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HIGHER DIMENSIONAL PAINLEVÉ INTEGRABLE MODELS FROM THE REAL NONLINEAR EVOLUTION EQUATIONS*

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A conformal invariant asymptotic expansion approach is proposed to solve any nonlinear integrable and nonintegrable models with any dimension. Many new Painlevé integrable models with the same dimensions can be obtained at the same time. Taking the (2+1)-dimensional KdV–Burgers (KdVB) equation and the (3+1)-dimensional Zabolotskaya–Khokhlov and Kudomtsev–Petviashvili (ZKKP) equation as concrete examples, we obtain some new higher dimensional conformal invariant models with a Painlevé property and the approximate solutions of these models. In certain special cases, some of the approximate solutions become exact.

Keywords: higher dimension, conformal invariance, ZKKP equation
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I. INTRODUCTION

To find integrable models, especially in the (3+1) dimension is one of the most important problems in nonlinear science. Because the real physical spacetime is (3+1)-dimensional, many physicists and mathematicians have been trying to find some nontrivial (3+1)-dimensional integrable models[1,2] for a long time. However, there is no significant $(n+1)$-dimensional ($n \geq 3$) nontrivial nonlinear integrable model known except the "(2+2)"-dimensional self-dual Yang–Mills field equation.[1]

Recently, Lou *et al.*[3] pointed out some possible directions to find nontrivial integrable models in higher dimensions under certain special conditions. Owing to the fact that all the known (2+1)-dimensional integrable models possess a common generalized Virasoro-type symmetry algebra, a general method is proposed to obtain some integrable models under the condition that they possess the generalized Virasoro symmetry algebra[4]. It is also known that every (1+1)- and (2+1)-dimensional integrable model possesses a Schwartz form which is a conformal invariant. Based on these facts, Lou points out that starting from a conformal invariant form is one of the most convenient ways to obtain higher dimensional integrable models.[5] In Refs.[6] and [7], the authors have extended the Painlevé analysis approaches to a new form such that many higher dimensional Painlevé integrable models can be obtained from lower dimensional ones.

In the papers by Lou[6] and Lou and Xu,[7] some important questions need to be answered further. One of the most important questions is whether those higher dimensional Painlevé integrable models, obtained from the extended Painlevé analysis, have real physical significance. Can they describe real nonlinear physical phenomena? Generally, most of the known lower dimensional integrable models are obtained by ignoring some unimportant factors from the real physical models with the same and/or higher dimensions. This suggests that (3+1)-dimensional integrable models may actually be obtained by means of some reasonable approximations from higher dimensional models. We think that combining the conformal invariant theory with the extended Painlevé analysis approach should be one of the possible ways to find higher dimensional integrable models with real physical applications.

In this paper, we use the invariant Painlevé analysis approach to solve real higher dimensional nonlinear physical problems and to discover some higher dimensional Painlevé integrable models in the same dimensions which can be considered as some reasonable approximations of real physical problems.

In order to make this approach clear, we take the

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(2+1)-dimensional KdVB[8] equation and the (3+1)-dimensional ZKKP[9] equation as concrete examples.

The organization of the paper is as follows. In section Ⅱ, a general approximation method (conformal invariant expansion) is proposed to solve a real physical model with any dimensions. In this approach, a real model is solved approximately by means of Painlevé integrable models in the same dimensions. Taking the (2+1)-dimensional KdVB equation and the (3+1)-dimensional ZKKP equation as two simple examples, some new (2+1)-dimensional and (3+1)-dimensional conformal invariant models are derived in section Ⅲ. Section Ⅳ is devoted to a discussion of the Painlevé integrability of these resulting models. The approximate solutions of the (2+1)-dimensional KdVB equation and (3+1)-dimensional ZKKP equation are shown in section V. Section Ⅵ includes a summary and discussion.

Ⅱ. GENERAL THEORY

For a given n-dimensional N order PDE,

$$F(x_0 \equiv t, x_1, x_2, \cdots, x_n, u, u_{x_i}, u_{x_i x_j, \cdots}, u_{x_{i_1} x_{i_2} \cdots x_{i_N}}) = 0, \tag{1}$$

where the function F is a polynomial function of the field u and its derivatives. Expanding the solution of (1) near the singular manifold ϕ, we should have the form

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha}, \tag{2}$$

where α is a negative integer. Substituting (2) into (1), we can get a recursion relation to determine the expansion coefficients u_j

$$G(j)u_j = f_j(x_i, \phi_{x_i}, \phi_{x_{i_1} x_{i_2}}, \cdots, u_1, u_2, \cdots, u_{j-1}) \equiv f_j, \tag{3}$$

where $G(j)$ is a polynomial function j. Eq.(1) possesses the Painlevé property if $G(j) = (j+1)(j - j_1)(j - j_2)\cdots(j - j_{N-1})$, where $j_1,\ j_2,\ \cdots, j_{N-1}$ are all positive integers and the resonance conditions

$$f_{j_i} = 0 \quad (i = 1, 2, \cdots, N-1) \tag{4}$$

are satisfied identically. If Eq.(1) is not Painlevé integrable, Eq.(3) could still be taken as a recursion formula to get some special solutions of (1). In this paper, we obtain some approximate solutions of (1) and integrable models corresponding to (1) with the help of the recursion relation (3). Because the expansion (2) is valid only for small ϕ, we ignore terms higher than $\phi^{M+\alpha}$ and fix ϕ by $u_M = 0$; that is to say, the truncated expansion

$$u = \sum_{j=0}^{M} u_j \phi^{j+\alpha}, \qquad u_M = 0 \tag{5}$$

can be considered as an approximate solution of Eq.(1) up to $\phi^{M+\alpha}$. Obviously, the larger M is, the more exact solution (5) would be. The expansion coefficients u_j in (5) can be determined from (1). Substituting Eq.(5) into (1), one can obtain an equation in the power form of ϕ, say,

$$\phi^{-K}\left[\sum_{j=0}^{M} F_j \phi^j + O(\phi^{M+1})\right] = 0, \tag{3'}$$

