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FRACTAL VISCOUS FINGERING AND ITS SCALING STRUCTURE IN RANDOM SIERPINSKI CARPET*

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Viscous fingering (VF) in the random Sierpinski carpet is investigated by means of the successive over-relaxation technique and under the assumption that bond radii are of Rayleigh distribution. In the random Sierpinski network, the VF pattern of porous media in the limit $M \to \infty$ (M is the viscosity ratio and equals η_2/η_1 where η_1 and η_2 are the viscosities of the injected and displaced fluids, respectively) is found to be similar to the diffusion-limited aggregation (DLA) pattern. The interior of the cluster of the displacing fluid is compact on long length scales when M=1, and the pores in the interior of the cluster have been completely swept by the displacing fluid. For finite values of M, such as $M \geq 10$, the pores in the interior of the cluster have been only partly swept by the displacing fluid on short length scales. But for values of M in $1 < M \leq 5$, the pores in the interior of the cluster have been completely swept by the displacing fluid on short length scales. The symmetry of the growth of VF is broken by randomizing the positions of the holes. The fractal dimension for VF in fractal space is calculated. However, the sweep efficiency of the displacement processes mainly depends upon the length of the network system and also on the viscosity ratio M. The fractal dimension D can be reasonably regarded as a useful parameter to evaluate the sweep efficiencies. The topology and geometry of the porous media have a strong effect on the structure of VF and the displacement process. The distribution of velocities normal to the interface has been studied by means of multifractal theory. Results show that the distribution is consistent with the hypothesis that, for a system of size L, $L^{f(\alpha)}$ sites have velocities scaling as $L^{-\alpha}$; and the scaling function $f(\alpha)$ is measured and its variation with M is found.

Keywords: viscous fingering, fractal construction, Sierpinski carpet
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I. INTRODUCTION

Viscous fingering (VF) in porous media has been extensively studied in the past. Recently, it has become a field of very active research. The phenomenon is also of practical importance in the recovery of oil. Displacement of a viscous oil by a less viscous solvent is inherently unstable. Even when the porous media is homogeneous, VF will form and reduce sweep efficiency, leading to an early breakthrough of the injected fluid. There are several experimental and theoretical methods to characterize the phenomenon. The first theoretical analysis by Saffmann and Taylor was of the related problem of Hele–Shaw flow. The first analogy has been made between the diffusion-limited aggregation (DLA) model of Witten and Sander[1] and VF with a zero viscosity displacing fluid.[2] Since then many attempts have been made to modify DLA to include surface tension effects.[3] However, fewer attempts have been made to allow for a finite viscosity ratio,[4] and shell-by-shell[5,6] model, and other stochastic models have been used.[7] Oxaal *et al.*[8] studied VF in square lattice models with two types of bonds by using the percolation model. In the deterministic model, many authors have used the successive over-relaxation technique.[9–12] The geometric model used by these authors is either a square, hexagonal or three-dimensional network. Reservoir rocks are rarely homogeneous, however. Several studies have shown that the pore space of many sedimentary rocks has a fractal structure, at least on a certain length scale.[13] Hence, studies on VF are very important in a fractal network. Our previous work on the deterministic model, includes VF in the Sierpinski carpet, the random Sierpinski carpet without overlap,[14] and the percolation cluster.[15]

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In this paper, we first present a new method, the hole-position randomizing method (HPRM), to build the random Sierpinski carpet. Then we will focus on VF in the random Sierpinski carpet by means of successive over-relaxation techniques and based on the assumption that bond radii are non-uniformly distributed (Rayleigh distribution). We discuss how the topology and the geometry of the porous media affect the displacement processes. Also, the fractal dimension D for VF in the random Sierpinski carpet will be calculated. The sweep efficiency E of the displacement processes as a function of network L and the viscosity ratio M will be given. Finally, we will study the distribution of velocities normal to the VF interface in the random Sierpinski carpet.

