Vol 10 No 3, March 2001
1009-1963/2001/10(03)/0234-06

Chinese Physics

© 2001 Chin. Phys. Soc.
and IOP Publishing Ltd

QUASILOCAL ENERGY FOR STATIONARY AXISYMMETRIC EMDA BLACK HOLE

WANG SHI-LIANG(王世良)[a] and JING JI-LIANG(荆继良)[a,b]

[a] *Physics Department and Institute of Physics, Hunan Normal University, Changsha* 410081, *China*

[b] *Department of Astronomy and Applied Physics, University of Science and Technology of China, Hefei* 230026, *China*

(Received 19 August 2000)

By using Brown–York quasilocal energy theory we calculate the quasilocal energy of a stationary axisymmetic EMDA black hole and explore the universality of Martinez's conjecture in string theory. We show that the energy is positive and monotonically decreases to the ADM mass at spatial infinity, and the Martinez's conjecture, the Brown–York quasilocal energy at the outer horizon reduces to twice its irreducible mass, is still valid for stationary axisymmetric EMDA black hole. From the result we also find that the Kerr–Sen spacetime keeps up with Martinez's conjecture. This is different from the Bose–Naing result that the quasilocal energy of the Kerr–Sen spacetime does not approach the Martinez's conjecture.

Keywords: black hole, energy, general relativity
PACC: 9760L, 0470D

I. INTRODUCTION

General relativity accepts several alternative 'definitions' of quasilocal energy (QLE). Despite considerable effort being made in attempting to define QLE in general relativity, there has not been a general definite expression up to now. In fact, different proposals provided satisfied the definition for energy only for some special cases. Brown and York[1] proposed an interesting definition for QLE. One of the appealing features of the definition is that QLE can be obtained from the gravitational action and Hamiltonian in a bounded region. Consider a spacetime M foliated by a spacelike super surface Σ_t defined by the constant t, as well as a timelike boundary B. A two-dimensional (2D) surface ${}^2B$ is noted by the intersection of Σ_t with B. Assuming that Σ_t is orthogonal to B, one can then obtain a stress-energy-momentum defined locally on the three-boundary B, with respect to the boundary metric. Proper energy surface density is defined by the projection of the stress-tensor on a 2D surface ${}^2B$, and the total QLE is defined by the integration over the 2D surface ${}^2B$:

$$E = \varepsilon - \varepsilon_0 = \frac{1}{\kappa}\int_{{}^2B} \mathrm{d}^2x\sqrt{\sigma}(k - k_0), \qquad (1)$$

where σ is the determinant of the metric σ_{ij} on ${}^2B$, k is the trace of the extrinsic curvature of ${}^2B$ as embedded in Σ_t, and k_0 is the value of k in a flat spacetime. ε_0 is a reference term proposed in Ref.[3] for the normalization of the energy with respect to a reference spacetime. We choose the reference spacetime to be flat since the stationary axisymmetric spacetime that we will study is asymptotically flat. The units are chosen so that $G = C = \hbar = 1$ and $\kappa = 8\pi$. This expression for the energy also appears in the study of self-gravitating systems in thermal equilibrium, where it plays the role of the thermodynamical energy conjugate to inverse temperature.[4–5] For a detailed discussion of QLE proposals see Refs.[1,6] and references cited therein.

The definition (1) was first used to calculate static spacetimes in Refs.[1,7], where the negative contributions to the energy arising from gravitational boundary have been studied. Later, it was extended to analysis the stationary axisymmetric spacetime by Martinez.[8] He considered the boundary surface for Kerr slice as embedded in both the Kerr slice and flat three-dimensional space (required as a normalization of energy), and obtained the following conclusion: the energy is a positive, monotonically decreasing function of boundary surface radius and approaches the Arnowitt–Deser–Misner (ADM) mass at spatial infinity and reduces to (twice) the irreducible mass (or $\sqrt{A/4\pi}$, where A is its area) at the horizon of the Kerr black hole. Recently, Bose and Naing[9] studied the QLE of the Kerr–Newman black hole and the Kerr–Sen black hole. They showed that the Kerr–Newman black hole keeps up with Martinez's conjecture, but

the Kerr–Sen black hole does not.[9]

It is known that the mechanical laws of black holes in general relativity can be extended to the string-inspired four-dimensional low-energy effective theory[10--14]. Then, the question whether other identities related to black hole mechanics in general relativity continue to hold for the black hole in string theory is still open. The main purpose of this paper is to investigate the QLE for a stationary axisymmetric Einstein–Maxwell dilaton–axion (EMDA) black hole by using the Brown–York proposal,[1] and to see whether Martinez's conjecture is applicable for black hole arisen in string theory.

