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SYMMETRIES OF MECHANICAL SYSTEMS WITH NONLINEAR NONHOLONOMIC CONSTRAINTS*

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The dynamical symmetries and adjoint symmetries of nonlinear nonholonomic constrained mechanical systems are analysed in two kinds of geometrical frameworks whose evolution equations are Routh's equations and generalized Chaplygin's equations, respectively. The Lagrangian inverse problem and the interrelation between Noether's symmetries and dynamical symmetries are briefly concerned with. Finally an illustrative example is analysed.

Keywords: nonholonomic constraints, dynamical symmetries, adjoint symmetries, Noether symmetries
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I. INTRODUCTION

Classically, two kinds of mathematical models exist for nonholonomic constrained Lagrangian systems. One is based on the d'Alembert–Lagrange principle.[1−8] The other is dependent on a certain natural generalization of Hamilton's principle of stationary action, called vakonomic dynamics.[9,10] The former case is accepted in physics and the dynamical equations of the systems can be represented by generalized Chaplygin's equations or Routh's equations, according to the different methods of imposing constraint functions into the systems. So the symmetries of systems can be analysed with respect to this kind of differential equation.

Recently, Sarlet, Contrijn and Saunder[6,7] presented a geometrical framework for a mixed system of coupled first- and second-order equations treated on the same footing. The dynamical symmetries and adjoint symmetries are discussed as an application of the calculus along a specific projection. In fact, this work is an approach to differential geometry of nonholonomic Lagrangian systems in a broader perspective, which corresponds to generalized Chaplygin's equations. Alternatively, the symmetries of systems can also be viewed in another geometrical framework related to Routh's equations with Lagrangian multipliers, which is considered as a standard method for incorporating constraint functions into equations of motion. In this paper, we do not involve all aspects on symmetries as usual, e.g., dynamical covariant derivative and Jacobi endomorphism, etc. But we only pay attention to some aspects easily confused in the literature. In section II, we analyse the dynamical symmetries and adjoint symmetries in the framework of decomposition of two-jet bundles into an affine bundle and a vector bundle. Then the symmetries must satisfy particular conditions. In section III, a constraint sub-manifold is constructed and the invariant one-forms of dynamical vector field are proved to be taken as adjoint symmetries of the systems. The relations between Noether's symmetries and dynamical symmetries of the systems are given in section IV. Finally, we give an illustrative example to verify most of the results given in this paper.

Throughout the paper, Einstein's summation convention is used. The Latin indices i, j, s run from 1 to n and the Greek indices μ, ν, σ run from 1 to $\varepsilon = n - g$; α, β run from 1 to g.

II. SYMMETRIES OF ROUTH'S EQUATIONS

The framework for studying time-dependent mechanical systems is a configuration bundle $\pi : M \to R$ with an n-dimensional typical fibre Q, the configuration space, and the real line R as the base manifold.

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M is a manifold of dimension $n+1$ with local coordinates $\{t, q^s\}(s=1,2,\cdots,n)$. Let $J_1M \subset TM$ and VM denote the one-jet bundle and the vertical subspace of TM with respect to π, respectively. Suppose that the mechanical system is subject to admissible nonholonomic constraints of the type

$$F_\alpha(t, q^i, \dot{q}^i) = 0. \tag{1}$$

The dynamical vector field $\vec{Z}$ on J_1M:

$$\vec{Z} = \partial/\partial t + \dot{q}^i \partial/\partial q^i + f^i\left(t, q^j, \dot{q}^j\right)\partial/\partial \dot{q}^i \tag{2}$$

is compatible with the above constraints, i.e., $\vec{Z}(F_\alpha) = 0$. Then the two-jet bundle $J_2M \to J_1M$ can be decomposed into an affine bundle $j_2M \to J_1M$ of dynamical vector fields $\vec{Z}$ and a vector bundle $R \to J_1M$ [8], i.e.,

$$J_2M = j_2M \oplus_{J_1M} R. \tag{3}$$

If the constraints are ideal, the sections $r: J_1M \to R$ are constraint reaction forces. Furthermore, if the map $g: J_1M \to V^*M \otimes V^*M$ is restricted to be positive-definite, the dynamical vector field $\vec{Z}$ is uniquely determined. Such a dynamical vector field can be taken as a representation of Routh's equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = \lambda^\alpha \frac{\partial F_\alpha}{\partial \dot{q}^s} \tag{4}$$

for the dynamical function L of the system.