II. GENERATION OF RANDOM SIERPINSKI CARPET

The random Sierpinski carpet can be generated by HPRM. For a stage-n carpet, it has a $4^n \times 4^n$ site. In a stage-k carpet, the site $(I, J) \in [1, 4^n - 4^{n-k} + 1]$ is randomly chosen. Then the sites $(I, J), (I + 1, J), \cdots, (I + 4^{n-k} - 1, J), (I, J + 1), \cdots (I + 4^{n-k} - 1, J + 4^{n-k} - 1)$ are regarded as holes. This process is repeated $(4^2 - 1)^{k-1}$ times. Hence, we can construct an arbitrary stage of random Sierpinski carpet (e.g., stage-4 carpet with 256×256 lattice in Fig.1, which will be discussed in this paper), of which the fractal dimension is d_s=1.968. It must be pointed out that considering the centre injective, it is necessary to control the positions of the holes without cutting the centre and to guarantee the centre site to be a part of a connective big cluster of random Sierpinski carpets without being cut off totally. Because randomization permits overlapping of holes, the fractal dimension d_s of Fig.1 is bigger than the fractal dimension ($\ln 15/\ln 4$=1.953) of this particular Sierpinski carpet. This random Sierpinski carpet is similar to Mandelbrot's Swiss cheeses[16] and Foster and Jensen's fully penetrable polydisperse spheres (FPPS) model on porous media.[17]

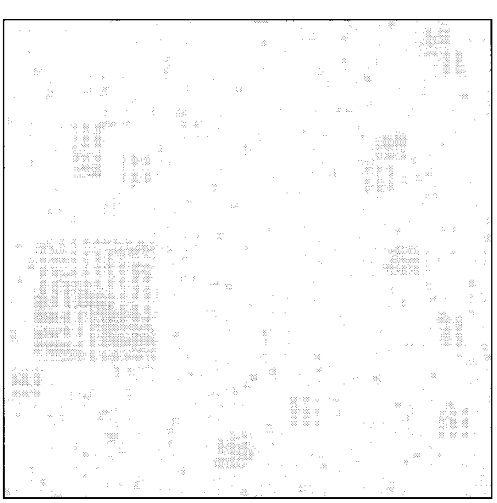


Fig.1. A stage-4 random Sierpinski carpet with a 256×256 lattice.

III. THROAT RADIUS DISTRIBUTION AND SIMULATION METHOD

The simulation of radial displacement used here has been described in great detail.[10] It is similar to the simulation of Chen and Wilkinson,[9] with an extension to a finite viscosity ratio. The random Sierpinski carpet corresponds to porous media, with the sites (nodes) as pores, the bond as throat.[9−11] Our model is different from the previous one.[9−11] The model that we used for a pore medium is on a square network or hexagonal network of cylindrical tubes of length L and radius r. The radii are taken uniformly from the interval $[(1-\lambda), (1+\lambda)]$ where λ is a disorder parameter.

The main characteristic of the model that we consider for pore media is that the random Sierpinski carpet and cylindrical tube radius share non-uniform distribution. The radii r are distributed according to a Rayleigh distribution[15,18]

$$f(r) = 2re^{-r^2}. \tag{1}$$

If i and j are two adjacent pores, the flow rate Q_{ij} between them is given by[9]

$$\begin{aligned} Q_{ij} &= \frac{\pi r_{ij}^4 (p_i - p_j - P_{ij})^+}{8[\eta_1 x_{ij} + \eta_2 (L - x_{ij})]} \\ &= g_{ij}(p_i - p_j - P_{ij})^+, \end{aligned} \tag{2}$$

where p_i and p_j are the pressure at ith and jth nodes, respectively, η_1 and η_2 are the viscosities of the injected and displaced fluid, respectively, the viscosity ratio is $M = \eta_2/\eta_1$, r_{ij} is the radius of the bond ij, and x_{ij} is the length of bond ij occupied by the injected fluid. P_{ij} is the capillary pressure, given by Laplace's law: $P_{ij} = 2\gamma \cos\theta / r_{ij}$, γ and θ are the interfacial tension and the contact angle, respectively. The quantity g_{ij} is the flow conductance of the bond ij, and is a function of time as the interfaces move. In this equation, the + denotes the positive part; this

