Vol 10 No 2, February 2001
1009-1963/2001/10(02)/0103-06
Chinese Physics
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# SUPERSYMMETRIC $SU(1|2)$ GAUDIN MODEL*

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(Received 2 July 2000; revised manuscript received 9 September 2000)

In this paper, we propose a supersymmetric $SU(1|2)$ Gaudin model and have derived its eigenvalues. We also present the well-defined eigenstates through the quasi-classical limit of the eigenstates in the supersymmetric t–J model.

**Keywords:** Gaudin model, quantum determinant, supersymmetric t–J model
**PACC:** 1240E, 2991, 7510D

## I. INTRODUCTION

The Gaudin model associated with the $SU(2)$ Lie algebra was first proposed by Gaudin in 1973.[1−3] Babujian and Flume[4] generalized it into the general Lie algebra $g$ case but did not list the solution and Bethe ansatz equations. Using the Wakimoto construction, Feigin *et al.*[5] obtained its Bethe ansatz equation and solution. Noticing the $R$-matrix structure of the $SU(2)$ Gaudin model, Sklyanin proposed in Ref.[6] that the spectrum and eigenfunctions of the spin-$\frac{1}{2}$ Gaudin models with rational and trigonometric interaction could be derived from XXX and XXZ chains.[7−9] Then, in Refs.[10-12], Sklyanin and Takebe obtained the arbitrary spin XYZ Gaudin model as a quasi-classical limit of the inhomogeneous higher spin generalization of the XYZ model.

The highly correlated electron models, such as the t–J model, were the subject of numerous studies in the context of high-$T_c$ superconductivity. The Hamiltonian of the t–J model includes the nearest-neighbour hopping (t) and antiferromagnetic exchange (J).[13,14]

The one-dimensional (1D) t–J model describes a quantum system, in which each site can be empty or occupied by one electron with spin-up or spin-down. The Hamiltonian can be written as

$$H = \sum_{j=1}^{N} \Big\{ -t\mathcal{P} \sum_{\sigma=\uparrow,\downarrow} (C_{j,\sigma}^{+} C_{j+1,\sigma} + \text{H.c.})\mathcal{P} + J(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} - \frac{1}{4} n_n n_{j+1}) \Big\}, \quad (1)$$

where $\mathcal{P} = \prod_{j=1}^{N}(1 - n_{j,\uparrow} n_{j,\downarrow})$ is the projecting operator which restricts the Hilbert space without double occupancy at one lattice point and $N$ is the length of the lattice. $C_{j,\sigma}^{+}(C_{j,\sigma})$ is the creation (annihilation) operator with spin $\sigma$ in the $j$th side, $n_j$ is the number of electron and $\boldsymbol{S}_j$ is the spin-$\frac{1}{2}$ operator. It was stated that this model is supersymmetric and integrable for $J = \pm 2t$,[15−18] and its solution was given in Refs.[19–21] (for a review, see Ref.[22]). This model was also considered in graded version. Essler and Korepin *et al.*[21] used the graded quantum inverse scattering method (QISM)[23,24] and obtained the eigenvalues and eigenvectors for the supersymmetric t–J model with periodic boundary conditions in three different backgrounds. For related works see, for example, Refs.[25–27].

Considering a new model by substituting the nearest-neighbour interaction with long range interaction, generally the new model is nonintegrable. Thus, how to keep the integrability becomes a key problem. In the present paper, we carry it out in the supersymmetric $SU(1|2)$ case. Starting from the graded monodromy matrix of the supersymmetric t–J model, we define the graded quantum determinant which commutes with the monodromy matrix. Then, taking the quasi-classical limit ($c \to 0$) of the graded quantum determinant and the monodromy matrix of supersymmetric t–J model, we obtain the Hamiltonians of the supersymmetric $SU(1|2)$ Gaudin model. Thanks to the eigenvectors and eigenvalues of the transfer matrix of the supersymmetric t–J model in Ref.[21], we get the eigenvectors and eigenvalues of the transfer matrix and Hamiltonians of the supersymmetric $SU(1|2)$ Gaudin model.