Similar to the case of unconstrained systems, the dynamical symmetry $X \in \chi(J_1M)$ of the constrained system can also be defined by $\left[\vec{Z}, X\right] = \lambda \vec{Z}$. However, $\vec{Z}$ is compatible with the admissible, ideal constraints, i.e., the constraints confine the motion to an affine subbundle $j_2M \to J_1M$ of the two-jet bundle $J_2M \to J_1M$. It is easy to verify that the compatibility of $\vec{Z}$ with constraints makes the dynamical symmetry X satisfy $\vec{Z}(X(F_\alpha)) = 0$.

Since the section $\vec{Z}$:$J_1M \to j_2M$ provides Routh's equations in an invariant form, we can assume that the dynamical symmetry X belongs to the sections of the bundle $j_2M \to J_1M$. Therefore, we have

$$i_X S(\mathrm{d}F_\alpha) = 0, \tag{5}$$

where $S = \partial/\partial \dot{q}^i \otimes \theta^i$ is the vertical endomorphism. Considering the basis $\left\{\vec{Z}, \partial/\partial q^i, \partial/\partial \dot{q}^i\right\}$ of vector fields on J_1M and its dual basic one-forms $\left\{\mathrm{d}t, \theta^i = \mathrm{d}q^i - \dot{q}^i \mathrm{d}t, \omega^i = \mathrm{d}\dot{q}^i - f^i \mathrm{d}t\right\}$,

$$X = \tau \vec{Z} + \xi^i \partial/\partial q^i + \eta^i \partial/\partial \dot{q}^i. \tag{6}$$

In local coordinates, we then have $\left(\partial F_\alpha/\partial \dot{q}^i\right)\xi^i = 0$.

Considering the positive-definite fibre metric $g: J_1M \to V^*M \otimes V^*M$ and the isomorphism of $VJ_1M \to J_1M$ and $VM \to M$, a vector can be defined by

$$Y_\alpha = g(S(\mathrm{d}F_\alpha)) = g^{ij}\frac{\partial F_\alpha}{\partial \dot{q}^i}\partial/\partial \dot{q}^j. \tag{7}$$

The fundamental two-form Ω on J_1M is

$$\Omega = g_{ij}\left(\omega^i + \Gamma^i_k \theta^k\right) \wedge \theta^j, \tag{8}$$

where $\Gamma^i_k = -\frac{1}{2}\left(\partial f^i/\partial \dot{q}^k\right)$ are determined by the dynamical vector field $\vec{Z}$.

Proposition 1 Suppose Ω is of maximal rank, then associated with a vector field X satisfying $i_X S(\mathrm{d}F_\alpha) = 0$, there corresponds to a one-form β annihilated by the dynamical vector field $\vec{Z}$, which satisfies $i_{Y_\alpha}\beta = 0$.

Proof. Since Ω is of maximal rank, there exists an equivalent class $[\beta]$ of one-forms on J_1M with the property of $i_{\mathbf{Z}}\beta = 0$, defined by $i_X\Omega = \beta$. Suppose $\beta = \sigma_i\theta^i + \pi_i\omega^i$, then the above relation leads to

$$\pi_i = -g_{ij}\xi^j, \quad \sigma_j = g_{ij}\left(\eta^i + \Gamma^i_k\xi^k\right) - g_{ik}\Gamma^i_j\xi^k. \tag{9}$$

It is straightforward to verify that

$$i_{Y_\alpha}\beta = -\frac{\partial F_\alpha}{\partial \dot{q}^i}\xi^i = -i_X S(\mathrm{d}F_\alpha) = 0. \tag{10}$$

The relations $i_X S(\mathrm{d}F_\alpha) = 0$ and $i_{Y_\alpha}\beta = 0$ are dual to each other. This duality is the result of the above decomposition of J_2M into a direct sum. β is an adjoint symmetry[11,12] if $\mathcal{L}_{\vec{Z}}\beta = 0$. It is evident that if $\beta = \mathrm{d}F$, F is a first integral of $\vec{Z}$ and vice versa.

