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NEW EXACTLY SOLVABLE SUPERSYMMETRIC PERIODIC POTENTIALS*

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Using the formalism of supersymmetric quantum mechanics, we give an exact solution for a family of one-dimensional periodic potentials, which are the supersymmetric partners of the potential proportional to the trigonometric function $\cos(2x)$ such that the Schrödinger equation for this potential is named the Mathieu equation mathematically. We show that the new potentials are distinctly different from their original ones. However, both have the same energy band structure. All the potentials obtained in this paper are free of singularities.

Keywords: periodic potential, exact solution, Mathieu's equation, supersymmetric quantum mechanics
PACC: 0365G, 1130P, 7320D

I. INTRODUCTION

The energy spectrum of electrons on a periodic potential of a lattice is of central importance in condensed matter physics. Unfortunately, there are very few exactly solvable periodic potentials in quantum mechanics, even in one dimension. The Lamé potential is one, and the Mathieu equation with the potential proportional to the trigonometric function cos $(2x)$ is another.

In recent years, the generation of a new potential from the Lamé potential has been studied in detail by many authors,[1−3] by using the well-known supersymmetric quantum mechanics (SUSY QM). The Lamé potential reads $V_-(x) = a(a+1)sn^2(x,m)$, where $sn(x,m)$ is the 'Jacobian elliptic sinus' with parameter $m = k^2(0 \le m \le 1)$, and $a = 1, 2, 3 \ldots$. By using the SUSY QM, the partner potential of $V_-(x)$ has been obtained. Many of the new potentials obtained in this way are different from the original one in shape, so a large number of new analytically solvable potentials have successfully been obtained.[1−3] The aim of this paper is to derive a new family of exactly solvable periodic potentials, which are the partner of the periodic potential expressed by trigonometric function.

Before actually solving the Schrödinger equation for the new period potential, let us make a few general comments. We shall choose units with $\hbar$=1, and take the particle mass to be 1/2 throughout this paper.

Now we consider the following one-dimensional trigonometric potential

$$V_-(x) = 2q\cos(2x). \tag{1}$$

The Schrödinger equation $-\partial_x^2\Psi + V_-(x)\Psi = E\Psi$ for this potential corresponds to an exactly solvable equation called the Mathieu equation

$$\partial_x^2\Psi + (E - 2q\cos(2x))\Psi = 0, \tag{2}$$

where E and q are parameters corresponding to the energy of a particle and the strength of the potential, respectively. The solution of the above equation is called the Mathieu function, which is discussed in detail below. Based on this equation, and by means of the SUSY QM transformation, we find that the exact solution for a family of new potentials can also be obtained, which are the partner of $V_-(x)$, denoted by $V_+(x)$

$$V_+(x) = -2q\cos(2x) + 2E_0 + 2\left(\frac{\partial_x(\mathrm{Ce}(E_0,q,x))}{\mathrm{Ce}(E_0,q,x)}\right)^2, \tag{3}$$

where E_0 is the ground-state energy, and $\mathrm{Ce}(E_0,q,x)$ is one of the Mathieu functions, defined to be even in x, with parameter $E = E_0$. The aim of this paper is to derive $V_+(x)$ from $V_-(x)$.

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This paper is arranged as follows. In the next section we discuss the SUSY QM for the period potential. In section III, we first discuss the solution of the Mathieu equation, and then derive the exact solution of the new potential expressed by Eq.(3).

II. SUPERSYMMETRIC QUANTUM MECHANICS FOR ONE-DIMENSIONAL PERIOD POTENTIAL

SUSY QM on the real line can be summarized as follows.[2] Two Hamiltonians, H_+ and H_- correspond to an isospectral pair of potentials $V_\pm(x)$ defined in terms of the 'superpotential', $W(x)$, as $(W'(x) = \mathrm{d}W(x)/\mathrm{d}x)$

$$V_\pm(x) = W^2(x) \pm W'(x).$$

The Hamiltonians may be factorized into the products of Hermitian conjugate operators

$$H_+ = AA^+, H_- = A^+A, \tag{4}$$

where A and A^+ are two operators defined by

$$A = \partial_x + W(x), A^+ = -\partial_x + W(x). \tag{5}$$

The ground-state eigenfunctions for $H_\pm$ are (non-normalized)

$$\Psi_0^{(\pm)} = \exp\left[\pm \int^x W(x')\mathrm{d}x'\right]. \tag{6}$$

The relations between the two excited eigenfunctions $\Psi_n^+(x)$ and $\Psi_n^-(x)$ $(n = 1, 2, 3\ldots)$ are

$$\Psi_n^{(+)}(x) = A\,\Psi_n^{(-)}(x). \tag{7}$$

Thus by starting from an exactly solvable periodic potential, say $V_-(x)$, one can derive another isospectral periodic potential $V_+(x)$. Therefore, if $V_+(x)$ is not self-isospectral,[2] then $V_+(x)$ is a new solvable periodic potential.