This paper is organized as follows. Section Ⅱ gives some description of the supersymmetric t–J model. In

*Project supported by the National Natural Science Foundation of China (Grant No. 19975036).

section Ⅲ, we will discuss the construction of the supersymmetric $SU(1|2)$ Gaudin model in detail. The explicit Hamiltonians, eigenstates and eigenvalues are presented. Section Ⅳ includes a brief summary and some discussion.

## Ⅱ. DESCRIPTION OF SUPERSYMMETRIC t–J MODEL

We first recall the supersymmtric t–J model and the graded version of the QISM. The 1D supersymmetric t–J model with J=2t is easily translated to the $spl(2,1)$-supersymmetric 15-vertex model.[28−30] Taking the following grading $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = 0$, the $R$-matrix is defined as

$$\hat{R}(u) = \frac{1}{u}\begin{pmatrix} \mathrm{i}c-u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}c & 0 & -u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}c & 0 & 0 & 0 & u & 0 & 0 \\ 0 & -u & 0 & \mathrm{i}c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i}c-u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i}c & 0 & u & 0 \\ 0 & 0 & u & 0 & 0 & 0 & \mathrm{i}c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & \mathrm{i}c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}c+u \end{pmatrix}.$$

The $L$ operators, $L_n(u)$, whose elements act nontrivially only on the $n$th component, can be constructed from the $R$-matrix as

$$L_n(u) = \frac{1}{u}\begin{pmatrix} u-\mathrm{i}cE_n^{11} & -\mathrm{i}cE_n^{21} & \mathrm{i}cE_n^{31} \\ -\mathrm{i}cE_n^{12} & u-\mathrm{i}cE_n^{22} & \mathrm{i}cE_n^{32} \\ \mathrm{i}cE_n^{13} & \mathrm{i}cE_n^{23} & u+\mathrm{i}cE_n^{33} \end{pmatrix}, \tag{2}$$

where $E_n^{ab}$ are quantum operators acting in the $n$th quantum space with matrix representation $(E_n^{ab})_{\alpha\beta} = \delta_{a\alpha}\delta_{b\beta}$.

As is well known, the Yang–Baxter[31,32] relation plays a key role for integrable models with periodic boundary conditions,

$$\begin{aligned} &\hat{R}_{12}(u-v)L_n(u)\otimes L_n(v) \\ =&L_n(v)\otimes L_n(u)\hat{R}_{12}(u-v), \end{aligned} \tag{3}$$

where the tensor product is in the graded sense $(F\otimes G)_{ac}^{bd} = F_{ab}G_{cd}(-1)^{\epsilon_c(\epsilon_a+\epsilon_b)}$. In the following, all tensor products are in this graded sense. We can also write the Yang–Baxter relation explicitly as

$$\begin{aligned} &R(u-v)_{a_1a_2}^{c_1c_2}L(u)_{c_1}^{b_1}L(v)_{c_2}^{b_2}(-1)^{\epsilon_{c_2}(\epsilon_{c_1}+\epsilon_{b_1})} \\ =&L(v)_{a_2}^{c_2}L(u)_{a_1}^{c_1}(-1)^{\epsilon_{a_1}(\epsilon_{a_2}+\epsilon_{c_2})}R(u-v)_{c_1c_2}^{b_1b_2}. \end{aligned} \tag{4}$$

