Vol 10 No 4, April 2001
1009-1963/2001/10(04)/0271-06
Chinese Physics
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BIRKHOFF'S EQUATIONS AND GEOMETRICAL THEORY OF ROTATIONAL RELATIVISTIC SYSTEM*

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(Received 4 August 2000; revised manuscript received 8 December 2000)

The Birkhoffian and Birkhoff's functions of a rotational relativistic system are constructed, the Pfaff action of rotational relativistic system is defined, the Pfaff–Birkhoff principle of a rotational relativistic system is given, and the Pfaff–Birkhoff–D'Alembert principles and Birkhoff's equations of rotational relativistic system are constructed. The geometrical description of a rotational relativistic system is studied, and the exact properties of Birkhoff's equations and their forms on $\vec{R}\times T^*M$ for a rotational relativistic system are obtained. The global analysis of Birkhoff's equations for a rotational relativistic system is studied, the global properties of autonomous, semi-autonomous and non-autonomous rotational relativistic Birkhoff's equations, and the geometrical properties of energy change for rotational relativistic Birkhoff's equations are given.

Keywords: rotational relativistic system, Birkhoff's equation, geometrical description, global analysis
PACC: 0320

I. INTRODUCTION

With the developments in science and technology, more and more experimental facts have been related to the problem of high-speed spinning[1]. In 1985, M Carmeli constructed the theory of rotational relativistic mechanics[2−5]. Since 1996, we have obtained the theory of analytic mechanics for the rotational relativistic system, and given its basic theoretical frame[6−9]. In recent years, this work has been generalized to the system with variable rotational inertia[10,11]. Mei Feng-xiang and his co-workers constructed the dynamics of the Birkhoffian system on the basis of Refs.[12, 13], and gave its theoretical frame[14−18]. The dynamics of the Birkhoffian system is more general than the Hamiltonian mechanics, and more and more attention is paid to it in the field of mathematics, mechanics and physics. The Hamiltonian mechanics has been extensively applied to the fields of modern physics[19,20]. So the dynamics of the Birkhoffian system will also be applied to the fields of modern physics. Recently, Ref.[21] gave relativistic Birkhoffian equations, and studied the Lie symmetries and conserved quantities. In this paper, the Birkhoffian dynamics of a rotational relativistic system is studied. The Pfaff–Birkhoff principle and the Birkhoffian equations of this system are given, along with the geometrical description and global analysis of these equations.

II. PFAFF–BIRKHOFF PRINCIPLE AND BIRKHOFF'S EQUATIONS OF A ROTATIONAL RELATIVISTIC SYSTEM

The repeated subscripts express making sum. Denote the Pfaff action by the integral

$$A=\int_{t_1}^{t_2}\{R_\nu(t,\boldsymbol{a})\dot{a}^\nu-B(t,\boldsymbol{a})\}\mathrm{d}t, \tag{1}$$

where $B(t,\boldsymbol{a})$ is the Birkhoffian and $R_\nu(t,\boldsymbol{a})(\nu=1,\cdots,2n)$ the Birkhoff's functions. Then

$$\delta A=0, \tag{2}$$

$$\mathrm{d}\delta a^\nu=\delta\mathrm{d}a^\nu \qquad (\nu=1,\cdots,2n), \tag{3}$$

$$\delta a^\nu|_{t=t_1}=\delta a^\nu|_{t=t_2}=0 \quad (\nu=1,\cdots,2n), \tag{4}$$

are called the Pfaff–Birkhoff principle[3].

Define the Pfaff action of a rotational relativistic system by

$$A^*=\int_{t_1}^{t_2}\{R_\nu^*(I_i(t,\boldsymbol{a}),t,\boldsymbol{a})\dot{a}^\nu - B^*(I_i(t,\boldsymbol{a}),t,\boldsymbol{a})\mathrm{d}t, \tag{5}$$

where B^* and R_ν^* are the Birkhoffian and Birkhoff's functions of a rotational relativistic system respectively, which include the relativistic rotational inertia as follows

$$I_i=\frac{I_{oi}}{\sqrt{1-\dot{\theta}_i^2/I_i^2}} \quad (i=1,\cdots,N), \tag{6}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 19972010) and the Natural Science Foundation of Henan Province (Grant Nos. 984053100, 998040080, 934060800)

where I_{oi} is the static rotational inertia of the ith body, θ_i is the angular coordinate, Γ_i is the limiting angular-velocity of the ith body[5,7].

