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BETHE ANSATZ FOR SUPERSYMMETRIC MODEL WITH $U_q[osp(1|2)]$ SYMMETRY*

YANG WEN-LI(杨文力)

Institute of Modern Physics, Northwest University, Xi'an 710069, China

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Using the algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix of the supersymmetric model with $U_q[osp(1|2)]$ symmetry under periodic boundary and twisted boundary conditions.

Keywords: Yang–Baxter equation, supersymmetric models, algebraic Bethe ansatz
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Integrable models with (quantum) superalgebra symmetries have been the focus of recent studies[1−5] in the context of strongly correlated fermion systems. The integrable models have been solved by the Bethe ansatz method only for $gl(m|n)$ $(U_q[gl(m|n)])$,[1−4] $osp(m|n)$[5] and $(U_q[osp(2|2)^{(2)}])$.[6] In this paper, we will study the integrable model with quantum superalgebra $U_q[osp(1|2)]$[7] by the algebraic Bethe ansatz method.

We start by considering the three-dimensional representation V of $U_q[osp(1|2)]$, which has basis $\{v_i\}_{i=1,2,3}$ with the Z_2-grading $[v_1] = [v_3] = 0$ and $[v_2] = 1$. Associated with this representation, there is a solution $R(z) \in End(V \otimes V)$ of the graded Yang–Baxter equation

$$\begin{aligned} &R_{12}(z)R_{13}(zw)R_{23}(w) \\ =&R_{23}(w)R_{13}(zw)R_{12}(z). \end{aligned} \tag{1}$$

The multiplication rule for the tensor product is Z_2-graded, namely

$$(a \otimes b)(a' \otimes b') = (-1)^{[b][a']}(aa' \otimes bb'), \tag{2}$$

for homogeneous elements $a,\ b,\ a',\ b'$ of $U_q[osp(1|2)]$, which extends to inhomogeneous elements through linearity.

It can be shown that the $R(z/w)$ with $U_q[osp(1|2)]$ symmetry has the following form:[7]

$$R(z/w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & c & 0 & r & 0 & 0 \\ 0 & f & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & b & 0 \\ 0 & 0 & s & 0 & g & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} a(z) &= \frac{q(z-1)}{zq^2-1}, \\ b(z) &= \frac{(q^2-1)}{zq^2-1}, \\ c(z) &= \frac{q^{1/2}(q^2-1)(z-1)}{(zq^2-1)(zq^3-1)}, \\ d(z) &= \frac{q^2(z-1)(zq-1)}{(zq^2-1)(zq^3-1)}, \\ e(z) &= a(z) - \frac{z(q^2-1)(q^3-1)}{(zq^2-1)(zq^3-1)}, \\ f(z) &= \frac{z(q^2-1)}{zq^2-1}, \\ g(z) &= -\frac{q^{5/2}z(q^2-1)(z-1)}{(zq^2-1)(zq^3-1)}, \\ r(z) &= \frac{(q^2-1)[q^3z+q(z-1)-1]}{(zq^2-1)(zq^3-1)}, \\ s(z) &= \frac{z(q^2-1)[q^3z+q^2(z-1)-1]}{(zq^2-1)(zq^3-1)}. \end{aligned} \tag{4}$$

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The standard row-to-row monodromy matrix for an N tensor space $\underbrace{V \otimes V \cdots \otimes V}_{N}$ is defined by

$$L_0(z) = R_{01}(z)R_{02}(z)\cdots R_{0N}(z). \tag{5}$$

It also satisfies the graded Yang–Baxter relation

$$R_{00'}(z/w)L_0(z)L_{0'}(w) = L_{0'}(w)L_0(z)R_{00'}(z/w). \tag{6}$$

This can be proved by repeatedly using the Yang–Baxter equation (1). Usually, the space V_0 in Eq.(5) is called the auxiliary space, and the space $V_1 \otimes V_2 \cdots \otimes V_N$ is called the quantum space. The transfer matrix $T(z)$ for the periodic boundary condition is defined as the supertrace of the row-to-row monodromy

$$\begin{aligned} T(z) &= \mathrm{str}(L(z)) = \sum_{i=1}^{3} L_i^i(-1)^{[v_i]} \\ &= L_1^1(z) - L_2^2(z) + L_3^3(z). \end{aligned} \tag{7}$$