where the positive integer K is determined by the leading order of the PDE (1) and F_j $(j = 0, 1, 2, ..., M)$ are polynomial functions of $u_j (j = 0, 1, 2, ..., M)$, and the derivatives of u_j and ϕ. Solving $F_j = 0, j = 0, 1, 2, ..., M$ for u_j which are equivalent to those given by (3), we know that the expansion coefficients u_j are functions of the derivatives of ϕ and $J \leq N-1$ arbitrary functions which are some of $u_{j_1},\ u_{j_2}, \cdots,\ u_{j_{N-1}}$. If the original PDE (1) possesses the Painlevé property, then J, the number of the arbitrary functions u_j, is equal to $N-1$. Otherwise $J < N-1$. So if $J = N-1$ and $M > j_{N-1}$,

$$u_M = 0 \tag{6}$$

shows an equation of ϕ and the arbitrary functions u_{j_1}, $u_{j_2}, \cdots u_{j_{N-1}}$. The solution of Eq.(6) determines the concrete form of (5) which can be considered as an approximate solution of (1) though some arbitrary functions have possibly been included because the original Eq.(1) is correct up to the order of ϕ^{M+1} (see Eq.(3')). Usually, N arbitrary functions are allowed to be included in the general solution of an N order PDE. In our approximate solution, only $N-1$ arbitrary functions have been included because another arbitrary function ϕ has been fixed by (6). If we take $M < j_{N-1}$, there will be fewer arbitrary functions in our approximate solution.

If the original PDE (1) is not Painlevé integrable, then $J < N-1$, which means that $N-1-J$ resonance conditions of (4) are not satisfied identically. Then, $N-1-J$ additional conditions are introduced for the J arbitrary functions and ϕ. In this case, we can still obtain some approximate solutions of (1) by solving these $N-1-J$ additional conditions and (6) for J arbitrary functions and ϕ.

It is worth pointing out that after substituting (5) into (1), one finds that there may be a special selection

$$u_0 = u_1 = \cdots = u_{-\alpha-1} = 0$$

for our approximate solution, which means that the approximate solution is analytical with respect to the manifold $\phi = 0$. In this case, we can always select $u_{-\alpha}$, $u_{-\alpha+1}$, $\cdots$ $u_{-\alpha+N-1}$ as arbitrary constants and fix other coefficients by the recursion relation. The approximate solution of (1) reads

$$u = \sum_{j=0}^{\alpha+M} u_{-\alpha+j}\phi^j, \qquad u_M = 0.$$

Finally, it is also worth pointing out that if we want to obtain some special solutions of (1) which are more exact, we may fix the arbitrary functions (and constants) in two different ways. (i) The J arbitrary functions may be fixed from (3) by $u_{M+1} = u_{M+2}\cdots = u_{M+J} = 0$ (or equivalently from (3') by $F_{M+1} = F_{M+2}\cdots = F_{M+J} = 0$ with $u_{M+1} = u_{M+2}\cdots = u_{M+J} = 0$) such that Eq.(3') and then (1) are valid up to the ϕ^{M+J}th order and the related approximate solution is valid up to the $\phi^{M+J+\alpha}$th order. (ii) The J arbitrary functions may be fixed appropriately from (3') such that the truncated solution (5) becomes an exact solution of the original PDE (1). More details on these points will be seen from the later concrete examples.

Because there are many particularly interesting properties of integrable models, we hope Eq.(6) is integrable at least under some special conditions. To our knowledge, all known Painlevé integrable models possess conformal invariant forms. Therefore, if Eq.(6) is conformal invariant, it may be a candidate for Painlevé integrable equation. Because of the singular manifold ϕ being arbitrary, it is possible to make (6) conformal invariant. Actually, in (1+1)-dimensions, Conte had changed the expansion function ϕ as $\xi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1}$ such that the expansion coefficients are conformal invariant.[10] Similarly, we can exchange for the expansion function ϕ

$$\xi = \left(\frac{\phi_{x_n}}{\phi} - \frac{\phi_{x_n x_n}}{2\phi_{x_n}}\right)^{-1}, \tag{7}$$

where x_n expresses any one of the variables $(x_0, x_1, x_2, \cdots, x_n)$, while the corresponding approximation expansion (5) is changed to

$$u = \sum_{j=0}^{M} u'_j \xi^{j+\alpha}. \tag{8}$$

From Eq.(7), one can easily prove the following identities:

$$\xi_{xi} = P_i - P_{ix_n}\xi + \frac{1}{2}(P_i S + P_{ix_n x_n})\xi^2 \tag{9}$$
$$i = 0, 1, 2, \cdots, n,$$

where the functions

$$P_i \equiv \frac{\phi_{x_i}}{\phi_{x_n}},$$
$$S \equiv \frac{\phi_{x_n x_n x_n}}{\phi_{x_n}} - \frac{3}{2}\left(\frac{\phi_{x_n x_n}}{\phi_{x_n}}\right)^2 \equiv \{\phi, x_n\}, \tag{10}$$

are all conformal invariant. In other words, P_i and S are invariant under the Möbious transformation

$$\phi \longrightarrow \frac{a + b\phi}{c + d\phi}, \quad ad \neq bc. \tag{11}$$

It is easy to see that all the expansion coefficients u'_j in (8) are conformal invariant because they are functions of P_i, S and arbitrary function $\{u'_{j_1}, u'_{j_2}, \cdots, u'_{j_{N-1}}\}$. Therefore, the equation related to the truncated expansion coefficient u'_M

$$u'_M = 0 \tag{12}$$

is conformal invariant. Whether Eq.(12) is integrable under the meaning that it possesses the Painlevé property should be verified further by means of Painlevé analysis.

According to the Lou's idea proposed in [5], the conformal invariant Eq.(12) may be Painlevé integrable. When (12) is proved to be Painlevé integrable, then Eq.(1) is solved approximately by a Painlevé integrable model in the same dimensions and the approximate solution (8) is valid up to $\xi^{M+\alpha}$.