Variating Eq.(5), and considering Eq.(6), we obtain

$$\begin{aligned}\delta A^* = \int_{t_1}^{t_2} \Bigg\{ & \left(\frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial R_\nu^*}{\partial a^\mu} \right. \\ & \left. - \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu} - \frac{\partial R_\mu^*}{\partial a^\nu} \right)\dot{a}^\nu \\ & - \left(\frac{-\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial B^*}{\partial a^\mu} \right. \\ & \left. + \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial t} + \frac{\partial R_\mu^*}{\partial t} \right) \Bigg\} \delta a^\mu \mathrm{d}t = 0, \\ & (\nu, \mu = 1, \cdots, 2n; i = 1, \cdots, N). \end{aligned} \tag{7}$$

Eq.(7) is called the Pfaff–Birkhoff principle of a rotational relativistic system. Here the exchanging relation (3) and the condition of endpoint (4) are considered. Let

$$\tilde{B}^* = \tilde{B}^*(t, \boldsymbol{a}) = B^*(I(t, \boldsymbol{a}), t, \boldsymbol{a}), \tag{8}$$

$$\tilde{R}_\nu^* = \tilde{R}_\nu^*(t, \boldsymbol{a}) = R_\nu^*(I(t, \boldsymbol{a}), t, \boldsymbol{a}). \tag{9}$$

We have

$$\begin{aligned} \frac{\partial \tilde{B}^*}{\partial a^\mu} &= \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial B^*}{\partial a^\mu}, \\ \frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} &= \frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial R_\nu^*}{\partial a^\mu}, \\ \frac{\partial \tilde{R}_\nu^*}{\partial t} &= \frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial t} + \frac{\partial R_\nu^*}{\partial t}, \end{aligned} \tag{10}$$

then the principle (7) can be expressed as

$$\begin{aligned}\delta A^* = \int_{t_1}^{t_2} \Bigg\{ & \left(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu} \right)\dot{a}^\nu \\ & - \left(\frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}_\mu^*}{\partial t} \right) \Bigg\} \delta a^\mu \mathrm{d}t = 0 \\ & (\nu, \mu = 1, \cdots, 2n). \end{aligned} \tag{11}$$

By the arbitrariness of the integral interval $[t_1, t_2]$, we obtain the Pfaff–Birkhoff–D'Alembert principle of a rotational relativistic system

$$\begin{aligned} & \left\{ \left(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu} \right)\dot{a}^\nu - \left(\frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}_\mu^*}{\partial t} \right) \right\} \delta a^\mu = 0 \\ & (\nu, \mu = 1, \cdots, 2n). \end{aligned} \tag{12}$$

If $\delta a^\mu (\mu = 1, \cdots, 2n)$ are independent of each other, from the principle (12), we can obtain Birkhoff's equations of a rotational relativistic system

$$\begin{aligned} & \tilde{\Omega}_{\mu\nu}^* \dot{q}^\nu - \left(\frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}_\mu^*}{\partial t} \right) = 0 \quad (\nu, \mu = 1, \cdots, 2n), \\ & \tilde{\Omega}_{\mu\nu}^* = \left(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu} \right). \end{aligned} \tag{13}$$

Eqs.(13) can be written in the contravariant form

$$\begin{aligned} & \dot{a}^\mu = \tilde{\Omega}^{*\mu\nu} \left(\frac{\partial \tilde{B}^*}{\partial a^\nu} + \frac{\partial \tilde{R}_\nu^*}{\partial t} \right) \quad (\nu, \mu = 1, \cdots, 2n), \\ & \tilde{\Omega}^{*\mu\nu} = \left(\left\| \frac{\partial \tilde{R}_\beta^*}{\partial a^\alpha} - \frac{\partial \tilde{R}_\alpha^*}{\partial a^\beta} \right\|^{-1} \right)^{\mu\nu} \quad (\alpha, \beta = 1, \cdots, 2n), \end{aligned} \tag{14}$$

where $\tilde{\Omega}^{*\mu\nu}$ are called the Birkhoff contravariant tensors of a rotational relativistic system. In general, suppose

$$\det(\tilde{\Omega}_{\mu\nu}^*) \neq 0. \tag{15}$$

If $\tilde{R}_\mu^*$ and $\tilde{B}^*$ do not include time t apparently, then the Birkhoff's equations (13) of a rotational relativistic system are autonomous, and have the following simple form

$$\tilde{\Omega}_{\mu\nu}^*(\boldsymbol{a})\dot{a}^\nu - \frac{\partial \tilde{B}^*(\boldsymbol{a})}{\partial a^\mu} = 0 \quad (\nu, \mu = 1, \cdots, 2n). \tag{16}$$