means that Q_{ij}=0 as long as $(p_i - p_j) < P_{ij}$, otherwise, the expression of Q_{ij} is that for a diphasic Poiseuille flow in a tube with a pressure jump at the interface. At each node we have the conservation equation

$$\sum_j Q_{ij} = 0. \tag{3}$$

The flow rate between two interfacial nodes no longer depends linearly on the pressure difference. To avoid the nonlinear problem, an approximately linear calculation similar to that in Ref.[11] is tried here. That is, within a displacing time step (the step equals the shortest time needed by the displacing front to reach the next node), Eq.(2) is regarded as linear. Then, by means of the successive over-relaxation algorithm (Gauss–Seidel iteration), the approximate calculation of the equation is

$$p_i^{(n+1} = (1-\omega)p_i^{(n)} + \frac{\omega\left(\sum_j g_{ij}p_j^{(n)} + \sum_j g_{ij}p_c\right)}{\sum_j g_{ij}}, \tag{4}$$

where ω is the over-relaxation parameter (approximately 1.66). Accordingly, over-relaxation calculation procedures are abstracted concerning the computer simulation of two-phase flow displacement process in the porous media. (i) Calculate the pressure field by Eq.(4). (ii) Judge the states of the bond. If $p_i - p_j > p_c$, the bond is conducted. Then the flow quantity can be calculated by $Q_{ij} = g_{ij}(p_i - p_j)$. If $p_i - p_j \le p_c$, the bond is blocked, and Q_{ij}=0. (iii) Calculate the time step $\Delta t_{ij} = (L - x_{ij})/(Q_{ij}/\pi R_{ij}^2)$ on the basis of Q_{ij} and limit Δt to guarantee that only one bond is completely occupied by fluid 1. (iv) Move the interface between the two fluids corresponding to Δt and introduce a new interface. (v) Modify x_{ij} and g_{ij} based on the new interface. (vi) Repeat procedures (i)–(v) until fluid 1 reaches the boundary of the lattice. Then ends the simulation.

IV. RESULTS AND DISCUSSION

We first use the deterministic model to simulate the process in the limit $M \to \infty$. The resulting patterns of VF are shown in Fig.2(a). Fig.2(b) is the DLA in the random Sierpinski carpet. The patterns of Fig.2(a) show that the results look much like a typical DLA simulation (see Fig.2(b)) in the Sierpinski carpet when $M \to \infty$ and their fractal dimension D is 1.620 and 1.621, respectively. This result identifies with simulation in uniform lattice.[1,9]

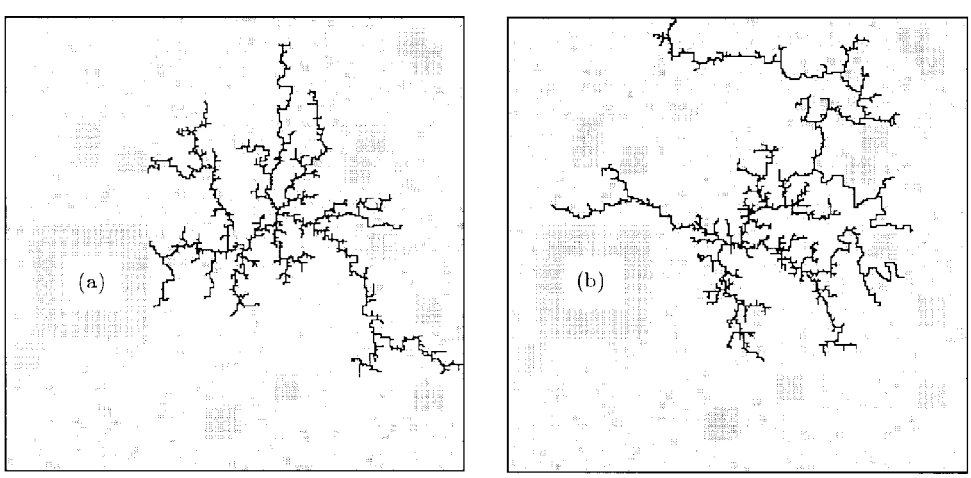


Fig.2. VF and DLA patterns in stage-4 random Sierpinski carpet: (a) VF patterns with uniform radii distribution when $M \to \infty$; (b) DLA.