Ⅲ. CONFORMAL INVARIANT EXPANSION FOR TWO MODELS

A. Conformal invariant expansion for (2+1)-dimensional KdVB equation

In this section, we apply the general theory of the last section to the (2+1)-dimensional nonintegrable KdVB equation[8]

$$u_t + puu_x + ru_{xx} - su_{xxx} + \epsilon u_{yy} = 0. \tag{13}$$

Substituting (8) into Eq.(13) and taking

$$\xi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1}, \tag{14}$$

we know that u of (13) can be expanded as

$$u = \sum_{j=0}^{M} u_j \xi^{j-2} \tag{15}$$

with

$$u_0 = \frac{12s}{p} \qquad \text{or} \qquad u_0 = 0, \tag{16}$$

from the leading order analysis. Starting from $u_0 = \frac{12s}{p}$, we can get the recursion relation about u_j. The first three u_j can be written

$$u_1 = \frac{12r}{5p}, \tag{17}$$

$$u_2 = -\frac{1}{25sp\phi_x^2}(25s\phi_t\phi_x - 100s^2\phi_{xxx}\phi_x + r^2\phi_x^2 + 150s^2\phi_{xx}^2 + 25\epsilon s\phi_y^2), \tag{18}$$

$$\begin{aligned} u_3 = &-\frac{1}{125p\phi_x^3}(125s\phi_x^2\phi_{xxxx} - 500s\phi_x\phi_{xxx}\phi_{xx} + 125\phi_x\phi_t\phi_{xx} + 375s\phi_{xx}^3 \\ &+ 75r\phi_x\phi_{xx}^2 - 50r\phi_x^2\phi_{xxx} - 125\phi_{xt}\phi_x^2 - r^3s^{-2}\phi_x^3 + 125\epsilon\phi_{xx}\phi_y^2 - 125\epsilon\phi_x^2\phi_{yy}). \end{aligned} \tag{19}$$

To get some approximate solutions of (13), we can take $u_M = 0$ for an appropriate M and neglect the terms higher than ξ^{M-2} for small ξ. For M=2, the ϕ equation is given by $u_2 = 0$, i.e.,

$$25s\phi_t\phi_x - 100s^2\phi_{xxx}\phi_x + r^2\phi_x^2 + 150s^2\phi_{xx}^2 + 25\epsilon s\phi_y^2 = 0, \tag{20}$$

and the corresponding approximate solution of (13) reads

$$u = \frac{u_0}{\xi^2} + \frac{u_1}{\xi}. \tag{21}$$

For $M = 3$, the function ϕ should satisfy

$$\begin{aligned} &125s\phi_x^2\phi_{xxxx} - 500s\phi_x\phi_{xxx}\phi_{xx} + 125\phi_x\phi_t\phi_{xx} + 375s\phi_{xx}^3 + 75r\phi_x\phi_{xx}^2 \\ &- 50r\phi_x^2\phi_{xxx} - 125\phi_{xt}\phi_x^2 - r^3s^{-2}\phi_x^3 + 125\epsilon\phi_{xx}\phi_y^2 - 125\epsilon\phi_x^2\phi_{yy} = 0, \end{aligned} \tag{22}$$

while the approximate solution of (13) is

$$u = \frac{u_0}{\xi^2} + \frac{u_1}{\xi} + u_2. \tag{23}$$

If we substitute (15) with $u_0 = 0$ into (13), we can find that $u_1 = 0$ and $u_2,\ u_3,\ u_4,\ u_5$ can be taken as arbitrary functions. Taking

$$u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4, \quad u_5 = a_5, \tag{24}$$

then the equation of $u_6 = 0$ reads

$$\begin{aligned} &2a_3^2p\phi_x^3 - 4\epsilon a_3\phi_x\phi_y\phi_{xy} + 2\epsilon a_3\phi_x^2\phi_{yy} + 2\epsilon a_3\phi_y^2\phi_{xx} - 3ra_3\phi_x\phi_{xx}^2 - 18sa_3\phi_{xx}^3 + 4a_4a_2p\phi_x^3 - 6sa_3\phi_x^2\phi_{xxxx} \\ &+ 24sa_3\phi_x\phi_{xx}\phi_{xxx} + 2ra_3\phi_x^2\phi_{xxx} + 24sa_4\phi_x\phi_{xx}^2 - 16sa_4\phi_x^2\phi_{xxx} + 4a_4\phi_t\phi_x^2 + 12ra_5\phi_x^3 + 4\epsilon a_4\phi_x\phi_y^2 = 0, \end{aligned} \tag{25}$$

where a_2, a_3, a_4 and a_5 are arbitrary constants. The corresponding approximate solution of (13) becomes

$$u = a_2 + a_3\xi + a_4\xi^2 + a_5\xi^3. \tag{26}$$

B. Conformal invariant expansion for (3+1)-dimensional ZKKP equation

The (3+1)-dimensional ZKKP equation can be written as

$$\partial_x(\partial_t u + c_0 u\partial_x u - c_1\partial_x^2 u + c_2\partial_x^3 u + c_3\partial_x^5 u) = -\frac{1}{2}c(\partial_y^2 u + \partial_z^2 u), \tag{27}$$

where c_2 and c_3 are dispersive coefficients having the same meaning as in the KdV and Kawahara equations, c_1 is the Burgers dissipative coefficient and the right-hand side is a diffraction term. Because of the nonintegrability of (27), it is difficult to find exact solutions of (27). Some authors treat it numerically.[9] Now, we use the method of conformal invariant expansion to study the (3+1)-dimensional nonintegrable ZKKP Eq.(27) approximately and analytically.

