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EXACT SOLUTION FOR EXTENDED ESSLER–KOREPIN–SCHOUTENS MODEL WITH OPEN BOUNDARY CONDITIONS*

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The eigenvalue of the transfer matrix is obtained for the extended Essler–Korepin–Schoutens model with open boundary conditions. The energy of the system and the Bethe equations of the model are given also.

Keywords: EKS model, open boundary condition, Bethe equations
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I. INTRODUCTION

Strongly correlated electronic systems are currently intensively studied in relation to high-T_c superconductivity. Apart from the two well-studied models, the Hubbard model[1] and the t–J model,[2] there are other solvable models, such as the Essler–Korepin–Schoutens model (EKS model).[3] This model combines and extends some of the interesting features of the Hubbard and t–J models. In a sense, it can be viewed as a modified Hubbard model with additional nearest-neighbour interactions similar to those in the t–J model. On one dimension, this model was solved by the algebraic Bethe ansatz method under periodic boundary conditions.[3]

Recently, much attention has been paid to the study of models with open or impurity boundaries.[4−22] The extended EKS model with an impurity boundary has been solved in Ref.[22]. In Ref.[22], the non-c-number boundary K-matrices are highly non-trivial so that it cannot be factorized into the product of a c-number boundary K-matrix and the corresponding local L-matrices. It may be interesting to solve the model with the c-number boundary K-matrix, from which one can construct the impurity model with the c-number boundary K-matrix dressed by the corresponding local L-matrices.

In this paper, we solve the open extended EKS model with non-trivial c-number boundary K-matrices. The eigenvalue of the transfer matrix for the model is obtained. The energy for the Hamiltonian and the corresponding Bethe ansatz equations are also derived. This paper is organized as follows. In section II, the model and integrable K-matrices are introduced. The final results are presented in section III. In section IV, a summary and some discussions are included.

II. THE OPEN EXTENDED EKS MODEL

The graded non-zero vertex weights $R(u)$ for the extended EKS model are defined as[23]

$$\begin{aligned} R^{aa}_{aa}(u) =& (-1)^{\epsilon_a}\sin(\eta+(-1)^{\epsilon_a}u), \\ R^{ab}_{ab}(u) =& \sin(u), \\ R^{ab}_{ba}(u) =& (-1)^{\epsilon_a\epsilon_b}\sin(\eta)\exp(\mathrm{i}u\mathrm{sign}(a-b)), \\ & 1\le a\ne b\le 4, \end{aligned} \tag{1}$$

where η is an anisotropic parameter, $\epsilon_a = 0$ when $a\in[1,2]$, while $a\in[3,4]$, $\epsilon_a = 1$. The R-matrix defined by Eq.(1) satisfies the graded Yang–Baxter equation

$$\begin{aligned} & R(u-v)^{a_1,a_2}_{b_1,b_2}R(u)^{b_1,a_3}_{c_1,b_3}R(v)^{b_2,b_3}_{c_2,c_3}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} \\ =& R(v)^{a_2,a_3}_{b_2,b_3}R(u)^{a_1,b_3}_{b_1,c_3}R(u-v)^{b_1,b_2}_{c_1,c_2}(-1)^{(\epsilon_{a_1}+\epsilon_{c_1})\epsilon_{b_2}}. \end{aligned} \tag{2}$$

Additionally, the R-matrix also fulfils the following properties (a) unitarity, $R_{12}(u)R_{21}(-u)=\rho(u)$, (b) crossing symmetry, $R^{\mathrm{st}_1}_{12}(u)M_1R^{\mathrm{st}_2}_{12}(-u)M_1^{-1} = \tilde{\rho}(u)$ with $\rho(u)$, $\tilde{\rho}(u)$ being some scalar functions. Here the superscript "st" means the super-transposition, $M=\mathrm{diag}(1,\exp(i2\eta),\exp(i2\eta),1)$.