Figure 3 shows the results of the simulation for the four kinds of viscosity ratio M; Figs.3(a)–(d) correspond to M=1, 5, 10 and 100, respectively. Our simulation shows that the interior of the cluster of the displacing fluid is compact for long length scales when M=1 (see Fig.3(a)). This means that the pores in the interior of the cluster have been completely swept by the displacing fluid, and at a large length scale, one

has a compact cluster with $D = d_s$. That is, the displacement process is stable and the sweep efficiency is 100%. For small values of M, such as $1 < M \leq 5$ (see Fig.3(b)), the interior of the cluster of the displacing fluid is compact for short length scales. This means that the pores in the interior of the cluster have been completely swept by the displacing fluid. But, for finite values of M such as $M \geq 10$ (see Figs.3(c) and (d)), the interior of the cluster of the displacing fluid is not compact for short length scales. This means that the pores in the interior of the cluster have been only partly swept by the displacing fluid. It is obvious that the patterns of VF gradually change from compact to interiorly-compact, then to an open branch structure when M gradually increases from 1 to ∞. The fractal dimension D as a function of viscosity ratio M is shown in Fig.4, which shows that D decreases as M increases. This result is the same as that of non-Newtonian fluid.[19]

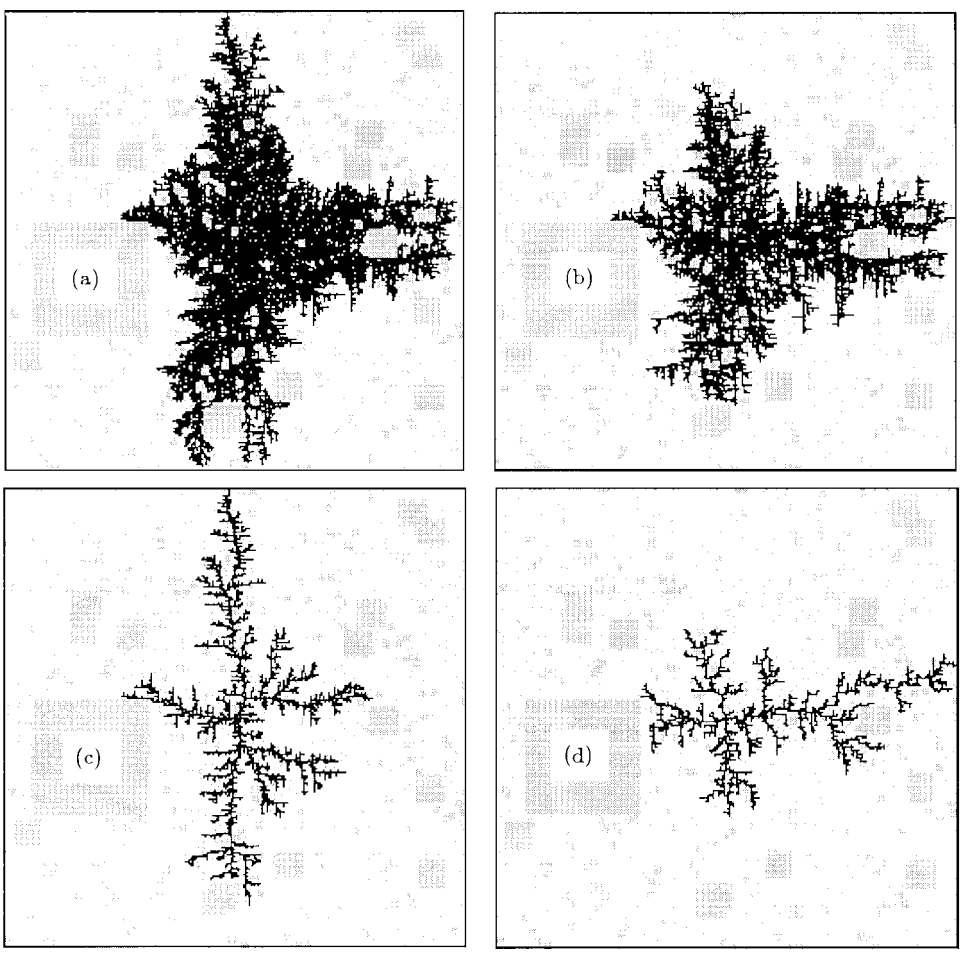