If $\tilde{R}_\mu^*$ do not include time t apparently, then the Birkhoff's equations (13) of a rotational relativistic system are semi-autonomous, and have the following form

$$\tilde{\Omega}_{\mu\nu}^*(\boldsymbol{a})\dot{a}^\nu - \frac{\partial \tilde{B}^*(t, \boldsymbol{a})}{\partial a^\mu} = 0 \quad (\nu, \mu = 1, \cdots, 2n). \tag{17}$$

If $\tilde{R}_\mu^*$ and B^* include time t apparently, then the Birkhoff's equations (13) of a rotational relativistic system are non-autonomous, and can be written in the following form

$$\begin{aligned} & \tilde{\Omega}_{\mu\nu}^*(t, \boldsymbol{a})\dot{a}^\nu - \frac{\partial \tilde{B}^*(t, \boldsymbol{a})}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*(t, \boldsymbol{a})}{\partial t} = 0 \\ & (\nu, \mu = 1, \cdots, 2n). \end{aligned} \tag{18}$$

III. GEOMETRICAL DESCRIPTION OF BIRKHOFF'S EQUATIONS OF A ROTATIONAL RELATIVISTIC SYSTEM

The well-known Dutch mechanist Koiter said that, for the further development of mechanics, we must use step-by-step more abstract and more accurate mathematics[22]. Poincaré pointed out that the differential manifold can become the phase space of a mechanical system instead of Euclidian space.

In this situation, Lagrange mechanics, Hamilton mechanics, nonholonomic mechanics and Birkhoff dynamics can all be described by modern differential geometry[22−24]. Now we give the geometrical theory of Birkhoffian equations of a rotational relativistic system.

A. Exact symplectic form of Birkhoff's equations of a rotational relativistic system in local coordinates

For autonomous Birkhoff's equations (16) and semi-autonomous Birkhoff's equations (17) of a rotational relativistic system, define a 1-form

$$R_1^* = R_\nu^*(I_i(\boldsymbol{a}), \boldsymbol{a})\mathrm{d}a^\nu. \tag{19}$$

Its exterior differentiation $\mathrm{d}R_1^*$ is a 2-form in the cotangent bundle T^*M, where M is an n-dimensional manifold with local coordinates $a^\mu(\mu = 1, \cdots, 2n)$.

$$\begin{aligned}\Omega^* =&\mathrm{d}R_1^* = \left(\frac{\partial R_\nu^*}{\partial a^\mu} + \frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\mathrm{d}a^\mu \wedge \mathrm{d}a^\nu\\ =&\frac{1}{2}\left(\frac{\partial R_\nu^*}{\partial a^\mu} + \frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right.\\ &\left. - \frac{\partial R_\mu^*}{\partial a^\nu} - \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\right)\mathrm{d}a^\mu \wedge \mathrm{d}a^\nu\\ =&\frac{1}{2}\Omega_{\mu\nu}^*\mathrm{d}a^\mu \wedge \mathrm{d}a^\nu.\end{aligned} \tag{20}$$

For a non-autonomous system, the Pfaff action of a rotational relativistic system can be written in the form

$$A^*(E) = \int_{t_1}^{t_2} \hat{R}_1^*(I(\hat{\boldsymbol{a}}), \hat{\boldsymbol{a}}), \quad \hat{\boldsymbol{a}} = (t, \boldsymbol{a}^\mu), \tag{21}$$

where the item inside the integral symbol is a 1-form on $R \times T^*M$

$$\hat{R}_1^*(I(\hat{\boldsymbol{a}}), \hat{\boldsymbol{a}}) = \hat{R}_\nu^*(I(\hat{\boldsymbol{a}}), \hat{\boldsymbol{a}})\mathrm{d}a^\nu. \tag{22}$$