The monodromy matrix $T(u)$ with shifts of the spectral parameter $\delta_m$ is defined as the matrix product over the $L$ operators on all sites of the lattice,

$$T(u) = L_N(u-\delta_N)L_{N-1}(u-\delta_{N-1})\cdots L_1(u-\delta_1), \tag{5}$$

satisfying the Yang–Baxter relation

$$\hat{R}(u-v)T(u)\otimes T(v) = T(v)\otimes T(u)\hat{R}(u-v), \tag{6}$$

where the tensor product is still in the graded sense. The elements of $T(u)$ can be written as

$$\begin{aligned} &\left\{[T(u)]^{ab}\right\}_{\alpha_1\alpha_2\cdots\alpha_N}^{\beta_1\beta_2\cdots\beta_N} \\ =&L_N(u-\delta_N)_{a\alpha_N}^{c_N\beta_N}L_{N-1}(u-\delta_{N-1})_{c_N\alpha_{N-1}}^{c_{N-1}\beta_{N-1}}\cdots \\ &L_1(u-\delta_1)_{c_2\alpha_1}^{b\beta_1}(-1)^{\sum_{j=2}^{N}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j}\epsilon_{\alpha_i}}. \end{aligned} \tag{7}$$

The transfer matrix $t(u)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space. It is defined as the following in the general case,

$$t(u) = \mathrm{str}T(u) = \sum(-1)^{\epsilon_a}[T(u)]^{aa}. \tag{8}$$

As a consequence of the Yang–Baxter relation, we can prove that the transfer matrices commute with each other for different spectrum parameters $[t(u), t(v)] = 0$. It has been proved that the Hamiltonian obtained by taking the first logarithmic derivative at the zero spectral parameter is equivalent to the Hamiltonian of the supersymmetric t–J model.[21]

Now, we try to define the supersymmetric quantum determinant which will play a key role in the construction of the supersymmetric $SU(1|2)$ Gaudin

model. In the graded case, the quantum determinant can be defined as

$$\Delta(u) \equiv \mathrm{str}_{123} P^{-}_{123} T_{01}(u) T_{02}(u - \mathrm{i}c) T_{03}(u - 2\mathrm{i}c), \quad (9)$$

where $P^{-}_{123}$ is a projector onto the space of completely anti-symmetric tensors,

$$P^{-}_{123} = \frac{1}{6}[1 - P_{12} - P_{13} - P_{23} + P_{123} + P_{132}], \quad (10)$$

with the graded permutation operator $[P_{12}]^{b_1 b_2}_{a_1 a_2} = \delta_{a_1 b_2} \delta_{a_2 b_1} (-1)^{\epsilon_{a_1} \epsilon_{a_2}}$. It can be shown $[\Delta(u), T(v)] = 0$.

We introduce the local reference state (boson background) $|\Omega\rangle_n = (0, 0, 1)^t$. Take the state $|\Omega\rangle = |\Omega\rangle_1 \otimes |\Omega\rangle_2 \cdots |\Omega\rangle_N$ as the reference state. The monodromy matrix acting on this state gives

$$\begin{aligned} T(u)^1_1 |\Omega\rangle &= T(u)^2_2 |\Omega\rangle = |\Omega\rangle, \\ T(u)^3_3 |\Omega\rangle &= \prod_{n=1}^{N} \frac{u - \delta_n + \mathrm{i}c}{u - \delta_n} |\Omega\rangle, \end{aligned} \quad (11)$$

$$\begin{aligned} T(u)^i_3 |\Omega\rangle &\neq 0 \qquad i = 1, 2, \\ T(u)^j_i |\Omega\rangle &= 0 \qquad i \neq j. \end{aligned} \quad (12)$$

Applying the quantum determinant to the reference state, we arrive at

$$\Delta |\Omega\rangle = \left\{ -4 + 3 \prod_{j=1}^{N} (1 + \frac{\mathrm{i}c}{u - \delta_j}) \right\} |\Omega\rangle, \quad (13)$$

where we have used the Yang–Baxter relations to simplify all calculations, such as,

$$\begin{aligned} & T(u)^3_1 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^1_3 |\Omega\rangle \\ =& - T(u)^3_3 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle \\ & + T(u)^1_1 T(u - \mathrm{i}c)^3_3 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle, \end{aligned} \quad (14)$$