The transfer matrix of the EKS model with open boundary conditions is constructed as follows:

$$\begin{aligned} t(u) &= \mathrm{str}K^+(u)U(u) \\ &= \sum_a(-1)^{\epsilon_a}K^+_a(u)U_{aa}(u), \end{aligned} \tag{3}$$

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with $U(u) = T(u)K^-(u)T^{-1}(-u)$ and $T(u) = L_N(u)\cdots L_1(u)$. The component expression of $T(u)$ is

$$\{T(u)^{a_1\cdots a_N}_{b_1\cdots b_N}\}_{ab} = R(u)^{aa_N}_{c_N b_N} R(u)^{c_N a_{N-1}}_{c_{N-1} b_{N-1}} \cdots R(u)^{c_2 a_1}_{b b_1} (-1)^{\sum_{j=2}^{N}(\epsilon_{a_j}+\epsilon_{b_j})\sum_{i=1}^{j-1}\epsilon_{a_i}}. \quad (4)$$

The reflecting matrices $K^{\pm}(u)$ must satisfy the following reflection equations[24] respectively:

$$\begin{aligned} &R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) \\ =&K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \end{aligned} \quad (5)$$

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)M_1^{-1}R_{21}(-u-v)K_2^+(v)M_2^{-1} \\ =&K_2^+(v)M_2^{-1}R_{12}(-u-v) \\ &\cdot K_1^+(u)M_1^{-1}R_{21}(-u+v). \end{aligned} \quad (6)$$

Solving Eqs.(5) and (6), we have the following non-trivial diagonal solution:[25]

$$K^-_{aa}(u,\xi) = \begin{cases} \exp(-\mathrm{i}u)\sin(\xi+u) & 1 \le a \le A, \\ \exp(\mathrm{i}u)\sin(\xi-u) & A < a \le 3; \end{cases} \quad (7)$$

$$K^+_{aa}(u,\tilde{\xi}) = \begin{cases} \exp(\mathrm{i}u)\sin(\tilde{\xi}-u)M_{aa} & 1 \le a \le B, \\ \exp(-\mathrm{i}u)\sin(\tilde{\xi}+u)M_{aa} & B < a \le 3, \end{cases} \quad (8)$$

with ξ and $\tilde{\xi}$ both being free parameters.

One can prove that $U(u)$ also satisfies Eq.(5). With the help of Eq.(6) and the properties of the R-matrix, we can show $[t(u), t(v)] = 0$. So the transfer matrix constitutes an one-parameter commutative family which ensures the integrability of the model. The Hamiltonian of the quantum chain with nearest-neighbour interaction and boundary terms can be obtained from the transfer matrix

$$\begin{aligned} H =& \sin(\eta)(t(u)|''_{u=0} - \mathrm{str}K^+(0)|''_{u=0})/(4\mathrm{str}K^+(u)|'_{u=0}) \\ =& \sum_{j=1}^{N-1} H_{j+1,j} + H_a + H_b + \text{constant}. \end{aligned} \quad (9)$$

The detailed calculation gives

$$\begin{aligned} H_{j+1,j} =& \cos(\eta)(c^+_{j,\uparrow}c_{j+1,\uparrow}(1-n_{j+1,\downarrow}-n_{j,\downarrow}) + c^+_{j+1,\uparrow}c_{j,\uparrow}(1-n_{j+1,\downarrow}-n_{j,\downarrow}) \\ &+ c^+_{j,\downarrow}c_{j+1,\downarrow}(1-n_{j+1,\uparrow}-n_{j,\uparrow}) + c^+_{j+1,\downarrow}c_{j,\downarrow}(1-n_{j+1,\uparrow}-n_{j,\uparrow}) + c^+_{j+1,\uparrow}c^+_{j+1,\downarrow}c_{j,\downarrow}c_{j,\uparrow} \\ &+ c_{j+1,\downarrow}c_{j+1,\uparrow}c^+_{j,\uparrow}c^+_{j,\downarrow} - c^+_{j+1,\downarrow}c_{j+1,\uparrow}c^+_{j,\uparrow}c_{j,\downarrow} - c^+_{j+1,\uparrow}c_{j+1,\downarrow}c^+_{j,\downarrow}c_{j,\uparrow}) + \frac{1}{2}(n_{j+1}-1)(n_j-1) \\ &- \frac{1}{2}(n_{j+1,\uparrow}-n_{j+1,\downarrow})(n_{j,\uparrow}-n_{j,\downarrow}) + (n_{j+1,\uparrow}-\frac{1}{2})(n_{j+1,\downarrow}-\frac{1}{2}) + (n_{j,\uparrow}-\frac{1}{2})(n_{j,\downarrow}-\frac{1}{2}) \\ &+ \mathrm{i}\sin(\eta)(2n_{j+1,\uparrow}n_{j,\uparrow}(n_{j+1,\downarrow}-n_{j,\downarrow}) + n_{j+1,\downarrow}n_{j,\uparrow} - n_{j+1,\uparrow}n_{j,\downarrow} \\ &+ n_{j+1,\uparrow} + n_{j+1,\downarrow} - n_{j+1,\uparrow}n_{j+1,\downarrow} - n_{j,\uparrow} - n_{j,\downarrow} + n_{j,\uparrow}n_{j,\downarrow}), \end{aligned} \quad (10)$$