Fig.3. VF patterns with the four kinds of viscosity ratio M in stage-4 random Sierpinski carpet: (a) M=1; (b) M=5; (c) M=10; (d) M=100.

Moreover, whatever the radii distribution function $f(r)$ of the throat and the viscosity ratio M, even if the radii are of uniform distribution and M=1, the pattern of VF has no cross-like structure. The reason is that the symmetry of growth of the VF is broken by randomizing the positions of the holes. The result shows that the topology and the geometry of the porous media have a strong effect on the structure of the VF and the displacement process.

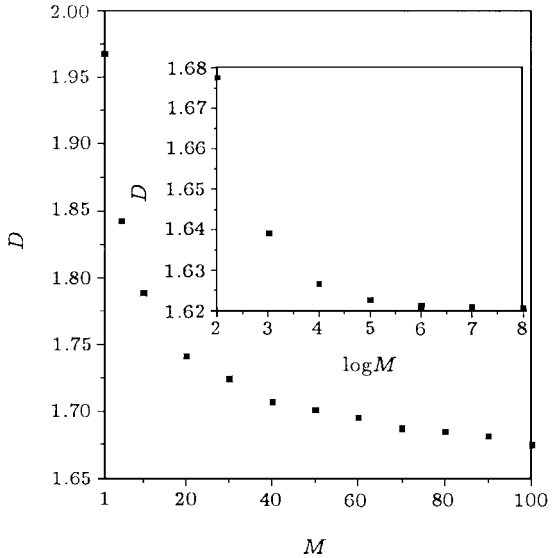


Fig.4. Fractal dimension D as a function of viscosity ratio M.

A particularly useful area is in the prediction of sweep efficiencies E. As can be seen, for a fixed value of M, the results can be fitted to a power law,

$$E \sim \frac{L^D}{L^{d_s}} = L^{D-d_s}, \tag{5}$$

where L is the network size. So, the greater the value of M, the less the fractal dimension D and the lower the sweep efficiency E. Hence, the parameters such as the fractal dimension D, which describes the complexity in a system, can be reasonably regarded as useful parameters to evaluate the sweep efficiency E. Fig.5 shows that E increases along with decreasing viscosity ratio M and network size L. It is apparent that $E \to 0$ as $L \to \infty$ if $M \gg 1$.

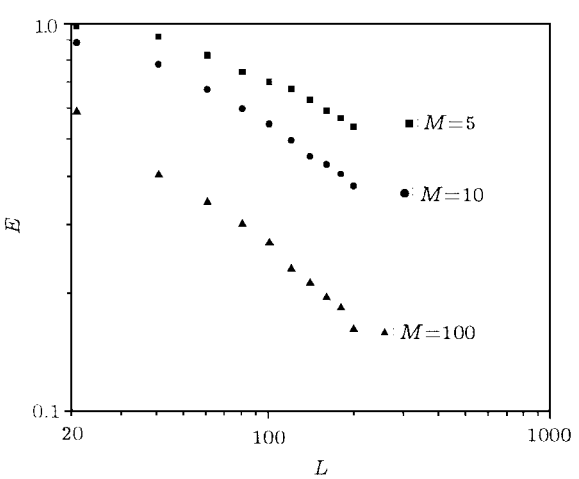


Fig.5. Sweep efficiency E as a function of network size L for various values of viscous ratio M.