The supersymmetric model with $U_q[osp(1|2)]$ symmetry is described by the following Hamiltonian[7]

$$H = z\frac{\mathrm{d}}{\mathrm{d}z}\left(T(z)\right)|_{z=1}. \tag{8}$$

By using the graded Yang–Baxter relation (6), one can prove that the transfer matrix with different spectral parameters commute with each other, namely, $[T(z), T(w)] = 0$. This ensures the integrability of the model defined by the Hamiltonian (8). In the following part of this paper, we will construct the eigenvectors and the corresponding eigenvalues of the transfer matrix $T(z)$ by the algebraic Bethe ansatz method.

As usual, we write the standard row-to-row monodromy matrix $L(z)$ as

$$\begin{aligned} L(z) &= \begin{pmatrix} L_1^1(z) & L_2^1(z) & L_3^1(z) \\ L_1^2(z) & L_2^2(z) & L_3^2(z) \\ L_1^3(z) & L_2^3(z) & L_3^3(z) \end{pmatrix} \\ &= \begin{pmatrix} A(z) & B_1(z) & B_2(z) \\ C_1(z) & D_1(z) & B_3(z) \\ C_2(z) & C_3(z) & D_2(z) \end{pmatrix}. \end{aligned} \tag{9}$$

Define the "bare" vacuum as

$$|0\rangle = \prod_{l=1}^{N} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

Recalling the definition of the row-to-row monodromy matrix, one has

$$\begin{aligned} &A(z)|0> = |0\rangle, \\ &D_1(z)|0\rangle = a^N(z)|0\rangle, \\ &D_2(z)|0\rangle = d^N(z)|0\rangle, \\ &C_i(z)|0\rangle = 0, \quad B_i(z)|0\rangle \neq 0 \quad i = 1,2,3. \end{aligned} \tag{11}$$

Then one can find that the "bare" vacuum $|0\rangle$ (10) is the eigenvector of the transfer matrix $T(z)$ and the corresponding eigenvalue $\tau^{(0)}(u)$ becomes

$$\begin{aligned} &\tau^{(0)}(z)|0\rangle = T(z)|0\rangle \\ \equiv &\{A(z) - D_1(z) + D_2(z)\}|0\rangle \\ = &\left(1 - a^N(z) + d^N(z)\right)|0\rangle. \end{aligned} \tag{12}$$

The one-particle excitation state $|w\rangle$ over the "bare" vacuum can be defined as

$$|w\rangle = B_1(w)|0\rangle, \tag{13}$$

and the corresponding eigenvalue of the transfer matrix on it is given in Eq.(17). In order to calculate the eigenvalue of the transfer matrix, we should find the commutation relations which come from the Yang–Baxter relation (6),

$$\begin{aligned} A(z)B_1(w) =& \frac{1}{a(w/z)}B_1(w)A(z) \\ &- \frac{b(w/z)}{a(w/z)}B_1(z)A(w), \\ D_1(z)B_1(w) =& \left(\frac{e(z/w)}{a(z/w)} - \frac{c(z/w)g(z/w)}{a(z/w)d(z/w)}\right) \\ &\cdot B_1(w)D_1(z) + \frac{f(z/w)}{a(z/w)}B_1(z)D_1(w) \\ &- \frac{g(z/w)}{d(z/w)}B_3(z)A(w) \\ &+ \left(\frac{g(z/w)r(z/w)}{a(z/w)d(z/w)} - \frac{c(z/w)}{a(z/w)}\right) \\ &\cdot B_2(w)C_1(z) \\ &- \frac{f(z/w)g(z/w)}{a(z/w)d(z/w)}B_2(z)C_1(w), \\ D_2(z)B_1(w) =& \frac{a(z/w)}{d(z/w)}B_1(w)D_2(z) \\ &+ \frac{b(z/w)}{d(z/w)}B_2(w)C_3(z) \\ &- \frac{s(z/w)}{d(z/w)}B_2(z)C_3(w) \\ &- \frac{g(z/w)}{d(z/w)}B_3(z)D_1(w). \end{aligned}$$