$$\begin{aligned} & T(u)^3_1 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^1_3 |\Omega\rangle \\ =& - \frac{1}{2} T(u)^3_3 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle \\ & + \frac{1}{2} T(u)^1_1 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^3_3 |\Omega\rangle, \end{aligned} \quad (15)$$

$$\begin{aligned} & T(u)^3_1 T(u - \mathrm{i}c)^3_3 T(u - 2\mathrm{i}c)^1_3 |\Omega\rangle \\ =& - T(u)^3_3 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^3_3 |\Omega\rangle \\ & + T(u)^1_1 T(u - \mathrm{i}c)^3_3 T(u - 2\mathrm{i}c)^3_3 |\Omega\rangle, \end{aligned} \quad (16)$$

$$\begin{aligned} & T(u)^3_1 T(u - \mathrm{i}c)^1_2 T(u - 2\mathrm{i}c)^2_3 |\Omega\rangle \\ =& - T(u)^3_3 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^2_2 |\Omega\rangle \\ & + T(u)^1_1 T(u - \mathrm{i}c)^3_3 T(u - 2\mathrm{i}c)^2_2 |\Omega\rangle \\ & + \frac{1}{2} T(u)^3_3 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle \\ & - \frac{1}{2} T(u)^1_1 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^3_3 |\Omega\rangle, \end{aligned} \quad (17)$$

$$\begin{aligned} & T(u)^3_2 T(u - \mathrm{i}c)^2_1 T(u - 2\mathrm{i}c)^1_3 |\Omega\rangle \\ =& - T(u)^3_3 T(u - \mathrm{i}c)^2_2 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle \\ & + T(u)^2_2 T(u - \mathrm{i}c)^3_3 T(u - 2\mathrm{i}c)^1_1 |\Omega\rangle \\ & + \frac{1}{2} T(u)^3_3 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^2_2 |\Omega\rangle \\ & - \frac{1}{2} T(u)^2_2 T(u - \mathrm{i}c)^1_1 T(u - 2\mathrm{i}c)^3_3 |\Omega\rangle. \end{aligned} \quad (18)$$

## III. THE SUPERSYMMETRIC $SU(1|2)$ GAUDIN MODEL

We define the supersymmetric $SU(1|2)$ Gaudin model as a quasi-classical limit of the supersymmetric t–J model defined above. Let us examine the asymptotic behaviour of the operators in the previous section when $c$ tends to 0. The $L$ operator, the monodromy matrix and the $R$-matrix are expanded as

$$\begin{aligned} \hat{R}_{12}(u) &= P_{12} + \mathrm{i}c P_{12} r_{12}(u), \\ L_n(u) &= 1 + \mathrm{i}c \mathcal{L}_n(u), \end{aligned} \quad (19)$$

$$\begin{aligned} T(u) &= 1 + \mathrm{i}c \mathcal{T}(u) + (\mathrm{i}c)^2 \mathcal{T}_2(u) + O((\mathrm{i}c)^3), \\ \mathcal{T}(u) &= \sum_{n=1}^{N} \mathcal{L}_n(u - \delta_n). \end{aligned} \quad (20)$$

Then, $\mathcal{L}(u)$ and $\mathcal{T}(u)$ satisfy the modified Yang–Baxter relation,

$$\begin{aligned} & [\mathcal{L}_{01}(u_1), \mathcal{L}_{02}(u_2)] \\ =& [\mathcal{L}_{01}(u_1) + \mathcal{L}_{02}(u_2), r_{12}(u_1 - u_2)], \\ & [\mathcal{T}_{01}(u_1), \mathcal{T}_{02}(u_2)] \\ =& [\mathcal{T}_{01}(u_1) + \mathcal{T}_{02}(u_2), r_{12}(u_1 - u_2)]. \end{aligned} \quad (21)$$