III. ADJOINT SYMMETRIES AS AN INVARIANT ONE-FORM OF DYNAMICAL VECTOR FIELD

We solve Eq.(1) to obtain g velocity coordinates (denoted by $\dot{q}^{\varepsilon+\beta}$) in terms of the other velocity coordinates $\varepsilon = n - g$ (denoted by $\dot{q}^\sigma$), i.e.,

$$\dot{q}^{\varepsilon+\beta} = \varphi^\beta\left(t, q^s, \dot{q}^\sigma\right). \tag{11}$$

This construction provides a canonical projection τ: $M \to M_0$, where $M_0 \subset M$ is a sub-manifold of dimension $\varepsilon+1$ with local coordinates $\{t, q^\sigma\}$.[6] Let $V_\tau M$ denote the vertical subspace of TM with respect to the projection $\tau: M \to M_0$ and J_1M_0 a one-jet manifold with respect to $\pi_0: M_0 \to R$.

$\tau^*(J_1M_0) = M \times_{M_0} J_1M_0$ is the pull-back bundle over M.

Let $i : \tau^*(J_1M_0) \to J_1M$ be an embedding defined by the above constraints, then $N = i(\tau^*(J_1M_0))$ is called constraint sub-manifold of J_1M with local coordinates $\{t, q^\sigma, \dot{q}^\sigma, q^{\varepsilon+\beta}\}$. The dynamical vector field of the mechanical system on N can be represented by

$$\hat{\vec{Z}} = \partial/\partial t + \dot{q}^\sigma \partial/\partial q^\sigma + \varphi^\beta(t, q^s, \dot{q}^\mu)\partial/\partial q^{\varepsilon+\beta} + f^\sigma(t, q^s, \dot{q}^\mu)\partial/\partial \dot{q}^\sigma. \tag{12}$$

Dual to this vector field are two one-forms on the manifold N: $\theta^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \mathrm{d}t$ and $\omega^\sigma = \mathrm{d}\dot{q}^\sigma - f^\sigma \mathrm{d}t$ which will be taken as basic one-forms on N.

If $V_\rho N$ denotes the vertical subspace of TN with respect to the projection $\rho : N \to J_1M_0$ and $\rho^*(T(J_1M_0)) = N \times_{J_1M_0} T(J_1M_0)$ the pull-back bundle over N, then the following exact sequence

$$0 \to V_\rho N \to TN \to \rho^*(T(J_1M_0)) \to 0 \tag{13}$$

determines a horizontal distribution on the bundle $\rho : N \to J_1M_0$, which can be spanned, in local coordinates, by

$$\begin{aligned} H_0 &= \partial/\partial t + \Gamma_0^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \\ H_\sigma &= \partial/\partial q^\sigma + \Gamma_\sigma^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \\ H_{\dot{\sigma}} &= \partial/\partial \dot{q}^\sigma + \Gamma_{\dot{\sigma}}^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \end{aligned} \tag{14}$$

where the connection coefficients are determined by

$$\begin{aligned} \Gamma_0^{\varepsilon+\beta} &= \varphi^\beta - \dot{q}^\sigma \partial\varphi^\beta/\partial\dot{q}^\sigma, \\ \Gamma_\sigma^{\varepsilon+\beta} &= \partial\varphi^\beta/\partial\dot{q}^\sigma, \quad \Gamma_{\dot{\sigma}}^{\varepsilon+\beta} = 0. \end{aligned} \tag{15}$$