The exterior differentiation $\mathrm{d}\hat{R}_1^*$ of the 1-form $\hat{R}_1^*$ expresses a 2-form on $\boldsymbol{R} \times T^*M$

$$\begin{aligned}\hat{\Omega}_\nu^* =&\mathrm{d}\hat{R}_1^* = \frac{1}{2}\left(\frac{\partial \hat{R}_\nu^*}{\partial \hat{a}^\mu} + \frac{\partial \hat{R}_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial \hat{a}^\mu}\right.\\ &\left. - \frac{\partial \hat{R}_\mu^*}{\partial \hat{a}^\nu} - \frac{\partial \hat{R}_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial \hat{a}^\mu}\right)\mathrm{d}\hat{a}^\mu \wedge \mathrm{d}\hat{a}^\nu\\ \triangleq&\frac{1}{2}\Omega_{\mu\nu}^*(I(\hat{\boldsymbol{a}}), \hat{\boldsymbol{a}})\mathrm{d}\hat{a}^\mu \wedge \mathrm{d}\hat{a}^\nu,\end{aligned} \tag{23}$$

which has the following properties:

(1) $\hat{\Omega}_2^*$ is the most possible local expression of an exact 2-form in $\vec{R} \times T^*M$, because the 1-form $\hat{R}_1^*$ is related to the biggest admitted function.

(2) $\hat{\Omega}_2^*$ is a contact 2-form, because its rank is $2n$, and symplectic for T^*M.

(3) $\hat{\Omega}_2^*$ characterizes the geometrical structure of Birkhoff's equations of a rotational relativistic system, because

$$\Omega_{\mu\nu}^* = \begin{bmatrix} 0 & \frac{\partial B^*}{\partial a^\mu} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial R_\mu^*}{\partial t} + \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial t} \\ -\frac{\partial R_\mu^*}{\partial t} - \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial t} - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} & \frac{\partial R_\nu^*}{\partial a^\mu} + \frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu} - \frac{\partial R_\mu^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu} \end{bmatrix} \tag{24}$$

and on $\vec{R} \times T^*M$ the Birkhoff's equations of a rotational relativistic system are expressed as

$$\hat{\Omega}_{\mu\nu}^*(I(\hat{\boldsymbol{a}}), \hat{\boldsymbol{a}})\dot{\hat{a}}^\nu = 0 \quad (\mu, \nu = 1, \cdots, 2n), \tag{25}$$

or the apparent form

$$\hat{\Omega}_{\mu\nu}^*(I(\hat{a}), \hat{a})\dot{\hat{a}}^\nu = \begin{cases} \left(\dfrac{\partial B^*}{\partial a^\mu} + \dfrac{\partial B^*}{\partial I_i}\dfrac{\partial I_i}{\partial a^\mu} + \dfrac{\partial R_\nu^*}{\partial t} + \dfrac{\partial R_\nu^*}{\partial I_i}\dfrac{\partial I_i}{\partial t}\right)\dot{a}^\nu = 0 \quad (\nu = 0), \\ \dfrac{\partial B^*}{\partial a^\mu} + \dfrac{\partial B^*}{\partial I_i}\dfrac{\partial I_i}{\partial a^\mu} + \dfrac{\partial R_\mu^*}{\partial t} + \dfrac{\partial R_\mu^*}{\partial I_i}\dfrac{\partial I_i}{\partial t} - \left(\dfrac{\partial R_\nu^*}{\partial a^\mu} + \dfrac{\partial R_\nu}{\partial I_i}\dfrac{\partial I_i}{\partial a^\mu}\right. \\ \qquad \left. - \dfrac{\partial R_\mu^*}{\partial a^\nu} - \dfrac{\partial R_\mu^*}{\partial I_i}\dfrac{\partial I_i}{\partial a^\nu}\right)\dot{a}^\nu = 0 \quad (\nu, \mu = 1, \cdots, 2n); \end{cases} \tag{26}$$

i. e.

$$\hat{\tilde{\Omega}}_{\mu\nu}^*(\hat{\boldsymbol{a}})\dot{\boldsymbol{a}}^\nu = \begin{cases} \left(\dfrac{\partial \tilde{B}^*}{\partial a^\nu} + \dfrac{\partial \tilde{R}_\nu^*}{\partial t}\right)\dot{a}^\nu = 0 \qquad (\nu = 0), \\ \dfrac{\partial \tilde{B}^*}{\partial a^\mu} + \dfrac{\partial \tilde{R}_\mu^*}{\partial t} - \left(\dfrac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \dfrac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu = 0 \quad (\mu, \nu = 1, \cdots, 2n). \end{cases} \tag{27}$$

B. Self-adjointness of rotational relativistic Birkhoff's equations

Similar to Ref.[13], we have the following proposition.