where $c^+_{j,\sigma}(c_{j,\sigma})$ are fermonic creation and annihilation operators with spin $\sigma = \uparrow, \downarrow$. $n_{i,\sigma} = c^+_{i,\sigma}c^+_{i,\sigma}, n_i = n_{i,\uparrow} + n_{i,\downarrow}$. H_a and H_b are boundary terms,

$$H_a = \begin{cases} \exp(\mathrm{i}\xi)\sin(\eta)/\sin(\xi)(n_{1,\uparrow}n_{1,\downarrow} - n_{1,\uparrow} - n_{1,\downarrow}), & A = 1, \\ \exp(\mathrm{i}\xi)\sin(\eta)/\sin(\xi)(2n_{1,\uparrow}n_{1,\downarrow} - n_{1,\uparrow} - n_{1,\downarrow}), & A = 2, \\ \exp(\mathrm{i}\xi)\sin(\eta)/\sin(\xi)(n_{1,\uparrow}n_{1,\downarrow} - n_{1,\downarrow}), & A = 3; \end{cases} \quad (11)$$

$$H_b = \begin{cases} \exp(\mathrm{i}\eta)\sin(\eta)(n_{N,\uparrow}n_{N,\downarrow} - n_{N,\uparrow} - n_{N,\downarrow}), & B = 1, \\ \exp(\mathrm{i}2\eta)(2n_{N,\uparrow}n_{N,\downarrow} - n_{N,\uparrow} - n_{N,\downarrow})/\cot(\eta), & B = 2, \\ \exp(\mathrm{i}\eta)\sin(\eta)(n_{N,\uparrow}n_{N,\downarrow} - n_{N,\downarrow}). & B = 3; \end{cases} \quad (12)$$

III. EXACT SOLUTION FOR THE MODEL

In order to find the eigenvalue of $t(u)$, we have to construct the eigenstates. As usual, we can apply the creating operators to the reference states and compute the eigenvalue. Here, we omit the tedious calculation and give the final results of the model. (For simplicity, we take $\tilde{\xi} = 0$ in Eq.(8).)

$$\begin{aligned}\Lambda(u) =& S_1(u)\chi_1(u)(\frac{\sin(\eta+u)}{\sin(\eta-u)})^N \prod_{r=\pm}\prod_{j=1}^{P_1}\frac{\sin(u-\eta/2-rw_j^{(1)})}{\sin(u+\eta/2-rw_j^{(1)})}\\ &+\frac{\sin(2u)}{\sin(2u+\eta)}\frac{\gamma_1 S_2(u)\chi_2(u)\sin(u)^{2N}}{(\sin(\eta+u)\sin(\eta-u))^N}\prod_{r=\pm}\prod_{j=1}^{P_1}\frac{\sin(u+3\eta/2-rw_j^{(1)})}{\sin(u+\eta/2-rw_j^{(1)})}\prod_{j=1}^{P_2}\frac{\sin(u-rw_j^{(2)})}{\sin(u+\eta-rw_j^{(2)})}\\ &+\frac{\sin(2u)}{\sin(2u+2\eta)}\frac{\gamma_1\gamma_2 S_3(u)\chi_3(u)\sin(u)^{2N}}{(\sin(\eta+u)\sin(\eta-u))^N}\prod_{r=\pm}\prod_{j=1}^{P_2}\frac{\sin(u-rw_j^{(2)})}{\sin(u+\eta-rw_j^{(2)})}\prod_{j=1}^{P_3}\frac{\sin(u+3\eta/2-rw_j^{(3)})}{\sin(u+\eta/2-rw_j^{(3)})}\\ &+\frac{\sin(2u)}{\sin(2u+\eta)}\frac{\gamma_1\gamma_2\gamma_3 S_4(u)\chi_4(u)\sin(u)^{2N}}{(\sin(\eta+u)\sin(\eta-u))^N}\prod_{r=\pm}\prod_{j=1}^{P_3}\frac{\sin(r(u+\eta/2)-rw_j^{(3)}+\eta)}{\sin(u+\eta/2-rw_j^{(3)})},\end{aligned}\tag{13}$$