A natural tetrad basis of vector fields on the bundle $\rho : N \to J_1M_0$ and its dual basic one-forms can be taken as $\left\{\hat{\vec{Z}},\ H_\sigma,\ \partial/\partial\dot{q}^\sigma,\ \partial/\partial q^{\varepsilon+\beta}\right\}$ and $\{\mathrm{d}t,\ \theta^\sigma,\ \omega^\sigma, \xi^{\varepsilon+\beta}\}$, respectively, where $\xi^{\varepsilon+\beta} = \mathrm{d}q^{\varepsilon+\beta} - \Gamma_\sigma^{\varepsilon+\beta}\mathrm{d}q^\sigma - \Gamma_0^{\varepsilon+\beta}\mathrm{d}t$ correspond to the quasi-coordinates or quasi-velocities in classical nonholonomic mechanics, which will be exact one-forms if the constraints are integrable in the Frobenius sense. With the help of this kind of basic vector field and basic one-form on N, the horizontal and vertical projections with respect to ρ can be obtained

$$\begin{aligned} P_H &= \hat{\vec{Z}} \otimes \mathrm{d}t + H_\sigma \otimes \theta^\sigma + \partial/\partial\dot{q}^\sigma \otimes \omega^\sigma, \\ P_V &= \partial/\partial q^{\varepsilon+\beta} \otimes \xi^{\varepsilon+\beta}. \end{aligned} \tag{16}$$

The vertical projection $P_V : TN \to V_\rho N$ can be taken as a connection one-form on the fibre bundle $\rho : N \to J_1M_0$. Its exterior covariant differential, i.e., the curvature two-form $R_\rho = \nabla P_V = -\mathrm{d}P_V \cdot P_H$ represents the non-involution of horizontal distributions on the bundle ρ which also determines the non-integrability of differential constraints in the Frobenius sense. By means of the above tetrad basis and its dual basic one-forms, the curvature two-form R_ρ can be represented by

$$\begin{aligned} R_\rho = \Big\{ & T_\sigma^{\varepsilon+\beta}\mathrm{d}t \wedge \theta^\sigma + \frac{1}{2}\big[H_\sigma\left(\Gamma_\mu^{\varepsilon+\beta}\right) \\ & - H_\mu\left(\Gamma_\sigma^{\varepsilon+\beta}\right)\big]\theta^\sigma \wedge \theta^\mu \\ & - \frac{\partial\Gamma_\sigma^{\varepsilon+\beta}}{\partial\dot{q}^\mu}\theta^\sigma \wedge \omega^\mu \Big\} \otimes \partial/\partial q^{\varepsilon+\beta}, \end{aligned} \tag{17}$$

where $T_\sigma^{\varepsilon+\beta} = \hat{\vec{Z}}\left(\Gamma_\sigma^{\varepsilon+\beta}\right) - H_\sigma\left(\varphi^\beta\right)$. Because of the duality of the tetrad and the basic one-forms, we have

$$R\left(\hat{\vec{Z}}, \xi^{\varepsilon+\beta}\right) = T_\sigma^{\varepsilon+\beta}\theta^\sigma, \tag{18}$$

which could serve as a geometric interpretation of mathematically unclear commutation relations concerning the total derivative operator d and the variational operator δ in classical nonholonomic mechanics:[13,14]

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}, \delta\right] q^{\varepsilon+\beta} = T_\sigma^{\varepsilon+\beta}\delta q^\sigma. \tag{19}$$

It is evident that if $T_\sigma^{\varepsilon+\beta} = 0$, the nonlinear constraints reduce to linear integrable constraints, and vice versa.

A dynamical symmetry of the mechanical system is a one-parameter invariance group on the manifold N which permutes the integral curves of dynamical vector field $\hat{\vec{Z}}$. It is generated locally by a vector field X with the property $\left[\hat{\vec{Z}}, X\right] = h\hat{\vec{Z}}$. For the constrained mechanical systems adjoint symmetries can be regarded essentially as invariant one-forms $\hat{\beta} \in \wedge^1(N)$, satisfying $i_{\hat{\vec{Z}}}\hat{\beta} = 0$ and $\mathcal{L}_{\hat{\vec{Z}}}\hat{\beta} = 0$.