In the star-shaped region $\tilde{R}^*$ near the regular point of $\vec{R}\times T^*M$, the analytic, regular general non-autonomous rotational relativistic equations of first order are

$$C^*_{\mu\nu}(I,(t,\boldsymbol{a}),t,\boldsymbol{a})\dot{a}^\nu + D^*_\mu(I(t,\boldsymbol{a}),t,\boldsymbol{a}) = 0 \quad (\mu,\nu=1,\cdots,2n). \tag{28}$$

The sufficient and necessary condition for the above equations to be self-adjoint in $\tilde{R}^*$ is that they are of rotational relativistic Birkhoff's type, i.e.,

$$\begin{aligned} C^*_{\mu\nu}\dot{a}^\nu + D^*_\mu \equiv & \left(\frac{\partial R^*_\nu}{\partial a^\mu} + \frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} - \frac{\partial R^*_\mu}{\partial a^\nu} - \frac{\partial R^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\right)\dot{a}^\nu \\ & - \left(\frac{\partial B^*}{\partial a^\mu} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial R^*_\mu}{\partial t} + \frac{\partial R^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial t}\right) = 0. \end{aligned} \tag{29}$$

IV. GLOBAL ANALYSIS OF BIRKHOFF'S EQUATIONS OF A ROTATIONAL RELATIVISTIC SYSTEM

A. Global property of autonomous rotational relativistic Birkhoff's equations

If there exists a non-degenerate closed exact 2-form Ω^* on T^*M, then T^*M is called an exact symplectic manifold, and is expressed as (T^*M,Ω^*). It shows that

$$\Omega^* = \mathrm{d}R^*, \quad \mathrm{d}\Omega^* = 0. \tag{30}$$

If an arbitrary vector field X on T^*M, for some function B^* on T^*M, satisfies the following condition

$$i_X\Omega^* \equiv X\lrcorner\,\Omega^* = -\mathrm{d}B^*, \tag{31}$$

then it is called the global rotational relativistic Birkhoffian vector field. Eq.(31) is the global formulation of autonomous rotational relativistic Birkhoff's equations. In local coordinates,

$$X\lrcorner\,\Omega^* = \Omega^*_{\mu_1\mu_2}X^{\mu_1}\mathrm{d}a^{\mu_2} = X_\mu \mathrm{d}a^\mu = -\mathrm{d}B^* \tag{32}$$

and

$$i_X\Omega^* = \frac{1}{2}\Omega^*_{\mu\nu}X^\nu \mathrm{d}a^\mu. \tag{33}$$

If the vector field X is of rotational relativistic Birkhoff's type, it must have the apparent form

$$\begin{aligned} X =& \Omega^{*\mu\nu}\left(\frac{\partial B^*}{\partial a^\nu}\frac{\partial}{\partial a^\mu} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\frac{\partial}{\partial a^\mu}\right), \\ & (\Omega^{*\mu\nu}) = (\Omega^*_{\mu\nu})^{-1}. \end{aligned} \tag{34}$$

So the autonomous rotational relativistic Birkhoff's equations with contravariant form are

$$\begin{aligned} \dot{a}^\mu =& \Omega^{*\mu\nu}(I_i(\boldsymbol{a}),\boldsymbol{a}) \\ & \cdot\left(\frac{\partial B^*(I_i(\boldsymbol{a}),\boldsymbol{a})}{\partial a^\nu} + \frac{\partial B^*(I_i(\boldsymbol{a}),\boldsymbol{a})}{\partial I_i(\boldsymbol{a})}\frac{\partial I_i(\boldsymbol{a})}{\partial a^\nu}\right). \end{aligned} \tag{35}$$

We can see that, from the viewpoint of global analysis, the difference between the rotational relativistic Hamilton vector field and the rotational relativistic Birkhoff vector field vanishes. That a rotational relativistic vector field is Hamiltonian or Birkhoffian depends on whether the symplectic structure in local coordinates is a fundamental one or general exact one.