From the above equations on the "bare" vacuum, we find

$$\begin{aligned} A(z)B_1(w)|0\rangle =&\frac{1}{a(w/z)}B_1(w)A(z)|0\rangle \\ &-\frac{b(w/z)}{a(w/z)}B_1(z)A(w)|0\rangle, \qquad (14)\\ D_1(z)B_1(w) =&\left(\frac{e(z/w)}{a(z/w)}-\frac{c(z/w)g(z/w)}{a(z/w)d(z/w)}\right)\\ &\cdot B_1(w)D_1(z)|0\rangle \\ &+\frac{f(z/w)}{a(z/w)}B_1(z)D_1(w)|0\rangle \\ &-\frac{g(z/w)}{d(z/w)}B_3(z)A(w)|0\rangle, \qquad (15)\\ D_2(z)B_1(w) =&\frac{a(z/w)}{d(z/w)}B_1(w)D_2(z)|0\rangle \\ &-\frac{g(z/w)}{d(z/w)}B_3(z)D_1(w)|0\rangle. \qquad (16) \end{aligned}$$

The first terms on the right-hand sides of Eqs.(14)–(16) are the so-called "wanted" terms, which will be dedicated to the eigenvalue of the transfer matrix; the other terms in the right-hand sides of Eqs.(14)–(16) are "unwanted" terms, which should be cancelled by the Bethe ansatz equations. Note that

$$-\frac{b(z/w)}{a(z/w)}=\frac{f(w/z)}{a(w/z)}=-\frac{w(q^2-1)}{q(z-w)},$$

the eigenvalue of the transfer matrix on a one-particle excited state Eq.(13), is

$$\begin{aligned} T(z)|w\rangle =&\tau^{(1)}(z|w)|w\rangle \\ =&\left\{\frac{q^{-1}z-qw}{z-w}-\frac{(qz-q^{-1}w)(z-qw)}{(z-w)(zq-w)}a^N(z)\right.\\ &\left.+\frac{zq^{3/2}-wq^{-3/2}}{zq^{1/2}-wq^{-1/2}}d^N(z)\right\}|w\rangle, \qquad (17) \end{aligned}$$

provided that

$$1-a^N(w)=0, \quad \text{namely,} \quad \left\{\frac{q^{-1}-qw}{1-w}\right\}^N=1. \qquad (18)$$

Equation (18) is the so-called Bethe ansatz equation for a one-particle excited state, and it ensures that the "unwanted" terms vanish. The condition that the eigenvalue should be an entire function also leads to Eq.(18).

Noting that

$$\begin{aligned} &\frac{c(z/w)d(z/w)-r(z/w)g(z/w)}{e(z/w)d(z/w)-c(z/w)g(z/w)}\\ =&\frac{q^{1/2}(q^2-1)w}{(z-qw)}=\frac{g(w/z)}{d(w/z)}, \end{aligned}$$

and from the Yang–Baxter relation (6), we can find

$$\begin{aligned} &B_1(z)B_1(w)+\frac{q^{1/2}(q^2-1)z}{(w-qz)}B_2(z)A(w)\\ =&\frac{c(z/w)g(z/w)-e(z/w)d(\frac{z}{w})}{d(\frac{z}{w})}\\ &\cdot\left\{B_1(w)B_1(z)+\frac{q^{1/2}(q^2-1)w}{(z-qw)}B_2(w)A(z)\right\}. \end{aligned}$$