If an adjoint symmetry $\hat{\beta}$ of $\hat{\vec{Z}}$ is exact, i.e., $\hat{\beta} = \mathrm{d}F$ for some function F, then F is a first integral of $\hat{\vec{Z}}$. Conversely, $\hat{\beta} = \mathrm{d}F$ defines a $\hat{\vec{Z}}$-basic one-form if F is a first integral. If the adjoint symmetries of $\hat{\vec{Z}}$ take the form

$$\hat{\beta} = \mathrm{d}F - \hat{S}\left(\mathrm{d}\hat{\vec{Z}}(F)\right) - \hat{\vec{Z}}(F)\,\mathrm{d}t \tag{20}$$

for $F \in C^\infty(N)$, then $\mathcal{L}_{\hat{\vec{Z}}}\hat{\beta} = 0$ leads to

$$\hat{\vec{Z}}\left(\frac{\partial\hat{\vec{Z}}(F)}{\partial\dot{q}^\sigma}\right) - \frac{\partial\hat{\vec{Z}}(F)}{\partial q^\sigma} = 0, \quad \frac{\partial\hat{\vec{Z}}(F)}{\partial q^{\varepsilon+\beta}} = 0. \tag{21}$$

Obviously, $L = \hat{\vec{Z}}(F)$, if regular, can be taken as a new Lagrangian for the systems. On the contrary, if $L = \hat{\vec{Z}}(F)$ is a Lagrangian of $\hat{\vec{Z}}$ for some function $F \in C^\infty(N)$, then it is easy to verify that $\hat{\beta} = \mathrm{d}F - \hat{S}\left(\mathrm{d}\hat{\vec{Z}}(F)\right) - \hat{\vec{Z}}(F)\,\mathrm{d}t$ is an adjoint symmetry of $\hat{\vec{Z}}$. In this case there exists a pre-symplectic submanifold $R \times TM_0$ of constraint manifold N, whose pre-symplectic structure is determined by the dynamical function L. The fundamental two-form Ω on $R \times TM_0$ is locally exact, i.e., $\Omega = -\mathrm{d}\theta_L$ where $\theta_L = \hat{S}\left(\mathrm{d}\hat{\vec{Z}}(F)\right) + \hat{\vec{Z}}(F)\,\mathrm{d}t$.

IV. NOETHER'S SYMMETRIES AND DYNAMICAL SYMMETRIES

Noether's symmetries are essentially different from the dynamical symmetries. The former is an invariant variation problem of the dynamical systems and the latter is a kind of Lie theory of differential equations of the systems. However, they may be consistent with each other for some Lagrangian systems. Because of this, many researchers have confused the concepts, when concerned with the first integrals.

Now we discuss Noether's symmetries of the constrained mechanical systems. For this we need a fundamental two-form $\hat{\Omega}$ on the manifold N, which is not closed in general. In the same way as in Ref.[16] we can construct it from a Lagrangian L and constraints by $\hat{\Omega} = \mathrm{d}\theta_{L'} + \Psi \wedge \mathrm{d}t$ such that $i_{\hat{\vec{Z}}}\hat{\Omega} = 0$, where $L' = i^*L$ $(i : N \to J_1M)$ is a Lagrangian embedded constraint and

$$\begin{aligned} &\theta_{L'} = S\left(\mathrm{d}L'\right) + L'\mathrm{d}t, \\ &\Psi = -i^*\left(\dot{R}_\rho\left(\hat{\vec{Z}}, \mathrm{d}L\right)\right) - P_V\left(\mathrm{d}L'\right). \end{aligned} \tag{22}$$

Here $\dot{R}_\rho$ is the vertical lift of R_ρ to TN. Usually, the geometrical properties of Lagrangian systems are described by the condition $\mathrm{d}\hat{\Omega} = 0$ or $\mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega} = 0$.