where $\Lambda(u)$ is the eigenvalue of $t(u)$,

$$S_k(u)=\begin{cases}-\exp(\mathrm{i}((k-2+B')\eta+(-1)^{\epsilon_k}u))\sin((-1)^{\epsilon_k}B'\eta+u)\dfrac{\sin(2u)}{\sin(2u+k'\eta)} & 1\le k\le B,\\ (-1)^{\epsilon_k}\exp(\mathrm{i}((k-2)\eta-u))\sin(u)\dfrac{\sin(2u)}{\sin(2u+k'\eta)} & B<k,\end{cases}\tag{14}$$

where $k'=\sum_{j=1}^{k}(-1)^{\epsilon_j}$ and $B'=\sum_{j=1}^{B}(-1)^{\epsilon_j}$.

$$\chi_k(u)=\begin{cases}\exp(-\mathrm{i}(u+(k-1)\eta/2))\sin(\xi+u) & 1\le k\le A,\\ \exp(\mathrm{i}(u+(k-1)\eta/2))\sin(\xi-A\eta-u) & A<k;\end{cases}\tag{15}$$

$$\gamma_k=\begin{cases}\exp(-(-1)^{\epsilon_k}\mathrm{i}\eta/2) & 1\le k\le A,\\ \exp(-(-1)^{\epsilon_k}\mathrm{i}3\eta/2) & A<k.\end{cases}\tag{16}$$

The Bethe ansatz equations take the form

$$\begin{aligned}&\prod_{r=\pm1}\prod_{j=1}^{P_{k-1}}\frac{\sin(w_l^{(k)}-rw_j^{(k-1)}-\eta/2)}{\sin(w_l^{(k)}-rw_j^{(k-1)}+\eta/2)}\prod_{j=1}^{P_{k+1}}\frac{\sin(w_l^{(k)}-rw_j^{(k+1)}-\eta/2)}{\sin(w_l^{(k)}-rw_j^{(k+1)}+\eta/2)}\\ &\cdot\prod_{j=1,\neq l}^{P_k}\frac{\sin(w_l^{(k)}-rw_j^{(k)}+\eta)}{\sin(w_l^{(k)}-rw_j^{(k)}-\eta)}=\theta_k(w_l^{(k)})\qquad(1\le k\le 3\quad k\neq 2),\end{aligned}\tag{17}$$

$$\prod_{r=\pm1}\prod_{j=1}^{P_1}\frac{\sin(w_l^{(2)}-rw_j^{(1)}-\eta/2)}{\sin(w_l^{(2)}-rw_j^{(1)}+\eta/2)}\prod_{j=1}^{P_3}\frac{\sin(w_l^{(2)}-rw_j^{(3)}+\eta/2)}{\sin(w_l^{(2)}-rw_j^{(3)}-\eta/2)}=\theta_2(w_l^{(2)}).\tag{18}$$

where $\theta_k(w_l^{(k)})=(-1)^{\epsilon_{k+1}}\phi_k(w_l^{(k)})\exp(-\mathrm{i}\eta)$ for $1\le k\le 2$ and $\theta_3(w_l^{(3)})=-\phi_3^{-1}(w_l^{(3)})\exp(-\mathrm{i}\eta)$ with

$$\phi_k(w_l^{(k)})=\frac{T_k(w_l^{(k)}+k'\eta/2)\chi_k(w_l^{(k)}+k'\eta/2)}{\sin(2w_l^{(k)}+k'\eta+\eta)S_{k+1}(w_l^{(k)}+k'\eta/2)\chi_{k+1}(w_l^{(k)}+k'\eta/2)\gamma_k},\tag{19}$$

$$T_k(u)=\begin{cases}-\exp(\mathrm{i}((k'-1+B')\eta+u))\sin(B'\eta+u)\sin(2u) & 1\le k\le B,\\ \exp(\mathrm{i}((k'-1)\eta-u))\sin(u)\sin(2u) & B<k.\end{cases}\tag{20}$$

In the above representation, $w_j^{(0)}=0\quad(j\in[1,N])$, $P_0=N$, $P_4=0$ are assumed.