In order to obtain the generating function of integrals of motion of the supersymmetric $SU(1|2)$ Gaudin model, we expand the transfer matrix, supertrace of the monodromy matrix, up to the order of $(\mathrm{i}c)^2$,

$$t(u) = -1 + \mathrm{i}c \mathrm{str} \mathcal{T}(u) + (\mathrm{i}c)^2 \mathrm{str} \mathcal{T}_2(u) + O((\mathrm{i}c)^3). \quad (22)$$

For quantum determinant $\Delta(u)$, we also substitute it into the expression,

$$\begin{aligned} \Delta(u) =& -1 + 3\mathrm{i}c \mathrm{str} \mathcal{T}(u) + (\mathrm{i}c)^2 [\frac{3}{2} \mathrm{str} \mathcal{T}^2(u) \\ & - \frac{3}{2} (\mathrm{str} \mathcal{T}(u))^2 + 3\mathrm{str} \mathcal{T}^{(2)}(u) \\ & - 3\mathrm{str} \mathcal{T}^{(1)}(u)] + O((\mathrm{i}c)^3), \end{aligned} \quad (23)$$

where $\mathcal{T}^{(1)}(u)$ means the first-order deviation of $\mathcal{T}(u)$.

We note that in the subtraction $\Delta(u)-3t(u)$, the term with $\mathrm{str}\mathcal{T}^{(2)}(u)$ disappears

$$\Delta(u)-3t(u)=2+(\mathrm{ic})^2 \cdot\left\{\frac{3}{2}\mathrm{str}\mathcal{T}^2(u)-\frac{3}{2}(\mathrm{str}\mathcal{T}(u))^2-3\mathrm{str}\mathcal{T}^{(1)}(u)\right\}. \quad (24)$$

Thus, the supersymmetric $SU(1|2)$ Gaudin model can be defined by specifying its generating function of the integrals of motion as,

$$\hat{\tau}(u)\equiv\frac{3}{2}\mathrm{str}\mathcal{T}^2(u). \quad (25)$$

We can show $[\hat{\tau}(u),\hat{\tau}(v)]=0$. Thus, the supersymmetric $SU(1|2)$ Gaudin model is integrable, and $\hat{\tau}(u)$ can also be explicitly written down in the following form:

$$\hat{\tau}(u)=-\frac{3}{2}\sum_{n=1}^{N}\frac{1}{(u-\delta_n)^2}+\sum_{n=1}^{N}\frac{\hat{H}_n}{u-\delta_n}. \quad (26)$$

The $N-1$ independent commuting Hamiltonians of the supersymmetric $SU(1|2)$ Gaudin model are

$$\hat{H}_n=\sum_{m=1,m\neq n}^{N}\sum_{\alpha_1,\alpha_2=1}^{3}\frac{3(-1)^{\alpha_2}E_n^{\alpha_1\alpha_2}E_m^{\alpha_2\alpha_1}}{\delta_n-\delta_m},$$
$$\sum_{n=1}^{N}\hat{H}_n=0. \quad (27)$$

Using the creation and annihilation operators, the Hamiltonians can also be expressed as,

$$\hat{H}_n=\sum_{m=1,m\neq n}^{N}\sum_{\sigma=\uparrow,\downarrow}\frac{3}{\delta_n-\delta_m} \cdot\left\{(C_{n\sigma}^{+}C_{m\sigma}+C_{m\sigma}^{+}C_{n\sigma})+1-n_n-n_m -2S_n^{+}S_m^{-}-2S_n^{-}S_m^{+}-2S_n^{z}S_m^{z}+\frac{1}{2}n_nn_m\right\} \quad (28)$$

where

$$S_j^{+}=\frac{1}{\sqrt{2}}C_{j,\uparrow}^{+}C_{j,\downarrow},$$
$$S_j^{-}=\frac{1}{\sqrt{2}}C_{j,\downarrow}^{+}C_{j,\uparrow},$$
$$S_j^{z}=\frac{1}{2}(C_{j,\uparrow}^{+}C_{j,\uparrow}-C_{j,\downarrow}^{+}C_{j,\downarrow}). \quad (29)$$