This paper is organized as follows. The metric of stationary axisymmetric EMDA black hole is introduced in section II. In section III, we calculate the QLE for stationary axisymmetric EMDA black hole and discuss the contributions due to the gravitational binding, as well as the universality of Martinez's conjecture. We summarize our conclusion in section IV. Finally, in the appendix, we deduce the general expressions for the integrals involved in the QLE for arbitrary stationary axisymmetric spaces.

II. STATIONARY AXISYMMETRIC EMDA BLACK HOLE

The four-dimensional low-energy Lagrangian obtained from string theory is[2]

$$\begin{aligned} S =& \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\Big(R - 2g^{\mu\nu}\,\nabla_\mu\,\Phi\,\nabla_\nu\,\Phi \\ & - \frac{1}{2}\mathrm{e}^{4\Phi}g^{\mu\nu}\,\nabla_\mu\,K_{\mathrm{a}}\,\nabla_\nu\,K_a \\ & - \mathrm{e}^{-2\Phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} - K_aF_{\mu\nu}\tilde{F}^{\mu\nu}\Big), \end{aligned} \tag{2}$$

with $\tilde{F}_{\mu\nu} = -\sqrt{-g}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$, where R is the scalar Riemann curvature, $g^{\mu\nu}$ is the metric four-dimensional tensor, Φ is the massless dilaton field, $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor field, and K_{a} is the axion field due to the three-index antisymmetric tensor field $H = -\exp(4\Phi)\cdot \mathrm{d}K_{\mathrm{a}}/4$.

The stationary axisymmetric EMDA black hole solution (we take the solution $b = 0$ in Eq.(14) in Ref.[2]; the reason we use this solution is that the solution $b \neq 0$ cannot be interpreted properly as a black hole) is given[2] by

$$\begin{aligned} \mathrm{d}s^2 =& -\frac{\Sigma - a^2\sin^2\theta}{\Delta}\mathrm{d}t^2 \\ & + \frac{\Delta}{\Sigma}\mathrm{d}r^2 + \Delta\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Delta} \\ & \cdot\left[(r^2 + a^2 - 2Dr)^2 - \Sigma a^2\sin^2\theta\right]\mathrm{d}\phi^2 \\ & - \frac{2a\sin^2\theta}{\Delta}\left[(r^2 + a^2 - 2Dr) - \Sigma\right]\mathrm{d}t\mathrm{d}\phi, \end{aligned} \tag{3}$$

with

$$\begin{aligned} \Sigma &= r^2 - 2mr + a^2, \\ \Delta &= r^2 - 2Dr + a^2\cos^2\theta, \end{aligned} \tag{4}$$

and

$$\mathrm{e}^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2\cos^2\theta), \quad \omega = \mathrm{e}^{2\Phi_0}, \tag{5}$$

$$K_a = K_0 + \frac{2aD\cos\theta}{W}, \tag{6}$$

$$\begin{aligned} A_t &= \frac{1}{\Delta}(Qr - \mathrm{g}a\cos\theta), \qquad A_r = A_\theta = 0, \\ A_\phi &= \frac{1}{a\Delta}(-Qra^2\sin^2\theta + \mathrm{g}(r^2 + a^2)a\cos\theta). \end{aligned} \tag{7}$$

The mass M, angular momentum J, electric charge Q, and magnetic charge P of the black hole are given by

$$\begin{aligned} & M = m - D, \quad J = a(m - D), \\ & Q = \sqrt{2\omega D(D - m)}, \quad P = g. \end{aligned} \tag{8}$$