Now, we will study the distribution of velocities normal to the interface in a random Sierpinski carpet. The method is described in detail in Refs.[20] and [21]. After each group of sites have been added, all the interfacial velocities are analysed, the root-mean-square cluster radius R is measured, and the total number of filled sites and surface sites is recorded. Surface sites are nodes filled with the displacing fluid, but with one or more nearest neighbours occupied by the displaced fluid. At each stage, we make a histogram of the velocity distribution, sorting the velocities into equally spaced logarithmic bins. The number N in each bin, assuming the scaling hypothesis, will be given by

$$N\mathrm{d}[\ln(V)] = R^{f(\alpha)}\mathrm{d}(-\alpha), \tag{6}$$

where the velocity V is equal to $R^{-\alpha}$ and $\mathrm{d}[\ln(V)]$ is the width of the logarithmic bin. Hence, $\alpha = -\ln(V)/\ln(R)$ and $\mathrm{d}[\ln(V)] = \mathrm{d}(-\alpha)\ln(R)$. This gives us a scaling function

$$f(\alpha) = \frac{\ln[N\ln(R)]}{\ln(R)}. \tag{7}$$

Figure 6 shows the effect of randomness on the velocity distribution in the random Sierpinski carpet. The tail of the distribution (large a) is longer for large M. When M is very small, the distribution is very sharp. And when $M \geq 1000$, the $f(\alpha)$ curve is almost fully a coincidence for all M. This means that the velocity distribution $f(\alpha)$ becomes stable when $M \geq 1000$. Thus, the viscosity ratio has strong effects on the displacement processes and the structure.

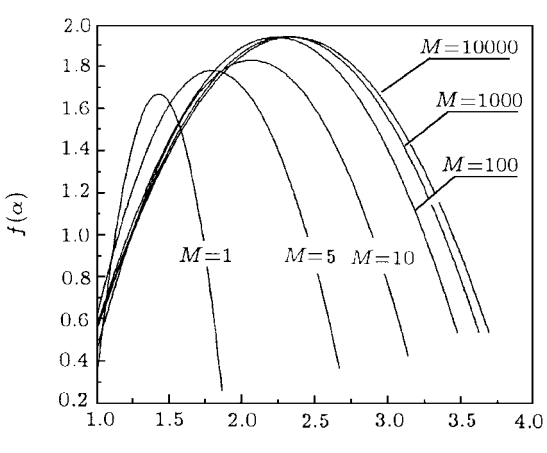


Fig.6. Scaling function $f(\alpha)$ and the effect of viscosity ratio M.

V. CONCLUSIONS

We have constructed the random Sierpinski carpet by HPRM. The VF in a random Sierpinski carpet, based on the assumption that the bond radii are of Rayleigh distribution, has been investigated by means of successive over-relaxation techniques. The interior of the cluster of the displacing fluid is compact for long length scales when M=1, and the pores in the interior of the cluster have been completely swept by the displacing fluid; and, at a large length scale, one has a compact cluster with $D = d_s$. The displacement process is stable and the sweep efficiency is 100. For finite values of M such as $M \approx 10$, the pores in the interior of the cluster have been only partly swept by the displacing fluid for short length scales. But, for smaller values of M such as $1 < M \leq 5$, the pores in the interior of the cluster have been completely swept by the displacing fluid for short length scales. The VF pattern of porous media in the limit $M \to \infty$ is found to be similar to the DLA pattern. The symmetry of growth of the VF is broken by randomizing the positions of the holes. The fractal dimension D and the sweep efficiency E have been calculated for various values of viscous ratio M. The result shows that the fractal dimension D can be reasonably regarded as a useful parameter to evaluate the sweep efficiencies and oil recoveries. It is found that the topology and geometry of the porous media have a strong effect on the structure of the FV and the displacement process. For larger M, the tail of the distribution (large a) is longer. When M is very small, the distribution is very sharp. And when $M \geq 1000$, $f(\alpha)$ curves are almost coincident with one another for all M. The current method can be used to solve VF problems in complex structures if the structures are statistically self-similar, or if they can be reduced to statistically self-similar structure.

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