B. Global property of semi-autonomous rotational relativistic Birkhoff's equations

We generalize the cotangent bundle T^*M to the contact manifold $\vec{R}\times T^*M$. If the maximal rank, closed exact 2-form, exists then the manifold of $2n+1$ dimensions is called the exact contact manifold. If Ω^* is an exact symplectic form on T^*M, using the mapping $\pi:\vec{R}\times T^*M\to T^*M$, notation $\pi^*\Omega^* = \tilde{\Omega}^*$ constructs a contact form. For some function B^* on $\vec{R}\times T^*M$, the 2-form

$$\Omega^*_B = \tilde{\Omega}^* - \mathrm{d}B^*\wedge\mathrm{d}t \tag{36}$$

is an exact contact form.

With set $(\vec{R}\times T^*M,\Omega^*)$, any vector field $\tilde{X}$ satisfying

$$i_{\tilde{X}}\Omega^*_B \equiv \tilde{X}\lrcorner\,\Omega^*_B = 0, \tag{37}$$

$$\mathrm{d}t(\tilde{X}) = 1 \tag{38}$$

is a semi-autonomous global relativistic Birkhoffian vector field. By Eq.(36), the rotational relativistic Birkhoffian depends apparently on time, but the substructure $\tilde{\Omega}^*$ does not depend on time in local coordinates.

C. Global property of non-autonomous rotational relativistic Birkhoff's equations

Introduce an arbitrary 1-form $\hat{R}^*$ on $\vec{R}^*\times T^*M$ such that the self-adjoint 2-form from exterior differentiation

$$\tilde{\Omega}^* = \mathrm{d}\tilde{R}^* \tag{39}$$

has the maximal rank. Any non-autonomous vector field $\tilde{X}$ on $\boldsymbol{R}\times T^*M$ satisfying

$$i_{\bar{X}}d\tilde{R}^*\equiv\tilde{X}_{\lrcorner}d\hat{R}^*=0, \tag{40}$$

$$\mathrm{d}t(\tilde{X})=1 \tag{41}$$

is called the non-autonomous global rotational relativistic Birkhoffian vector field. Eqs.(40) and (41) are global formulations of non-autonomous rotational relativistic Birkhoff's equations.

In fact, in local coordinates, the 1-form $\hat{R}^*$ can be written as

$$\begin{aligned}\hat{R}^*=&R^*_\mu(I_i(\boldsymbol{a}),\boldsymbol{a})\mathrm{d}\hat{a}^\mu\\ =&R^*_\mu(I_i(t,\boldsymbol{a}),,t,\boldsymbol{a})\mathrm{d}a^\mu-B^*(I_i(t,\boldsymbol{a}),t,\boldsymbol{a})\mathrm{d}t,\\ B^*=&-R^*_0;\end{aligned} \tag{42}$$

and its exterior differentiation

$$\begin{aligned}\mathrm{d}\hat{R}^*=&\left(\frac{\partial\hat{R}^*_\mu}{\partial\hat{a}^\nu}+\frac{\partial\hat{R}^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial\hat{a}^\nu}\right)\mathrm{d}\hat{a}^\nu\wedge\mathrm{d}\hat{a}^\mu\\ =&\frac{1}{2}\left(\frac{\partial\hat{R}^*_\mu}{\partial\hat{a}^\nu}+\frac{\partial\hat{R}^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial\hat{a}^\nu}\right.\\ &\left.-\frac{\partial\hat{R}^*_\nu}{\partial a^\mu}-\frac{\partial\hat{R}^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\mathrm{d}\hat{a}^\nu\wedge\mathrm{d}a^\mu\\ =&\frac{1}{2}\tilde{\Omega}^*_{\nu\mu}\mathrm{d}\hat{a}^\nu\wedge\mathrm{d}\hat{a}^\mu.\end{aligned} \tag{43}$$