The above results show that the stationary axisymmetric EMDA black hole differs considerably from the Kerr–Newman black hole. As a result the stationary axisymmetric EMDA black hole shows several different properties as compared to the Kerr–Newman black hole: (a) the two horizons of the Kerr–Newman black hole are given by $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, whereas in the case of the stationary axisymmetric EMDA black hole we have $r_\pm = (M - (Q^2/2\omega M)) \pm \sqrt{(M - (Q^2/2\omega M))^2 - a^2}$; (b) the Kerr–Newman metric has singularities at $r^2 + \cos^2\theta = 0$, whereas the stationary axisymmetric EMDA black hole has singularities at $r^2 - 2Dr + \cos^2\theta = 0$. On the other hand, by setting in the above expressions $D = -m\sinh^2(\alpha/2)$ and $\omega = 1$, Eq.(8) reduces to the parameters characterizing the Sen solution

$$\begin{aligned} & 2M = m(1 + \cosh\alpha), \quad 2J = ma(1 + \cosh\alpha), \\ & \sqrt{2}Q = m\sinh\alpha, \qquad \sqrt{2}\mu = ma\sinh\alpha, \end{aligned} \tag{9}$$

(see Eqs.(16) and (17) of Ref.[5]) and the metric structure (3) can be considered as a generalized Sen solution for the EMDA field equations in the string gravity.

III. QLE FOR THE STATIONARY AXISYMMETRIC EMDA BLACK HOLE

We first evaluate the energy within the surface ${}^2B$ embedded in Σ_t, defined by $r = r_B$ =const. We assume that $r_B \geq r_+$, where r_+ represents the outer horizon of stationary axisymmetric EMDA black hole. So the line element on this surface in the slow-rotation approximation (i.e. $a \ll r_B$) becomes

$$\mathrm{d}s^2 = r_B^2\left(1-\frac{2D}{r_B}+\frac{a^2}{r_B^2}\cos^2\theta\right)\mathrm{d}\theta^2 + r_B^2\left\{\left(1-\frac{2D}{r_B}\right) + \frac{a^2}{r_B^2}\left[1+\frac{2(m-D)}{r_B\left(1-\frac{2D}{r_B}\right)}\sin^2\theta\right]\right\}\mathrm{d}\phi^2. \tag{10}$$

Terms of order $O(a^4/r_B^4)$ and higher have been neglected.

Using Eq.(A6) of the appendix, the integral ε for the surface $r = r_B$ can be written explicitly as

$$\varepsilon = -\frac{r_B}{2}\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\int_0^\pi \mathrm{d}\theta\sin\theta\frac{1}{\sqrt{1-\frac{2D}{r_B}+\frac{a^2}{r_B^2}\cos^2\theta}} \cdot\frac{\left(1-\frac{D}{r_B}\right)\left(1-\frac{2D}{r_B}+\frac{a^2}{r_B^2}\right)-\frac{a^2}{2r_B}\left(1-\frac{M}{r_B}\right)\sin^2\theta}{\sqrt{\left(1-\frac{2D}{r_B}\right)^2+\frac{a^2}{r_B^2}\left[1+\cos^2\theta-\frac{4D}{r_B}+\left(\frac{2m}{r_B}+\frac{a^2}{r_B^2}\right)\sin^2\theta\right]}}. \tag{11}$$

Obviously, the integral cannot be expressed in simple functions. But when we consider the conditions of slow rotation defined by $a/r_B \ll 1$, the above integral approximately becomes

$$\varepsilon = -\frac{r_B}{2}\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\int_0^\pi \mathrm{d}\theta\sin\theta\left\{\frac{1-\frac{D}{r_B}}{\sqrt{1-\frac{2D}{r_B}}} - \frac{a^2}{2r_B^2}\frac{\left(1-\frac{2D}{r_B}\right)\left(1-\frac{D}{r_B}\right)\cos^2\theta+\frac{m-D}{r_B}\sin^2\theta}{\left(1-\frac{2D}{r_B}\right)^{5/2}} + O\left(\frac{a^4}{r_B^4}\right)\right\}, \tag{12}$$

where only leading terms in the expansion around the small parameter a/r_B are considered. Then, the integration can be performed in a straight way, so we get

$$\varepsilon = -r_B\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\left\{\frac{1-\frac{D}{r_B}}{\sqrt{1-\frac{2D}{r_B}}} - \frac{a^2}{6r_B^2}\cdot\left[\frac{1}{\left(1-\frac{2D}{r_B}\right)^{3/2}}+\frac{2m-D}{r_B\left(1-\frac{2D}{r_B}\right)^{5/2}}\right] + O\left(\frac{a^4}{r_B^4}\right)\right\}, \tag{13}$$

which is the QLE without a reference term for the stationary axisymmetric EMDA black hole. We should note that no approximation is made in the square root. In the limit $D = -\dfrac{Q^2}{2M\omega} \to 0$, Eq.(13) reduces to

$$\varepsilon = -r_B\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\cdot\left\{1-\frac{a^2}{6r_B^2}\left[1+\frac{2m}{r_B}+O\left(\frac{a^4}{r_B^4}\right)\right]\right\}. \tag{14}$$