Let $X_\eta = \lambda^\sigma H_\sigma + \gamma^\beta \partial/\partial q^{\varepsilon+\beta}$ be a vector field along $\eta : N \to M$. If its prolongation $X_\eta^{(1)}$ satisfies

$$\mathcal{L}_{X_\eta^{(1)}}\theta_{L'} = -i_{X_\eta^{(1)}}\left(\Psi \wedge \mathrm{d}t\right) + \mathrm{d}f, \text{for } f \in C^\infty(N), \tag{23}$$

then $X_\eta^{(1)} = \lambda^\sigma H_\sigma + \gamma^\beta H_{\varepsilon+\beta} + \hat{\vec{Z}}\left(\lambda^\sigma\right)\partial/\partial\dot{q}^\sigma$ is called a Noether's symmetry which yields a conserved quantity $F = f - i_{X_\eta^{(1)}}\theta_{L'}$.

Proposition 2 A Noether's symmetry $X_\eta^{(1)}$ of a constrained mechanical system is a dynamical symmetry of the system if and only if the constrained systems are locally Lagrangian. Conversely, a dynamical symmetry is also a Noether's symmetry only if the system is locally Lagrangian and the one-form $i_{X_\eta^{(1)}}\hat{\Omega}$ is exact.

Proof. By definition, the Noether's symmetry $X_\eta^{(1)}$ satisfies $i_{X_\eta^{(1)}}\hat{\Omega} = \mathrm{d}F$, so $\mathcal{L}_{\hat{\vec{Z}}}\, i_{X_\eta^{(1)}}\hat{\Omega} = 0$. Thus

$$i_{[\hat{\vec{Z}},\ X_\eta^{(1)}]}\hat{\Omega} = -i_{X_\eta^{(1)}}\ \mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega}. \tag{24}$$

Considering $Ker\hat{\Omega} = \mathrm{Span}\left\{\kappa\hat{\vec{Z}} | \kappa \in C^\infty(N)\right\}$, we obtain that $X_\eta^{(1)}$ is a dynamical symmetry if and only if $i_{X_\eta^{(1)}}\ \mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega} = 0$. As the situation of mechanical systems without constraints, this condition is equivalent to $\mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega} = 0$.

On the contrary, for the dynamical symmetry $X_\eta^{(1)}$, $\mathcal{L}_{\hat{\vec{Z}}}\, i_{X_\eta^{(1)}}\hat{\Omega} = i_{X_\eta^{(1)}}\ \mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega}$. If $X_\eta^{(1)}$ is a Noether's symmetry, i.e., $i_{X_\eta^{(1)}}\hat{\Omega} = \mathrm{d}F$, then $i_{X_\eta^{(1)}}\ \mathcal{L}_{\hat{\vec{Z}}}\hat{\Omega} = 0$. So the system is Lagrangian. Obviously, if $i_{X_\eta^{(1)}}\hat{\Omega}$ is exact for the Lagrangian system, the dynamical symmetry is Noether's symmetry.

V. AN ILLUSTRATIVE EXAMPLE

Consider a unit mass moving in a uniform gravitational field g. Let the local coordinates of the configuration manifold M be $x = q^1$, $y = q^2$, $z = q^3$. The dynamical function is $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - gz$. Suppose the particle is subject to a nonholonomic constraint

$$\dot{z}^2 - a^2\left(\dot{x}^2 + \dot{y}^2\right) = 0.$$

The fibre metric on the bundle $TM \to M$ takes $g_{ij} = g^{ij} = \delta_{ij}$. The vectors $\vec{Y}$ and $\vec{Z}$ are

$$\begin{aligned} \vec{Y} =& 2\left[\dot{z}\partial/\partial\dot{z} - a\left(\dot{x}\partial/\partial\dot{x} + \dot{y}\partial/\partial\dot{y}\right)\right], \\ \vec{Z} =& \dot{x}\partial/\partial x + \dot{y}\partial/\partial y + \dot{z}\partial/\partial z \\ &- \frac{ag\dot{x}}{\left(1+a^2\right)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{x} \\ &+ \frac{ag\dot{y}}{\left(1+a^2\right)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{y} - g\left(\frac{a}{1+a^2} - 1\right)\partial/\partial\dot{z}. \end{aligned}$$