The energy spectrum of the one-dimensional (1D) quantum system defined by Eq.(9) can be derived from $\Lambda^{(1)}(u)$. It is

$$\begin{aligned}E=&-\sum_{k=1}^{P_1=L}\frac{\sin(\eta)^2}{\cos(2w_1^{(1)})-\cos(\eta)}-\cot(\eta)/2+N\cos(\eta)/2\\ &+\frac{\exp(-\mathrm{i}\xi)\sin(\eta)}{4\sin(\xi)}-\frac{\exp(\mathrm{i}B'\eta)\sin(\eta)}{4\sin(B'\eta)}.\end{aligned}\tag{21}$$

IV.SUMMARY

We obtain the Bethe ansatz equations and the eigenvalue of the transfer matrix for the extended EKS model with open boundary conditions. The corresponding energy spectrum of the 1D quantum system defined by Eq.(9) is derived. Using the Bethe ansatz equations and energy spectrum, we can study the boundary field contributions to the thermodynamic quantities, such as magnetic susceptibility and specific heat, etc.

In fact, $\tilde{\xi}$ in Eq.(8) can take a non-zero number. Correspondingly, $\tilde{\xi}$ will appear in energy, Eq.(21), and Bethe equations (17) and (18). However, the Hamiltonian (9) is independent of $\tilde{\xi}$. When we solve the final energy by using the Bethe equations, $\tilde{\xi}$ should disappear in the final result. It may be hard work for us to prove this conclusion.

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References

[1] Hubbard J 1963 *Proc. R. Soc.* **276** 238
[2] Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759
[3] Essler F L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960
[4] Sklyanin E K 1988 *J. Phys.* A **21** 2375
[5] foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611
[6] Gonzalez A 1994 *Nucl. Phys.* B **424** [FS], 468
[7] Andrei N, Furuya K and Lowenstein J A 1983 *Rev. Mod. Phys.* **55** 331
[8] Kane C L and Fisher M P A 1992 *Phys. Rev. Lett.* **68** 1220
[9] Affleck I 1990 *Nucl. Phys.* B **336** 517
[10] Andrei N and Johannesson H 1984 *Phys. Lett.* A **100** 108
[11] Bedürfitg G, Eßler F H L and Frahm H 1996 *Phys. Rev. Lett.* **77** 5098
[12] Bedürfitg G, Eßler F H L and Frahm H 1997 *Nucl. Phys.* B **489** 697
[13] Schlottmann P and Zvyagin A A 1997 *Phys. Rev.* B **55** 5027
[14] Foerster A, Links J and Tonel A 1999 *Nucl. Phys.* B **552**[FS] 707
[15] Wang Y P 1997 *Phys. Rev.* B **56** 14045
[16] Wang Y P, Dai J H, Hu Z N and Pu F C 1997 *Phys. Rev. Lett.* **79** 1901
[17] Hu Z N, Pu F C and Wang Y P 1998 *J. Phys.* A **31** 5241
[18] Hu Z N and Pu F C 1998 *Phys. Rev.* B **58** R2959
[19] Links J and Foerster A 1999 *J. Phys.* A **32** 147
[20] Bedürfitg G and Frahm H 1999 *J. Phys.* A **32** 4585
[21] Hou B Y, Shi K J, Rue R H and Zhao S Y 2000 *Chin. Phys.* **9** 203
[22] Zhou H Q, Ge X Y and Gould M D 1999 *J. Phys.* A **32** 5383
[23] Perk J H and Schultz C L 1981 *Phys. Lett.* A **84** 407
[24] Mezincescue L and Nepomechie R I 1991 *J. Phys.* A **24** L19; 1991 *Int. J. Mod. Phys. Lett.* A **6** 2497
[25] Yue R H and Liang H 1996 *High Energy Phys. Nucl. Phys.* **20** 514 (in Chinese)