Substituting (8) with

$$\xi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1} \tag{28}$$

into (27), u of (27) can be expanded as

$$u = \sum_{j=1}^{M} u_j \xi^{j-4}, \tag{29}$$

where u_0 should be taken as

$$u_0 = -1680\frac{c_3}{c_0} \qquad \text{or} \qquad u_0 = 0. \tag{30}$$

Starting from $u_0 = -1680\frac{c_3}{c_0}$, we can get a recursion relation about u_j. The first four u_j can be written as

$$u_1 = 0, \tag{31}$$

$$u_2 = -\frac{280}{13c_0\phi_x^2}\left(52c_3\phi_{xxx}\phi_x - 78c_3\phi_{xx}^2 + c_2\phi_x^2\right), \tag{32}$$

$$\begin{aligned} u_3 =& \frac{280}{69c_0\phi_x^3}(207c_3\phi_{xx}^3 + c_1\phi_x^3 \\ & - 276c_3\phi_x\phi_{xx}\phi_{xxx} + 69c_3\phi_x^2\phi_{xxxx}), \end{aligned} \tag{33}$$

$$\begin{aligned} u_4 =& \frac{1}{1014c_0\phi_x^4}(141960c_3\phi_{xx}\phi_{xxxx}\phi_x^2 - 283920c_3\phi_{xxx}\phi_x\phi_{xx}^2 - 3640c_2\phi_{xxx}\phi_x^3 + 106470c_3\phi_{xx}^4 - 507c\phi_y^2\phi_x^2 \\ & + 31c_2^2c_3^{-1}\phi_x^4 + 47320c_3\phi_{xxx}^2\phi_x^2 - 507c\phi_z^2\phi_x^2 + 507\phi_t\phi_x^3 - 28392c_3\phi_{xxxxx}\phi_x^3 + 5460c_2\phi_{xx}^2\phi_x^2). \end{aligned} \tag{34}$$

To find some approximate solutions of (27), we can take $u_M = 0$ for an appropriate M and neglect the terms higher than ξ^{M-4} for small ξ. If we take $M = 4$, then the approximate solution of (27) reads $u = u_0/\xi^4 + u_1/\xi^3 + u_2/\xi^2 + u_3/\xi$. And the ϕ equation is given by $u_4 = 0$, i.e.,

$$\begin{aligned} & 141960c_3\phi_{xx}\phi_{xxxx}\phi_x^2 - 283920c_3\phi_{xxx}\phi_x\phi_{xx}^2 - 3640c_2\phi_{xxx}\phi_x^3 + 5460c_2\phi_{xx}^2\phi_x^2 + 106470c_3\phi_{xx}^4 \\ & + 31c_2^2c_3^{-1}\phi_x^4 + 47320c_3\phi_{xxx}^2\phi_x^2 - 507c\phi_z^2\phi_x^2 + 507\phi_t\phi_x^3 - 507c\phi_y^2\phi_x^2 - 28392c_3\phi_{xxxxx}\phi_x^3 = 0. \end{aligned} \tag{35}$$

Substituting (29) with $u_0 = 0$ into (27), one can find that $u_1 = u_2 = u_3 = 0$ and u_4, u_5, u_6, $,u_7$, u_8, u_9 can be taken as arbitrary functions. Taking

$$u_4 = a_0, \quad u_5 = u_7 = u_9 = 0, \quad u_6 = a_2, \quad u_8 = a_4, \tag{36}$$

the equation of $u_{10} = 0$ reads

$$\begin{aligned} & c_3a_2(220\phi_x^2\phi_{xxx}^2 + 495\phi_{xx}^4 + 360\phi_x^2\phi_{xxxx}\phi_{xx} - 1020\phi_x\phi_{xx}^2 - 72\phi_x^3\phi_{xxxxx}) + 12c_2a_2\phi_x^2\phi_{xx}^2 - 24c_2a_4\phi_x^4 \\ & - 480c_3a_4\phi_x^3\phi_{xxx} + 720c_3a_4\phi_x^2\phi_{xx}^2 - 2a_2\phi_t\phi_x^3 - ca_2\phi_z^2\phi_x^2 - ca_2\phi_y^2\phi_x^2 - 8c_2a_2\phi_x^3\phi_{xxx} - 2c_0a_0a_2\phi_x^4 = 0, \end{aligned} \tag{37}$$

where a_0, a_2, a_4 are arbitrary constants. The corresponding approximate solution of (27), after neglecting the terms higher than ξ^6 for small ξ, becomes

$$u = a_0 + a_2\xi^2 + a_4\xi^4. \tag{38}$$

IV. PAINLEVÉ INTEGRABILITY OF RESULTING EQUATIONS

It is clear that (20), (22), (25), (35) and (37) are all conformal invariant. In Ref.[5], Lou had pointed out that many kinds of quite general conformal invariant forms are Painlevé integrable. Now we hope to find out whether the resulting equations from real physical models (20), (22), (25), (35) and (37) are Painlevé integrable or not. In order to prove the Painlevé integrability of (20), (22) (25), (35) and (37), we slightly enlarge them to some more general systems. From Eq.(20), a slightly enlarged system has the form

$$\begin{aligned} & r^2H + 25sLH + 25\epsilon sV^2 - 100sHH_{xx} \\ & + 50s^2H^4 + 150s^2H_x^2 = 0, \end{aligned} \tag{39}$$

$$H_t = L_x, \tag{40}$$

$$V_t = L_y. \tag{41}$$

Eq.(39) can be obtained directly from (20) by using the transformation,

$$\phi = \exp(f), \quad f_x = H, \quad f_y = V, \quad f_t = L \tag{42}$$

as in Ref.[5], and Eqs.(40)–(41) are two consistent conditions of the transformation (42). We call (40)–(41) an enlarged system of (20) because for the transformation (42) there exists another consistent condition, $H_y = V_x$, though the relation $H_{yt} = V_{xt}$ is guaranteed by (40) and (41). In other words, only a subset of the solutions of (39)–(41) are related to the solutions of (20).