This is the unreferenced QLE of the neutral Kerr black hole in Ref.[8]. As $r_B \to \infty$, from Eq.(13) we have $\varepsilon \to M - r_B$, which is divergent. Therefore, it requires a subtraction term to renormalize the energy.

In order to get the reference term in expression (1), we find a two-dimensional surface isometric to Eq.(10), which is embedded in a flat three-dimensional slice (see Refs.[8,9] for a detailed discussion) with the line element

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\mathrm{d}\Theta^2 + R^2\sin^2\Theta\mathrm{d}\Phi^2, \tag{15}$$

where R, Θ and Φ are the spherical polar coordinates. Let the two-dimensional surface embedded in the flat slice denoted by $R = f(\Theta)$, where f is a function of the azimuthal angle Θ and the parameters (M, a, Q, r_B).

We assume that $\Theta = \Theta(\theta)$ and $\Phi = \phi$. So on the two-dimensional surface ${}^2B$, R is also a function of θ. Thus the line element on ${}^2B$ becomes

$$\mathrm{d}s^2 = (\dot{R}^2 + R^2\dot{\Theta}^2)\mathrm{d}\theta^2 + R^2\sin^2\Theta\mathrm{d}\phi^2, \tag{16}$$

where the overdot denotes derivative with respect to θ.

The requirement of the above line element to be isometric to Eq.(10) indicates that the following equations should be satisfied

$$\dot{R}^2 + R^2\dot{\Theta}^2 = r_B^2\left(1 - \frac{2D}{r_B} + \frac{a^2}{r_B^2}\cos^2\theta\right), \tag{17}$$

$$R^2\sin^2\Theta = r_B^2\sin^2\theta\left\{\left(1 - \frac{2D}{r_B}\right) + \frac{a^2}{r_B^2}\left[1 + \frac{2(m-D)\sin^2\theta}{r_B\left(1 - \frac{2D}{r_B}\right)}\right]\right\}. \tag{18}$$

Assuming that $\dot{R}^2 \simeq O\left(a^4/r_B^4\right)$ (which will be justified below) and combining the two equations, one can get the first-order ordinary differential equation

$$\frac{\mathrm{d}\Theta}{\sin\Theta} = \frac{\mathrm{d}\theta}{\sin\theta}\left\{1 - \frac{a^2\sin^2\theta}{2r_B^2\left(1 - \frac{2D}{r_B}\right)}\cdot\left[1 + \frac{2(m-D)}{r_B\left(1 - \frac{2D}{r_B}\right)}\right]\right\}. \tag{19}$$

The solution of the above equation is

$$\sin\Theta = \sin\theta\left\{1 + \frac{a^2\cos^2\theta}{2r_B^2\left(1 - \frac{2D}{r_B}\right)}\left(1 + \frac{2(m-D)}{1 - \frac{2D}{r_B}}\right)\right\}. \tag{20}$$

Putting the solution back into Eq.(19), we get

$$f(\Theta) = r_B\left[\sqrt{1 - \frac{2D}{r_B}} + \frac{a^2\sin^2\Theta}{2r_B^2\sqrt{1 - \frac{2D}{r_B}}} - \frac{a^2\cos 2\Theta\left(m/r_B - \frac{D}{r_B}\right)}{r_B^2\left(1 - \frac{2D}{r_B}\right)^{3/2}}\right] + r_B O\left(\frac{a^4}{r_B^2}\right), \tag{21}$$

where we substitute for θ in terms of Θ in order to differ from Eq.(10). The expression implies that the two-surface $R = f(\Theta)$ describes a distorted sphere, and it can also be used to prove that the above assuming $\dot{R}^2 \simeq O(a^4/r_B^4)$.