So we can construct the ansatz of a two-particle state, like that in Ref.[8] as

$$\begin{aligned} |w_1,w_2\rangle =&B_1(w_1)B_1(w_2)\\ &+\frac{q^{1/2}(q^2-1)w_1}{(w_2-qw_1)}B_2(w_1)A(w_2)|0\rangle, \qquad (19) \end{aligned}$$

which is symmetric for w_1, w_2 up to a whole factor. When the transfer matrix acts on the eigenvector (19), one can find the eigenvalue and many "unwanted" terms. From the condition that the eigenvalue is an entire function, which can be seen from the definition of the transfer matrix, one can obtain the corresponding Bethe ansatz equations. The demand that the eigenvalue is an entire function is the so-called functional Bethe ansatz.[9] Moreover, the above results can be generalized for any m-particle excitation.[8−10] Thus the final results are as follows. The eigenvalue of the transfer matrix is obtained as

$$\begin{aligned} &\tau^{(m)}(z|w_1,w_2,\cdots,w_m)\\ =&\prod_{i=1}^{m}\left\{\frac{q^{-1}z-qw_i}{z-w_i}\right\}\\ &+a^N(z)\prod_{i=1}^{m}\left\{-\frac{(qz-q^{-1}w_i)(z-qw_i)}{(z-w_i)(zq-w_i)}\right\}\\ &+d^N(z)\prod_{i=1}^{m}\left\{\frac{zq^{3/2}-w_iq^{-3/2}}{zq^{1/2}-w_iq^{-1/2}}\right\}, \qquad (20) \end{aligned}$$

where $w_1, w_2, \cdots, w_m$ should satisfy the following Bethe ansatz equations

$$a^N(w_i)=\prod_{j\neq i}^{m}\left\{-\frac{(q^{-1}w_i-qw_j)(qw_i-w_j)}{(qw_i-q^{-1}w_j)(w_i-qw_j)}\right\}. \qquad (21)$$

From now on, we will discuss the twisted boundary conditions for the model (8). One can find that a diagonal matrix g

$$g=\begin{pmatrix}\lambda & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda^{-1}\end{pmatrix},$$

with λ being an arbitrary parmeter, has the following property:

$$(g\otimes g)R_{12}(z)(g^{-1}\otimes g^{-1})=R_{12}(z). \qquad (22)$$

The monodromy matrix for the twisted boundary condition is defined as $L'(z) = gL(z)$. Thus, the corresponding transfer matrix becomes

$$T'(z) = \mathrm{str}(L'(z)) = \lambda A(z) - D_1(z) + \lambda^{-1} D_2(z). \quad (23)$$

It can easily be proved that $L'(z)$ still satisfies the graded Yang–Baxter relation (6) from the property (22).

Using the results obtained in studying the periodic boundary condition, one can easily find the eigenvalues of the transfer matrix with the twisted boundary condition and the corresponding Bethe ansatz equations.

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References

[1] Anderson P W 1987 *Science* **235** 1196; 1990 *Phys. Rev. Lett.* **65** 2306; Essler F H L and Korepin V E 1992 *Phys. Rev.* B **46** 9147; Foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611

[2] Essler F H L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960

[3] Bracken A J, Gould M D, Links J R and Zhang Y Z 1995 *Phys. Rev. Lett.* **74** 2768

[4] Perk J H and Shultz C L 1981 *Phys. Lett.* A **84** 3759; Yue R H, Fan H and Hou B Y 1996 *Nucl. Phys.* B **462** 167; Fan H, Hou B Y and Shi K J 1999 *Nucl. Phys.* B **541** 483

[5] Deguchi T, Fujii A and Ito K 1990 *Phys. Lett.* B **238** 242; Martins M J and Ramos P B 1997 *Phys. Rev.* B **561** 6376

[6] Yang W L, Bethe Ansatz for Supersymmetric Model Constructed from $U_q[osp(2|2)^{(2)}]$ R-matrix

[7] Gould M D and Zhang Y Z 1998 *Lett. Math. Phys.* **44** 291; 2000 *Nucl. Phys.* B **556** 529

[8] Fan H 1997 *Nucl. Phys.* B **488** 409

[9] Baxter R J 1982 *Exactly Solved Methods in Statistical Mechanics* (London: Academic)

[10] Tarasov V A 1988 *Theor. Math. Phys.* **56** 793