In the same way, the corresponding enlarged systems of (22) and (25) have the forms

$$\begin{aligned}&375s^3H_x^3 - 500s^3HH_xH_{xx} - 125s^3H^4H_x + 125s^2\epsilon V^2H_x - 125\epsilon s^2H^2V_y + 75s^2rHH_x^2\\&+ 25s^2rH^5 - 125s^2H^2H_t + 125s^2HLH_x + 125s^3H^2H_{xxx} - 50s^2rH^2H_{xx} - H^3r^3 = 0,\end{aligned} \tag{43}$$

$$H_t = L_x, \tag{44}$$

$$V_t = L_y \tag{45}$$

and

$$\begin{aligned}&- 4a_4a_2pH^3 + 16sa_4H^2H_{xx} - 6sa_3H^4H_x - 2\epsilon a_3V^2H_x + 18sa_3H_x^3 + 3ra_3HH_x^2\\&+ ra_3H^5 - 24sa_4HH_x^2 - 2ra_3H^2H_{xx} - 8sa_4H^5 - 2\epsilon a_3H^2V_y + 4\epsilon a_3VHH_y\\&- 24sa_3HH_xH_{xx} + 6sa_3H^2H_{xxx} - 2a_3^2pH^3 - 4a_4LH^2 - 4\epsilon a_4V^2H - 12a_5rH^3 = 0,\end{aligned} \tag{46}$$

$$H_t = L_x, \tag{47}$$

$$V_t = L_y. \tag{48}$$

And the corresponding enlarged systems of (35) and (37) have the forms

$$\begin{aligned}&H^2(141960c_3H_x^2H^2 + 507cW^2 - 283920c_3H_xH_{xxx} - 10920c_2H_x^2 - 94640c_3H_{xx}^2 + 507cV^2)\\&H^3(56784c_3H_{xxxx} + 7280c_2H_{xx} + 1014L + 33124c_3H^5 - 62c_2^2Hc_3^{-1})\\&- 212940c_3H_x^4 - 189280c_3H^5H_{xx} - 3640c_2H^6 + 567840c_3HH_{xx}H_x^2 = 0,\end{aligned} \tag{49}$$

$$H_t = L_x, \tag{50}$$

$$V_t = L_y, \tag{51}$$

$$W_t = L_z \tag{52}$$

and

$$\begin{aligned}&- 24c_2a_4H^4 - 2a_2LH^3 - ca_2V^2H^2 - 1020c_3a_2HH_xH_{xx} - 8c_2a_2H^3H_{xx} + 4c_2a_2H^6\\&+ c_3a_2(140H^5H_{xx} - 72H^3H_{xxxx} - 30H^4H_x^2 - 17H^8 + 360H^2H_{xxx}H_x + 495H_x^4 + 220H^2H_{xx}^2)\\&+ H^2(720c_3a_4H_x^2 - 480c_3a_4HH_{xx} - 2c_0a_0a_2H^2 - ca_2W^2 + 12c_2a_2H_x^2 + 240c_3a_4H^4) = 0,\end{aligned} \tag{53}$$

$$H_t = L_x, \tag{54}$$

$$V_t = L_y, \tag{55}$$

$$W_t = L_z. \tag{56}$$

For the same reason, Eqs.(49) and (53) can be obtained directly from (35) and (37) by using the transformation,

$$\begin{aligned}&\phi = \exp(f), \quad f_x = H,\\&f_y = V, \quad f_z = W, \quad f_t = L,\end{aligned} \tag{57}$$

and Eqs.(50)–(52) and (54)–(56) are three consistent conditions of the transformation (57). Now using the standard Painlevé analysis, we can prove that all the enlarged systems related to (20), (22), (25), (35) and (37) possess the Painlevé property.

After finishing the leading order analysis of (39)–

(41), we see that $H,\ L,\ V$, should be expanded as

$$H=\sum_{j=0}^{\infty}C_j\phi^{j-1},\quad L=\sum_{j=0}^{\infty}L_j\phi^{j-1},\quad V=\sum_{j=0}^{\infty}V_j\phi^{j-1},\tag{58}$$

with

$$H_0=\pm\phi_x,\quad V_0=\pm\phi_y,\quad L_0=\pm\phi_t.\tag{59}$$

Substituting (58) into (39)–(41), we know that the resonances occur at

$$j=-1,\quad 1,\quad ,1,\quad ,1\tag{60}$$

and the resonance conditions at $j=1$ read

$$-H_{0x}\phi_x+2H_0^2H_1+H_0\phi_{xx}-2H_1\phi_x^2=0,\tag{61}$$

$$H_{0t}=L_{0x},\tag{62}$$

$$V_{0t}=L_{0y}.\tag{63}$$

Obviously, (61)–(63) are satisfied identically because of (59). So the equation system (39)–(41) possesses the Painlevé property, i.e., the system (39)–(41) is Painlevé integrable and then its reduced system (20) is naturally Painlevé integrable.

Substituting (58) into (43)–(45), we can see that the resonances occur at

$$j=-1,\quad 1,\quad 1,\quad 1,\quad 2\tag{64}$$

and the resonance conditions at $j=1,\quad 2$ read

$$\begin{aligned}&rH_0^3-5H_0^2sH_{0x}+20sH_0^2H_1\phi_x+10sH_0\phi_x\phi_{xx}\\&-rH_0\phi_x^2-5sH_{0x}\phi_x^2-20sH_1\phi_x^3=0,\end{aligned}\tag{65}$$

$$V_{0t}-L_{0y}=0,\tag{66}$$

$$H_{0t}-L_{0x}=0,\tag{67}$$

$$\begin{aligned}&-5sH_0^3H_{1x}+15sH_0^3H_2\phi_x+5rH_0^3H_1+30sH_0^2H_1\phi_x\\&-20sH_0^2H_1H_{0x}-5sH_0^2\phi_{xxx}+2sH_0^2\phi_{xx}\\&+5H_0^2\phi_t-15sH_0H_2\phi_x^3+5sH_0H_{0x}\phi_{xx}+5sH_0H_{0xx}\phi_x\\&+40sH_0H_1\phi_x\phi_{xx}+5sH_0\phi_x^2H_{1x}-5H_0L_0\phi_x\\&-5rH_0H_1\phi_x^2+5\epsilon H_0V_0\phi_y-2sH_0H_{0x}\phi_x-5sH_{0x}^2\phi_x\\&-20sH_1H_{0x}\phi_x^2-30s\phi_x^3H_1-5\epsilon V_0^2\phi_x=0.\end{aligned}\tag{68}$$

Considered (59), Eqs.(65)–(68) are all satisfied; that is to say, Eq.(22) is also a Painlevé integrable model.