From the previous analysis, in the slow-rotation approximation we get the intrinsic metric on ${}^2B$, as embedded in a flat space

$$\mathrm{d}s^2 \simeq r_B^2\left[(1 - 2D/r_B) + \frac{a^2}{r_B^2}\cdot\left(\sin^2\Theta - \frac{2(m-D)}{r_B(1-2D/r_B)}\cos 2\Theta\right)\right]\cdot(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\Phi^2). \tag{22}$$

The extrinsic curvature k_0 and its proper integral can be calculated by using Eqs.(A5) and (A6). A tedious computation gives the reference term in Eq.(1)

$$\varepsilon_0 = \frac{1}{\kappa}\int_{{}^2B} k_0\sqrt{\sigma}\mathrm{d}\Theta\mathrm{d}\Phi \simeq -r_B\left\{\sqrt{1 - \frac{2D}{r_B}} + \frac{a^2}{3r_B^2\sqrt{1 - \frac{2D}{r_B}}}\cdot\left[1 + \frac{m-D}{r_B(1-2D/r_B)}\right] + O\left(\frac{a^4}{r_B^4}\right)\right\}. \tag{23}$$

By substituting Eqs.(23) and (13) into Eq.(1), we have

$$E = r_B\left\{\sqrt{1 - \frac{2D}{r_B}} - \frac{1 - \frac{D}{r_B}}{\sqrt{1 - \frac{2D}{r_B}}}\sqrt{1 - \frac{2m}{r_B} + \frac{a^2}{r_B^2}}\right\} + \frac{a^2}{6r_B\sqrt{1 - 2D/r_B}}\left\{2 + \frac{2(m-D)}{r_B(1-2D/r_B)} + \left[\frac{1}{1 - \frac{2D}{r_B}} + \frac{2m - D}{r_B\left(1 - \frac{2D}{r_B}\right)^2}\right]\sqrt{1 - \frac{2m}{r_B} + \frac{a^2}{r_B^2}}\right\} + r_B O(a^4/r_B^4), \tag{24}$$

which is the QLE for the stationary axisymmetric EMDA black hole with the three-dimensional super surface Σ_t bounded by a two-dimensional surface of constant radial r_B.

It is worthwhile to note several features of the above expression. In the asymptotic limit $r_B \to \infty$,

$$E \to m - D = M, \tag{25}$$

which is the ADM mass M[16] of the stationary axisymmetric EMDA black hole. For all values of $r_B > r_+$, the QLE (24) within the constant radius surface is positive, and can be expressed in a monotonically decreasing function of r_B. For vanishing rotation

$(a=0)$, Eq.(24) reduces to

$$E = r_B\left(\sqrt{1-\frac{2D}{r_B}} - \frac{1-D/r_B}{\sqrt{1-2D/r_B}}\sqrt{1-2m/r_B}\right), \quad (26)$$

which is the QLE of the static charged black hole in the string theory. When the electric charge vanishes, i.e. $D=0$, the energy reduces to

$$\begin{aligned} E =& r_B\left[1-\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\right]+\frac{a^2}{6r_B} \\ &\cdot\left[2+\frac{2m}{r_B}+\left(1+\frac{2m}{r_B}\right)\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\right] \\ &+ r_B O(a^4/r_B^4), \end{aligned} \quad (27)$$

which is the QLE of the Kerr black hole.[8] By setting $\omega=1$ and $D=-m\sinh^2(\alpha/2)$ into Eq.(23), we get the QLE for the Kerr–Sen black hole

$$\begin{aligned} E =& r_B\sqrt{\lambda}\left[1-\frac{r_B+m\sinh^2(\alpha/2)}{r_B\lambda}\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\right] \\ &+\frac{a^2}{6r_B\sqrt{\lambda}}\left\{2+\frac{2m\cosh^2(\alpha/2)}{r_B\lambda}\right. \\ &\left.+\left[1+\frac{m(1+\cosh^2(\alpha/2)}{r_B\lambda}\right]\sqrt{1-\frac{2m}{r_B}+\frac{a^2}{r_B^2}}\right\}, \end{aligned} \quad (28)$$

where $\lambda = 1+2m\sinh^2(\alpha/2)/r_B$. We note that the expression differs from that in Ref.[13]. Near the outer horizon $r_B=r_+$, the energy (24) becomes

$$\begin{aligned} &E(r_B=r_+) \\ =&r_+\left(\sqrt{1-\frac{2D}{r_+}}+\frac{1}{2\sqrt{1-2D/r_+}}\frac{a^2}{r_+^2}+O\left(\frac{a^4}{r_+^4}\right)\right). \end{aligned} \quad (29)$$