We assume the eigenstates of the generating function of the supersymmetric $SU(1|2)$ Gaudin model to be

$$\Phi^3=\sum_{\alpha_1,\alpha_2,\cdots,\alpha_m=1}^{2}\mathcal{C}_{\alpha_1}(\lambda_1)\mathcal{C}_{\alpha_2}(\lambda_2)\cdots \mathcal{C}_{\alpha_m}(\lambda_m)|\Lambda\rangle F^{\alpha_m\cdots\alpha_2\alpha_1}. \quad (30)$$

In the above summation, one should keep the number of $\alpha_j$ taking the value of 1 to be $m_1$, and the coefficient can be directly identified with

$$F^{\alpha_m\cdots\alpha_2\alpha_1}=\mathcal{C}(\lambda_1^{(1)})\mathcal{C}(\lambda_2^{(1)})\cdots\mathcal{C}(\lambda_{m_1}^{(1)})|\Omega^{(1)}\rangle, \quad (31)$$

with

$$\mathcal{C}_{\alpha_l}(\lambda_l)=\sum_{n=1}^{N}\frac{1}{(\lambda_l-\delta_n)}E_n^{\alpha_l 3},$$
$$\mathcal{C}(\lambda^{(1)})=\sum_{n=1}^{m_1}\frac{1}{(\lambda^{(1)}-\lambda_n)}e_n^{12}, \quad (32)$$

$$|\Omega^{(1)}\rangle_n=\begin{pmatrix}0\\1\end{pmatrix},$$

$$|\Omega^{(1)}\rangle=|\Omega^{(1)}\rangle_1\otimes|\Omega^{(1)}\rangle_2\cdots|\Omega^{(1)}\rangle_m. \quad (33)$$

Since our choice of $m$ and $m_1$ can be identified as the total number of electrons and the number of spin-up electrons respectively, i.e., $m=N_e=N_\uparrow+N_\downarrow$ and $m_1=N_\uparrow$. So, the eigenstates of the supersymmetric $SU(1|2)$ Gaudin model are

$$|\lambda_1,\lambda_2,\cdots,\lambda_m|F\rangle =\prod_{j=1}^{m}\sum_{n=1}^{N}\frac{1}{\lambda_j-\delta_m}E_n^{\alpha_j 3}|\Omega\rangle \cdot\prod_{k=1}^{m_1}\sum_{n=1}^{m}\frac{1}{\lambda_k^{(1)}-\lambda_n}\mathrm{e}_n^{12}|\Omega^{(1)}\rangle. \quad (34)$$

From the transfer matrix acting on eigenstates, one can obtain the eigenvalue of the supersymmetry t–J model in the FFB bases. In the limit of $c\to 0$, it can be written as

$$\Lambda_{t(u)}=-1+\sum_{n=1}^{N}\frac{\mathrm{ic}}{u-\delta_n}+(\mathrm{ic})^2 \cdot\left\{\sum_{n=1}^{N-1}\sum_{k=n+1}^{N}\frac{1}{(u-\delta_n)(u-\delta_k)} -\sum_{i=1}^{m_1-1}\sum_{j=i+1}^{m_1}\frac{2}{(\lambda_i^{(1)}-u)(\lambda_j^{(1)}-u)} +\sum_{n=1}^{N}\frac{1}{u-\delta_n}\sum_{j=1}^{m}\frac{1}{\lambda_j-u} -\sum_{i=1}^{m_1}\frac{1}{u-\lambda_i^{(1)}}\sum_{j=1}^{m}\frac{1}{\lambda_j-u}\right\}+O((\mathrm{ic})^3). \quad (35)$$