In the same way, substituting (58) into (46)–(48), one can see that all five resonances located at

$$j=-1,\quad 1,\quad 1,\quad 1,\quad 2\tag{69}$$

have the forms

$$\begin{aligned}&8sa_4H_0^3-ra_3H_0^3+6sa_3H_0^2H_{0x}\\&-24sa_3H_0^2H_1\phi_x+ra_3H_0\phi_x^2\\&-8sa_4H_0\phi_x^2+6sa_3\phi_x^2H_{0x}\\&+24sa_3H_1\phi_x^3-12sa_3H_0\phi_x\phi_{xx}=0,\end{aligned}\tag{70}$$

$$H_{0t}=L_{0x},\tag{71}$$

$$V_{0t}=L_{0y},\tag{72}$$

$$\begin{aligned}&-5ra_3H_0^3H_1-18sa_3H_0^3H_2\phi_x+40sa_4H_0^3H_1\\&+6sa_3H_0^3H_{1x}+24sa_3H_0^2H_1H_{0x}-36sa_3H_0^2H_1^2\phi_x\\&+6sa_3H_0^2\phi_{xxx}+16sa_4H_0^2\phi_{xx}-2ra_3H_0^2\phi_{xx}\\&-16sa_4H_0\phi_xH_{0x}-6sa_3H_0H_{0xx}\phi_x+2\epsilon a_3H_0V_0\phi_y\\&-48sa_3H_0H_1\phi_x\phi_{xx}-40sa_4H_0H_1\phi_x^2+2ra_3H_0\phi_xH_{0x}\\&-6sa_3H_0H_{0x}\phi_{xx}+18sa_3H_0H_2\phi_x^3+5ra_3H_0H_1\phi_x^2\\&+36sa_3H_1^2\phi_x^3-6sa_3H_0H_{1x}\phi_x^2+24sa_3H_1H_{0x}\phi_x^2\\&-2\epsilon a_3V_0^2\phi_x+6sa_3H_{0x}^2\phi_x=0.\end{aligned}\tag{73}$$

It is easy to see that (70)–(73) are all true. So, Eqs.(46)–(48) possess the Painlevé property, i.e., Eq.(25) is a Painlevé integrable model.

In order to check the Painlevé property of Eqs.(49)–(52) and (53)–(56), we need to expand variables $H,\quad L,\quad W,\quad V$ as

$$\begin{aligned}&H=\sum_{j=0}^{\infty}H_j\phi^{j-1},\quad L=\sum_{j=0}^{\infty}L_j\phi^{j-1},\\&W=\sum_{j=0}^{\infty}W_j\phi^{j-1},\quad V=\sum_{j=0}^{\infty}V_j\phi^{j-1},\end{aligned}\tag{74}$$

with

$$\begin{aligned}&H_0=\pm\phi_x,\quad V_0=\pm\phi_y,\\&W_0=\pm\phi_z,\quad \mathrm{L}_0=\pm\phi_t.\end{aligned}\tag{75}$$

Substituting (74) into (49)–(52), we see that the resonances occur at

$$j=-1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 2,\quad 3\tag{76}$$

and the resonance conditions at $j=1,\quad 2,\quad 3$ read

$$\begin{aligned}&14H_1H_0^4+10H_0^3\phi_{xx}-70H_0^2H_1\phi_x^2+5H_0^2H_{0x}\phi_x\\&-28H_0\phi_{xx}\phi_x^2+13H_{0x}\phi_x^3+56H_1\phi_x^4=0,\end{aligned}\tag{77}$$

$$H_{0t}=L_{0x},\tag{78}$$

$$V_{0t}=L_{0y},\tag{79}$$

$$W_{0t}=L_{0z},\tag{80}$$

$$364H_2H_0^5+1274H_0^4H_1^2-5c_2c_3^{-1}H_0^4-390H_0^3H_{1x}\phi_x+130H_0^3H_1\phi_{xx}-260H_0^3H_{0xx}-598H_0H_{0x}\phi_{xx}\phi_x$$
$$+H_0^2(1040H_1H_{0x}\phi_x+338\phi_{xx}^2+195H_{0x}^2+234\phi_x\phi_{xxx}+4030H_1^2\phi_x^2+5c_2c_3^{-1}\phi_x^2+390H_0^{-1}H_{1x}\phi_x^3$$
$$+\phi_x^2(1846H_0H_2\phi_x^2-3484H_0H_1\phi_{xx}+26H_2H_{0xx}-1144H_1H_{0x}\phi_x+65H_{0x}^2+2756H_1^2\phi_x^2-2210H_0^3H_2)=0, \tag{81}$$

$$H_0^2(390H_1H_{0x}^2+1430H_1^2H_{0x}\phi_x-390H_{1x}\phi_x\phi_{xx}+39H_{0x}\phi_{xxx}+455H_{2x}\phi_x^3+1378H_3\phi_x^4+130H_{1xx}\phi_x^2$$
$$-2002H_2\phi_{xx}\phi_x^2-104H_{0xx}\phi_{xx}-2210H_1^3\phi_x^3+572H_1\phi_{xx}^2+1274H_1^3H_0^2+936\phi_x^4H_1^3H_0^{-2}+182H_3H_0^4)$$
$$+H_0^3(-15c_2c_3^{-1}H_0H_1-39\phi_{xxxx}-195\phi_xH_{0x}^3H_0^{-3}-780H_1H_{1x}\phi_x+715H_{0x}H_2\phi_x-4810H_2H_1\phi_x^2)$$
$$+H_0(195H_0^2H_{0x}H_{1x}+15H_0H_1\phi_x^2+5c_2c_3^{-1}H_0H_{0x}\phi_x-650H_0^2H_1H_{0xx}+780H_1H_{1x}\phi_x^3-442H_{0x}H_1\phi_x\phi_{xx}$$
$$-3536H_1H_2\phi_x^4+195H_{0x}^2\phi_{xx}+104H_1H_{0xx}\phi_x^2-2782H_1^2\phi_x^2\phi_{xx}+195H_{0x}H_{1x}\phi_x-520\phi_x^2H_{0x}^2H_1H_0^{-1})$$
$$+52H_1^2H_{0x}\phi_x^3+767H_0H_2H_{0x}\phi_x^3+65H_0H_{0xx}H_{0x}\phi_x-1560H_0^4H_3\phi_x^2-455H_0^4H_{2x}\phi_x+520H_0^4H_2\phi_{xx}$$
$$+1274H_1H_2H_0^5+1300H_0^3H_1^2\phi_{xx}+39H_0^2H_{0xxx}\phi_x+546H_0^2H_1\phi_x\phi_{xxx}-5c_2c_3^{-1}H_0^3\phi_{xx}+130H_0^4H_{1xx}=0. \tag{82}$$