Combining the relation

$$\tilde{X}_{\lrcorner}\mathrm{d}\hat{R}^*=\Omega^*_{\mu\nu}X^\mu\mathrm{d}\hat{a}^\nu=\hat{\Omega}^*_{\mu\nu}d\hat{a}^\nu \tag{44}$$

with Eq.(40), we have

$$\tilde{\Omega}^*_{\mu\nu}\mathrm{d}\hat{a}^\nu=0,\quad \hat{a}=(t,a)\quad(\mu,\nu=0,1,\cdots,2n); \tag{45}$$

where

$$\begin{aligned}\tilde{\Omega}^*_{\mu\nu}=&\frac{\partial R^*_\nu}{\partial a^\mu}+\frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}-\frac{\partial R^*_\mu}{\partial a^\nu}-\frac{\partial R^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\\ &(\mu,\nu=1,\cdots,2n),\end{aligned} \tag{46}$$

$$\tilde{\Omega}^*_{o\nu}=\left(\frac{\partial B^*}{\partial a^\nu}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}+\frac{\partial R^*_\nu}{\partial t}+\frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial t}\right)=-\hat{\Omega}^*_{\nu o}$$
$$(\nu=1,\cdots,2n). \tag{47}$$

Eqs.(45) are just non-autonomous rotational relativistic Birkhoff's equations.

D. Geometrical property of energy change for rotational relativistic Birkhoffian system

In local coordinates, the Lie derivative of rotational relativistic Birkhoffian $B^*(I_i(t,\boldsymbol{a}),t,\boldsymbol{a})$ with respect to a global rotational relativistic Birkhoffian vector field $\tilde{X}$ can be expressed as

$$\begin{aligned}L_{\bar{X}}B^*=&\left(\frac{\partial B^*}{\partial a^\mu}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\dot{a}^\mu+\frac{\partial B^*}{\partial t}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial t}\\ =&\left(\frac{\partial B^*}{\partial a^\mu}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\Omega^*_{\mu\nu}\\ &\times\left(\frac{\partial B^*}{\partial a^\nu}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}+\frac{\partial R^*_\nu}{\partial t}+\frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial t}\right)\\ &+\frac{\partial B^*}{\partial t}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial t}\\ =&\left(\frac{\partial B^*}{\partial a^\mu}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\Omega^{*\mu\nu}\left(\frac{\partial R^*_\nu}{\partial t}+\frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial t}\right)\\ &+\frac{\partial B^*}{\partial t}+\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial t}.\end{aligned} \tag{48}$$

Choosing the rotational relativistic Birkhoffian as the total energy of the system, i.e., $B^*=E^*_{\mathrm{tot}}$, rule (48) gives the geometrical property of energy rate of variation in time

$$\begin{aligned}\dot{E}^*_{\mathrm{tot}}=&L_{\bar{X}}E^*_{\mathrm{tot}}\\ =&\left(\frac{\partial E^*_{\mathrm{tot}}}{\partial a^\mu}+\frac{\partial E^*_{\mathrm{tot}}}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\Omega^{*\mu\nu}(I_i(t,\boldsymbol{a}),t,\boldsymbol{a})\\ &\cdot\left(\frac{\partial R^*_\nu}{\partial t}+\frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial t}\right)+\frac{\partial E^*_{\mathrm{tot}}}{\partial t}+\frac{\partial E^*_{\mathrm{tot}}}{\partial I_i}\frac{\partial I_i}{\partial t}.\end{aligned} \tag{49}$$

Rule(49) does not have a Lie algebraic structure.

If Eq.(49) has the following form

$$\dot{E}^*_{\mathrm{tot}}(I(\boldsymbol{a}),\boldsymbol{a})=L_XE^*_{\mathrm{tot}}=[E^*_{\mathrm{tot}},E^*_{\mathrm{tot}}]^*_{(I(\boldsymbol{a}),\boldsymbol{a})}=0, \tag{50}$$

then it has a Lie algebraic structure.

V. CONCLUSION

The study of this paper has general meaning. In the classical approximation $\dot{\theta}_i\ll\Gamma_i$, the relativistic rotational inertia

$$I_i=\frac{I_{oi}}{\sqrt{1-\dot{\theta}^2_i/\Gamma^2_i}}\approx I_{oi} \tag{51}$$

becomes the classical rotational inertia. In this situation, this paper gives the classical rotational Pfaff–Birkhoff principle, Birkhoffian equations, and a geometrical description and global analysis of Birkhoff's equations. These results are novel in the field of classical physics.

ACKNOWLEDGMENT

We would like to thank Professor Mei Feng-xiang for his helpful guidance and support.

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