Noting the area of the outer event horizon

$$A=\int\sqrt{g_{\theta\theta}g_{\phi\phi}}\mathrm{d}\theta\mathrm{d}\phi = 4\pi(r_+^2+a^2-2Dr_+), \quad (30)$$

from Eqs.(29) and (30), we arrive at the result

$$E(r_B=r_+)\simeq\left[\frac{A}{4\pi}\right]^{1/2}. \quad (31)$$

The relation was first found by Martinez for the Kerr black hole,[12] and is valid for the Kerr–Newman black hole but not for the Kerr–Sen black hole in Ref.[13]. However, putting $r_B=r_+$ into Eq.(28), we get

$$\begin{aligned} E_{\mathrm{KS}}(r_B=r_+) &\simeq \left[\frac{A_{\mathrm{KS}}}{4\pi}\right]^{1/2} \\ &= r_+\cosh\frac{\alpha}{2}\left[1+\frac{a^2}{2r_+^2}+O(a^4/r_+^4)\right], \end{aligned} \quad (32)$$

for Kerr–Sen. Expression (32) shows that Martinez's conjecture is held for the Kerr–Sen black hole too. The conclusion differs from Bose–Naing's.[13] In order to illustrate the contributions of different parameters in the energy at the boundary (which is also useful in the study of black hole stable configurations in a gravitational microcanonical ensemble,[15,17] by inverting the energy equation (24) we obtain

$$M = E-\frac{E^2}{2r_B}+\frac{Q^2}{2r_B\omega}+\frac{D^2}{2r_B}, \quad (33)$$

where only the leading orders of $1/r_B$ are considered. In this case, the energy out r_B consists of three contributions: the negative Newton gravitational binding energy $-E^2/(2r_B^2)$ associated with building a spherical shell of radius r_B and m by bringing the individual particles together from infinity[18]; the positively coupled energy $+Q^2/(2\omega r_B)$, associated with charge Q, dilaton $\omega(=\mathrm{e}^{2\Phi_0})$ and r_B; and the third contribution $+D^2/(2r_B)$ comes from the dilaton charge D, which is also responsible for a long-range, attractive force between black holes and has the analogous behaviour as charge Q.[19]

IV. CONCLUSION

We have calculated the quasilocal energy of a stationary axisymmetric EMDA black hole in the slow-rotation regime, i.e. $a/r_B \ll 1$. The result gives relations of the quasilocal energy and the 'hair' of the black hole. We also show that, at the outer horizon, the QLE for such a black hole reduces to $\sqrt{A/4\pi}$, so does a Kerr–Sen black hole. The conclusion is different from that of Bose and Naing[13] that Martinez's conjecture dose not hold for the Kerr–Sen black hole in string theory. Since the energy (1) depends on the choice of a reference term, especially at the outer horizon, it is only dependent of the reference term. Whether the conjecture is applicable when the reference term is alternated will be discussed elsewhere.

APPENDIX

We deduce general expressions for the proper integrals involved in the quasilocal energy for arbitrary stationary axisymmetric spacetime. These expressions can be used to calculate ε and ε_0 in this paper. The detailed process is given in Ref.[8], and only in order to be complete we list the main result here.

A three-dimensional axisymmetric spacelike hypersurface Σ_t of a four-dimensional stationary axisymmetric spacetime is described by

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j = b^2\mathrm{d}r^2 + c^2\mathrm{d}\theta^2 + d^2\mathrm{d}\phi^2, \qquad \text{(A1)}$$

where $x^i(=r,\theta,\phi)$ denote arbitrary coordinates adapted to the symmetry, and the metric functions b, c and d depend only on the 'radial' coordinate r and the 'azimuthal' coordinate θ. An arbitrary two-dimensional axisymmetric surface ${}^2B$ (with the topology of a two-sphere) embedded in the three-dimensional space Σ_t is defined by the relation $r = R(\theta)$, where R is a function of the azimuthal angle and the parameters of the solution. Its two-dimensional line element is

$$\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b = (b^2\dot{R}^2 + c^2)\mathrm{d}\theta^2 + d^2\mathrm{d}\phi^2, \qquad \text{(A2)}$$

where an overdot denotes derivative with respect to the coordinate θ. The functions b, c and d in (A2) are evaluated at $r = R(\theta)$.