Eqs.(77)–(82) are also satisfied because of (75). So the (3+1)-dimensional conformal invariant model (35) possesses the Painlevé property. Using the same method as above, we can easily prove that Eq.(37) is also a Painlevé integrable model. Since the formula is too long, it is not given here.

Now, we have obtained three new (2+1)-dimensional models possessing the Painlevé integrability from the (2+1)-dimensional nonintegrable KdVB equation (13), and two new (3+1)-dimensional Painlevé integrable models from the (3+1)-dimensional nonintegrable ZKKP equation (27) through the conformal invariant expansion approach. The solutions of these models can be used to express the solutions of original equations approximately in some special ways.

V. SOLITARY WAVE SOLUTIONS

To find out some concrete approximate solutions of Eqs.(13) and (27), we should solve the Painlevé integrable models (20), (22), (25), (35) and (37). Because these models are conformal invariant, it is easy to verify that these conformal invariant equations all possess kink-type soliton solutions. For the model (20), the corresponding kink-type plane soliton solution has the form

$$\phi=\frac{A+B\exp(k_1x+k_2y+wt)}{C+D\exp(k_1x+k_2y+wt)}, \tag{83}$$

where

$$w=-\frac{1}{25sk_1}(r^2k_1^2+25s\epsilon k_2^2+50s^2k_1^4). \tag{84}$$

The corresponding ξ is given by

$$\xi=-\frac{2}{k_1}\frac{A+B\exp(k_1x+k_2y+wt)}{A-B\exp(k_1x+k_2y+wt)}. \tag{85}$$

The related approximate solution of (13) reads

$$u=\frac{u_0}{\xi^2}+\frac{u_1}{\xi}$$
$$=\frac{3sk_1^2}{p}\left(1-\operatorname{sech}^2\frac{1}{2}\left(k_1x+k_2y+wt+\ln\left(\frac{B}{A}\right)^{1/2}\right)\right)$$
$$+\frac{6rk_1}{5p}\tanh\frac{1}{2}\left(k_1x+k_2y+wt+\ln\left(\frac{B}{A}\right)^{1/2}\right). \tag{86}$$

If the arbitrary constant k_1 is taken as $r/5s$, u given by (86) becomes an exact solution of (13). We can also verify that

$$\phi=\frac{A+B\exp(\frac{r}{5s}x+k_2y+wt)}{C+D\exp(\frac{r}{5s}x+k_2y+wt)} \tag{87}$$

is a kink-type plane solitary wave solution of (22). The corresponding ξ is given by

$$\xi=-\frac{10s}{r}\frac{A+B\exp(\frac{r}{5s}x+k_2y+wt)}{A-B\exp(\frac{r}{5s}x+k_2y+wt)}. \tag{88}$$

The approximate solution of (13)

$$u=\frac{u_0}{\xi^2}+\frac{u_1}{\xi}+u_2$$
$$=-\frac{625\epsilon r^3k_2^2+3r^4+125rs^2w}{25sr^2p}$$
$$=\frac{3r^2}{25ps}\left(1-\operatorname{sech}^2\frac{1}{2}\left(\frac{r}{5s}x+k_2y+wt+\ln\left(\frac{B}{A}\right)^{1/2}\right)\right)$$
$$+\frac{6r^2}{25ps}\tanh\frac{1}{2}\left(\frac{r}{5s}x+k_2y+wt+\ln\left(\frac{B}{A}\right)^{1/2}\right) \tag{89}$$

is just an exact solution of Eq.(13). It is easy to check that

$$\phi = \frac{A - B\exp(k_1x + k_2y + wt)}{C + D\exp(k_1x + k_2y + wt)} \tag{90}$$

is a solution of (25) and the corresponding ξ is given by

$$\xi = -\frac{2}{k_1}\frac{A - B\exp(k_1x + k_2y + wt)}{A + B\exp(k_1x + k_2y + wt)}. \tag{91}$$

In this case, the approximate solution of Eq.(13) can be expressed as

$$\begin{aligned} u =& a_2 + a_3\xi + a_4\xi^2 + a_5\xi^3 \\ =& a_2 - \frac{2a_3}{k_1}\tanh\frac{1}{2}(k_1x + k_2y + wt) \\ &+ \frac{4a_4}{k_1^2}\tanh^2\frac{1}{2}(k_1x + k_2y + wt) \\ &+ \frac{8a_5}{k_1^3}\tanh^3\frac{1}{2}(k_1x + k_2y + wt). \end{aligned} \tag{92}$$

If the arbitrary constants in (92) are selected as

$$k_1 = \frac{r}{5s}, \quad a_5 = 0, \quad a_3 = \frac{3r^3}{125ps^2}, \quad a_4 = \frac{3r^4}{2500ps^3}, \tag{93}$$

then u given by (92) is an exact solution of (13). For the same reason, it is easy to prove that

$$\begin{aligned} \phi =& \frac{A + B\exp(k_1x + k_2y + k_3z + wt)}{C + D\exp(k_1x + k_2y + k_3z + wt)}, \\ w =& -\frac{1}{1014k_1}(507ck_3^2 + 33124c_3k_1^6 \\ &- 62c_2^2k_1^2c_3^{-1} - 3640c_2k_1^4 + 507ck_2^2) \end{aligned} \tag{94}$$

is a solution of the (3+1)-dimensional Painlevé integrable model (35) and the corresponding ξ is given by

$$\xi = -\frac{2}{k_1}\frac{A + B\exp(k_1x + k_2y + k_3z + wt)}{A - B\exp(k_1x + k_2y + k_3z + wt)}. \tag{95}$$