Obviously, the one-forms

$$\begin{aligned} &\beta = -\frac{\dot{y}}{\dot{x}^2}\mathrm{d}\dot{x} + \frac{1}{\dot{x}}\mathrm{d}\dot{y}, \\ &\beta' = \dot{x}\mathrm{d}x + \dot{y}\mathrm{d}y + \dot{z}\mathrm{d}z + g\mathrm{d}z \end{aligned}$$

are adjoint symmetries of $\vec{Z}$ by definition and satisfy $i_Y\bar{\beta} = i_Y\bar{\beta}' = 0$. Furthermore, the exact one-forms β and β' provide the following first integrals $\bar{F} = \dot{y}/\dot{x}$, $\bar{F}' = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + gz$.

Alternatively, the coordinates of constraint manifold N take $\{x, y, \dot{x}, \dot{y}\}$ and the dynamical function

embedded the constraint is $L' = (1+a^2)(\dot{x}^2+\dot{y}^2)/2 - gz$. Obviously, this system is not a Chaplygin's one. We can find out the dynamical vector field

$$\begin{aligned}\hat{\vec{Z}} =& \dot{x}\partial/\partial x + \dot{y}\partial/\partial y + a\sqrt{\dot{x}^2+\dot{y}^2}\partial/\partial z \\ &+ \frac{ag\dot{x}}{(1+a^2)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{x} \\ &+ \frac{ag\dot{y}}{(1+a^2)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{y}.\end{aligned}$$

At least two exact adjoint symmetries exist

$$\hat{\beta} = \mathrm{d}\left(\dot{y}/\dot{x}\right), \quad \hat{\beta}' = d\left[\frac{1}{2}\left(1+a^2\right)\left(\dot{x}^2+\dot{y}^2\right)+gz\right].$$

So there exist two first integrals $F = \dot{y}/\dot{x}$, $F' = \frac{1}{2}\left(1+a^2\right)\left(\dot{x}^2+\dot{y}^2\right)+gz$. We see that the two frameworks are equivalent for handling the symmetries of constrained systems.

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References

[1] Mei F X 1987 *The Study of Nonholonomic Dynamics* (Beijing: Beijing Institute of Technology Press) p50 (in Chinese)
[2] Mei F X 2000 *Acta Phys. Sin.* **49** 1207 (in Chinese)
[3] Zhang R C 2000 *Chin. Phys.* **9** 561
[4] Guo Y X, Yu Y and Huang H J 2001 *Chin. Phys.* **10** 1
[5] Morando P and Vignolo S 1998 *J. Phys. A: Math. Gen.* **31** 8233
[6] Sarlet W, Cantrijn F and Saunders D J 1995 *J. Phys. A: Math. Gen.* **28** 3253
[7] Saunders D J, Sarlet W and Cantrijn F 1996 *J. Phys. A: Math. Gen.* **29** 4265
[8] Giachetta G 1992 *J. Math. Phys.* **33** 1652
[9] Arnold V I, Kozlov V V and Nejshtadt A T 1988 *Dynamical Systems III* (*Encyclopedia of Mathematical Sciences)* (Berlin: Springer-Verlag) p31
[10] Cardin F and Favretti M 1996 *J. Geom. Phys.* **18** 295
[11] Sarlet W, Prince G and Crampin M 1990 *J. Phys. A: Math. Gen.* **23** 1335
[12] Sarlet W, Vandecasteele A and Cantrijn F 1995 *Diff. Geom. Appl.* **5** 171
[13] Guo Y X and Mei F X 1998 *Acta Mech. Sin.***14** 85
[14] Guo Y X, Wang J H and Mei F X 1998 In: *Proc. 3rd Int. Conf. Nonlinear Mech.* Editor-in-Chief: Chien Wei-zang (Shanghai: Shanghai University Press) 665
[15] Sarlet W and Cantrijn F 1981 *SIAM Rev.* **23** 467