Let n^i denote the unit outward-pointing spacelike normal to ${}^2B$ as embedded in the hyper surface Σ. In terms of metric functions, n^i can be defined as

$$(n^r, n^\theta, n^\phi) = \frac{1}{\sqrt{c^2 + b^2\dot{R}^2}}(c/b, -b\dot{R}/c, 0). \qquad \text{(A3)}$$

The extrinsic curvature of the two-surface ${}^2B$ as embedded in Σ_t is denoted by $k_{\mu\nu}$. Its trace k can be written as

$$k = -\frac{\partial_\mu(n^\mu\sqrt{h})}{\sqrt{h}}, \qquad \text{(A4)}$$

where h denotes the determinant of the three-metric $h_{\mu\nu}$. Using the coordinate components of the unit normal (A3), the trace (A4) can be written as

$$k = -\frac{1}{bcd}\left[(\alpha\gamma^{-1/2})_{,r} - (\beta\gamma^{-1/2})_{,\theta}\right]|_{r=R(\theta)}, \qquad \text{(A5)}$$

where $\alpha \equiv c^2 d, \beta \equiv b^2 d\dot{R}, \gamma \equiv c^2 + b^2\dot{R}^2$, and $\delta \equiv \ln\gamma$. Then, its proper surface integral becomes

$$\begin{aligned}&\frac{1}{\kappa}\int_{{}^2B} k\sqrt{\sigma}\mathrm{d}\theta\mathrm{d}\phi \\ &= -\frac{1}{4}\int_0^\pi \mathrm{d}\theta\frac{1}{bc}\left(\alpha_{,r} - \beta_{,\theta} - \frac{\alpha}{2}\delta_{,r} + \frac{\beta}{2}\delta_{,\theta}\right)|_{r=R(\theta)}.\end{aligned} \qquad \text{(A6)}$$

This integral is evaluated at the surface $r = R(\theta)$. Both ε and ε_0 in Eq.(1) can be calculated by the above expression in respect to different values for the functions b, c and d, as well as different coordinate definitions.

References

[1] Brown J D and Jr York J W 1993 *Phys. Rev.* D **47** 1407
[2] Garcia A, Galtsov K and Kechkin O 1995 *Phys. Rev. Lett.* **74** 1276
[3] Gibbons G W and Hawking S W 1977 *Phys. Rev.* D **15** 2752
[4] York Jr J W 1986 *Phys. Rev.* **33** 2091; Whiting B F and York Jr J W 1988 *Phys. Rev. Lett.* **61** 1336
[5] Brown J D, Martinez E A and Jr York J W 1991 *Phys. Rev. Lett.* **66** 2881
[6] Hayward S A 1994 *Phys. Rev.* D **49** 831
[7] Brown J D, Creington J D E and Mana R B 1994 *Phys. Rev.* D **50** 6394
[8] Martinez E A 1994 *Phys. Rev.* D **50** 4920
[9] Bose S and Naing T Z 1999 *Phys. Rev.* D **60** 104027
[10] Jing J L and Yan M L gr-qc/9904001 and 1999 *Phys. Rev.* D **60** 084015
[11] Jing J L and Yan M L gr-qc/9907011 and 2000 *Phys. Rev.* D **61** 044016
[12] Jing J L and Yan M L 2000 *Chin. Phys.* **9** 389
[13] Jing J L and Yan M L gr-qc/0004061 and 2000 *Phys. Rev.* D **62** 104013
[14] Jing J L and Yan M L 'Statistical Entropy of Stationary Dilaton Black Hole from Cardy Formulas' gr-qc/0005105 and 2000 *Phys. Rev.* D **62** to be published.
[15] Sen A 1992 *Phys. Rev. Lett.* **69** 1006
[16] Arnowitt R, Deser S and Misner C W 1962 *in Gravitation: An Introduction to Current Research* ed Witten L (New York: Wiley)
[17] Brown J D and Jr York J W *Phys. Rev.* D **47**, 1420
[18] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)
[19] Gibbons G W and Maeda K 1988 *Nucl. Phys.* B **298** 741