The approximate solution u of Eq.(27) can be written as

$$\begin{aligned} u =& \frac{u_0}{\xi^4} + \frac{u_1}{\xi^3} + \frac{u_2}{\xi^2} + \frac{u_3}{\xi} \\ =& -105c_3k_1^4\tanh^4\frac{1}{2}(k_1x + k_2y + k_3z + wt) \\ &+ \frac{70k_1^2(126c_3k_1^2 - c_2)}{13c_0}\tanh^2\frac{1}{2}(k_1x \\ &+ k_2y + k_3z + wt) + \frac{140c_1k_1}{69c_0}\tanh\frac{1}{2}(k_1x \\ &+ k_2y + k_3z + wt). \end{aligned} \tag{96}$$

Obviously,

$$\phi = \frac{A - B\exp(k_1x + k_2y + k_3z + wt)}{C + D\exp(k_1x + k_2y + k_3z + wt)}, \tag{97}$$

where

$$\begin{aligned} w =& -\frac{1}{2k_1a_2}(17c_3a_2k_1^6 - 4c_2a_2k_1^4 + 24c_2a_4k_1^2 \\ &- 240c_3a_4k_1^4 + ca_2k_2^2 + ca_2k_3^2 + 2c_0a_0a_2k_1^2) \end{aligned}$$

is a solution of Eq.(37). The corresponding ξ can be written as

$$\begin{aligned} \xi =& -\frac{2}{k_1}\frac{A - B\exp(k_1x + k_2y + k_3z + wt)}{A + B\exp(k_1x + k_2y + k_3z + wt)} \\ =& -\frac{2}{k_1}\tanh\frac{1}{2}(k_1x + k_2y + k_3z + wt). \end{aligned} \tag{98}$$

The related approximate solution of (27) reads

$$\begin{aligned} u =& a_0 + \frac{4a_2}{k_1^2}\tanh^2\frac{1}{2}(k_1x + k_2y + k_3z + wt) \\ &+ \frac{16a_4}{k_1^4}\tanh^4\frac{1}{2}(k_1x + k_2y + k_3z + wt). \end{aligned} \tag{99}$$

Generally, the approximate procedure is valid for ξ being small. In the plane soliton case, ξ given by (85) and (95) are small means u given by (86) and (96) are valid near the soliton centre, while the fact that ξ given by (91) and (98) are small means that u given by (92) and (99) are only valid far away from the soliton centre.

VI. SUMMARY AND DISCUSSION

In this paper, we have proposed a simple method to solve higher dimensional nonlinear problems by means of the Painlevé integrable models in the same dimensions. Taking the (2+1)-dimensional KdVB equation and (3+1)-dimensional ZKKP equation as concrete examples, we have obtained some new (2+1)-dimensional and (3+1)-dimensional Painlevé integrable models. Because the resulting equations possess conformal invariance, the plane solitary wave solutions of the model can be obtained easily. Generally the solitary waves are approximate for the original model. Certainly, if the arbitrary constants are selected suitably, the approximate solutions become exact.

We can obtain two forms of approximate solution according to the recursion relation starting from $u_0 \neq 0$ or $u_0 = 0$. The first form is valid near the soliton centre and the second is valid far away from the soliton centre. In fact, the second form of approximate solution can be considered as the usual tanh function expansion form when the solution of the resulting equation is taken as a solitary wave solution. For the (2+1)-dimensional KdVB equation, the exact

solution in some special case obtained here is the same as the exact solution obtained in Refs.[8] and [10] by using the homogeneous balance method and using an automated method.

In Refs.[5]–[7], Lou has proposed many types of conformal invariant equations which are Painlevé integrable. Two important problems are left in these papers: (i) Can one find physical applications of those models or can one find some types of models with conformal invariance and Painlevé property from real physical models? (ii) Are those Painlevé integrable models integrable under other meanings? In this paper, we have offered a positive answer for the first question. Using the conformal invariant expansion to any real physical model in any dimensions, we can get some approximate solutions by means of the Painlevé models. The second question is still open. It is worth studying further whether the models obtained here are integrable under other traditional meanings.

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References

[1] Ablowitz M J and Clarkson P A 1991 Solitons Nonlinear Evolution Equations and Inverse Scattering, *London Mathematical Society Lecture Note* Series 149 (Cambridge University Press); Ward R S 1985 *Phil. Trans. R. Soc. Lond.* A **315** 451

[2] Newton R 1976 *Phys. Rev. Lett.* **43** 541; Newton R 1982 *J. Math. Phys.* **23** 693; Faddeev L D 1966 *Soviet. Phys. Dokl.* **11** 209; Nachman A I and Ablowitz M J 1984 *Stud. Appl. Math.* **71** 243; Beals R and Coifman R R 1986 *Physica* D **18** 242; Gu C H 1992 *Lett. Math. Phys.* **26** 199

[3] David D, Kamran N, Levi D and Winternitz P 1986 *J. Math. Phys.* **27** 1225; Lou S Y and Hu X B 1994 *J. Phys.* A **27** L207; Lou S Y 1993 *Phys. Rev. Lett.* **71** 4099

[4] Lou S Y, Yu J and Lin J 1995 *J. Phys.* A **28** L191; Lou S Y, Lin J and Yu J 1995 *Phys. Lett.* A **201** 47

[5] Lou S Y 1998 *J. Math. Phys.* **39** 2112

[6] Lou S Y 1998 *Phys. Rev. Lett.* **80** 5027

[7] Lou S Y and Xu J J 1998 *J. Math. Phys.* **39** 5364

[8] Wang M G 1996 *Phys. Lett.* A **213** 279

[9] Karpman V I and Belashov V Yu 1991 *Phys. Lett.* A **154** 140

[10] Conte R 1989 *Phys. Lett.* A **140** 383