In the same limit, the Bethe ansatz equations will change into

$$\sum_{n=1}^{N}\frac{1}{\lambda_k-\delta_n}=\sum_{j=1}^{m_1}\frac{1}{\lambda_k-\lambda_j^{(1)}},$$
$$\sum_{k=1}^{m}\frac{1}{\lambda_k-\lambda_p^{(1)}}=\sum_{j=1,j\neq p}^{m_1}\frac{2}{\lambda_j^{(1)}-\lambda_p^{(1)}}. \quad (36)$$

Using the same method, we also expanded the eigenvalue of the quantum determinant up to the second order,

$$\Lambda_{\Delta(u)}=-1+3\sum_{n=1}^{N}\frac{\mathrm{i}c}{u-\delta_n}+3\sum_{n=1}^{N-1}\sum_{l=n+1}^{N}\frac{(\mathrm{i}c)^2}{(u-\delta_n)(u-\delta_l)}+O((\mathrm{i}c)^3). \quad (37)$$

Then the eigenvalues of the supersymmetric $SU(1|2)$ Gaudin model can be written as

$$\Lambda_{\hat{\tau}(u)}=3\sum_{i=1}^{m_1-1}\sum_{j=i+1}^{m_1}\frac{2}{(\lambda_i^{(1)}-u)(\lambda_j^{(1)}-u)}-3\sum_{n=1}^{N}\frac{1}{u-\delta_n}\sum_{j=1}^{m}\frac{1}{\lambda_j-u}+3\sum_{i=1}^{m_1}\frac{1}{u-\lambda_i^{(1)}}\sum_{j=1}^{m}\frac{1}{\lambda_j-u}+\frac{3}{2}\left(\sum_{n=1}^{N}\frac{1}{u-\delta_n}\right)^2-3\sum_{n=1}^{N}\frac{1}{(u-\delta_n)^2}. \quad (38)$$

With the help of the Bethe ansatz equations (35), the eigenvalues of the supersymmetric $SU(1|2)$ Gaudin model are

$$\Lambda_{\hat{\tau}(u)}=-\sum_{n=1}^{N}\frac{3/2}{(u-\delta_n)^2}+\sum_{n=1}^{N}\frac{1}{u-\delta_n}\cdot\left\{-\sum_{j=1}^{m}\frac{3}{\lambda_j-\delta_n}+\sum_{m=1,m\neq n}^{N}\frac{3}{\delta_n-\delta_m}\right\}. \quad (39)$$

For Hamiltonians acting on the eigenstates, we can obtain the eigenvalues

$$\Lambda_{\hat{H}_n}=-\sum_{j=1}^{m}\frac{3}{\lambda_j-\delta_n}+\sum_{m=1,m\neq n}^{N}\frac{3}{\delta_n-\delta_m}. \quad (40)$$

## IV. DISCUSSION

We have constructed the Hamiltonians of the supersymmetric $SU(1|2)$ Gaudin model corresponding to the supersymmetric t–J model. We proved that, for the graded case, the Hamiltonian can also be obtained from the transfer matrix, supertrace of the monodromy matrix and the graded quantum determinant. We give the quantum determinant in the graded case, which is very different from the nongraded one. Meanwhile, we give the eigenstates and eigenvalues of the transfer matrix of the supersymmetric $SU(1|2)$ Gaudin model. We also give the eigenvalues of the Hamiltonians.

In this paper, we only consider the supersymmetric t–J model. Based on physical considerations, one can extend these results to the generalized t–J model by using the same method. It is very interesting to study the boundary behaviour of the quantum integrable model in many kinds of boundary conditions. Hikami[33] has applied Sklyanin's method to the open Gaudin model by taking the quasi-classical limit of the transfer matrix for the inhomogeneous open spin-$\frac{1}{2}$ XXZ chain. That is to say, for the open boundary case, the Hamiltonians of open Gaudin model can be obtained by expanding the transfer matrix around $c\to 0$. So one can study the supersymmetric $SU(1|2)$ Gaudin model under some open boundary conditions. We will give these results in future papers.

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