathrm{d}}{\mathrm{d}x}[\ln F]\right) H_n(x) - 2n H_{n-1}(x)\right\},\end{aligned} \tag{21}$$

where $H_n(x)$ is the Hermit polynomial of degree n. So far we have obtained the complete solutions of the even parity potential $V_-(x)$ expressed by Eq.(13).

Other solutions to the asymmetric potential $V_-(x)$ can be obtained by replacing F by $f(x)$ in Eq.(12) with $C_2 \neq 0$. If one looks for the regular solutions, or the singular free solutions, one can restrict the domain of C_2 to guarantee that $f(x)$ has no zeros for any $x \in \mathcal{R}$. We are not going to discuss this case in detail here.

In summary, we have obtained the exact solutions of a continuous family of anharmonic potentials in which the ground-state energy can be any real value lower than the excited-state energy. The main mathematical feature in obtaining the new solution is that we transform the Riccati equation into the second-order differential equation which is exactly solvable. This exactly solvable family of potentials is more general than that obtained before. The shapes of the potential can be classified into three types; one is the single well, and the others are the multi-well potentials including two-well and three-well potentials. All the potentials are the SUSY partners of the linear harmonic potential.

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