III. TRIGONOMETRIC POTENTIAL AND ITS SUPERSYMMETRIC PARTNER

The supersymmetric quantum mechanics formalism discussed above will now be applied to the periodic potential $V_-(x)$=$2q\cos(2x)$, and its supersymmetric partner potential can be readily computed. As mentioned above, the Schrödinger equation for the potential in Eq.(1) is expressed by Eq.(2), and the eigenfunctions $\Psi^{(-)}$ are just the solutions of the Mathieu equation,[4]

$$\Psi^{(-)} = C_1\mathrm{Ce}(E, q, x) + C_2\mathrm{Se}(E, q, x), \tag{8}$$

where C_1 and C_2 are two arbitrary constants. The function $\mathrm{Ce}(E, q, x)$ is defined to be even in x, while $\mathrm{Se}(E, q, x)$ is odd; both of these are called Mathieu functions. Some of the solutions, for certain values of E, are stable, while others are unstable. The lines marking the boundaries between stable and unstable regions are shown in Fig.1. These boundary solutions are called the 'Mathieu characteristic values' denoted by $\mathrm{ce}_n(x, q)$ and $\mathrm{se}_n(x, q)$ with n being an integer or rational number, which are often denoted in short by a_n and b_n, respectively.

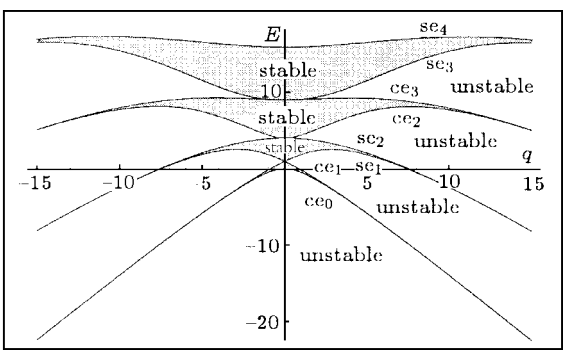


Fig.1. The q-E plane showing the character of the solutions of the Mathieu equation for different values of q.

The existence of unstable solutions for the Mathieu equation implies that energies for the stable states have a band structure, that is, the stable solutions are distributed in some intervals of E. The bands for $q > 0$ are arranged in the series from below to top as $[a_0,\ a_1]$, $[b_1,\ b_2]$,...$[a_j,\ a_{j+1}]$, $[b_{j+1},\ b_{j+2}]$,...,$(j = 0,1,2...)$. These values a_j and b_j are related to the value of the parameter q, which can be found in many mathematical physics books,[5] or can be calculated by a numerical method.

Now, we first calculate the superpotential $W(x)$. By defining a new function $K_-(x) \equiv a_0 - V_-(x)$=$E_0 - 2q\cos(2x)$, one can rewrite the Mathieu equation (2) as

$$\partial_x^2\,\Psi + K_-(x)\,\Psi = 0. \tag{9}$$

The superpotential $W(x)$ for this equation with the 'effective potential' $K_-(x)$ satisfies the Riccati equation

$$-W'(x) + W^2(x) = K_-(x). \tag{10}$$

To solve this equation, one can substitute $W(x)$ by a new function $y(x)$ defined by the following differential equation

$$W(x) = -(\partial_x y)/y.$$

Substituting this into Eq.(10), for $y(x) \neq 0$, one finally obtains the following equation

$$\partial_x^2 y(x) + K_-(x)y(x) = 0.$$

By recalling the definition $K_-(x) \equiv E_0 - 2q\cos(2x)$, one finds that the above equation is just the Mathieu equation for the new valuable $y(x)$ and with $E = E_0$, one can therefore write down the solution of the above equation straightforwardly, i.e. $y(x) = C_1\mathrm{Ce}(E_0, q, x) + C_2\mathrm{Se}(E_0, q, x)$. Thus, we finally obtain the explicit expression of $W(x)$,

$$W(x) = -\frac{\partial_x(C_1\mathrm{Ce}(E_0, q, x) + C_2\mathrm{Se}(E_0, q, x))}{C_1\mathrm{Ce}(E_0, q, x) + C_2\mathrm{Se}(E_0, q, x)}. \quad (11)$$

This method of obtaining this form of solution was recently discussed in detail by one of the authors (K J Liu), and one can find this detailed discussion in Ref.[6].

Now we are able to derive the new potential $V_+(x)$. To simplify our discussion, we shall discuss $W(x)$ in Eq.(11) with C_2=0, that is

$$W(x) = -\partial_x(\mathrm{Ce}(E_0, q, x))/\mathrm{Ce}(E_0, q, x). \quad (11-1)$$

The partner potential $V_+(x)$ can be calculated as follows. By defining $K_+(x) \equiv E_0 - V_+(x)$, and $K_+(x) = \partial_x W + W^2(x)$, one finally obtains a new exactly solvable periodic potential

$$V_+(x) = V_-(x) + 2W'(x) = -2q\cos(2x) + 2E_0 + 2\left(\frac{\partial_x(\mathrm{Ce}(E_0, q, x))}{\mathrm{Ce}(E_0, q, x)}\right)^2. \quad (12)$$

The shapes of the two potentials, with q=4/5 and q=4/3, are shown in Fig.2, which are different from that of the original potential $V_-(x) = 2q\cos(2x)$; however, both have the same energy spectrum.

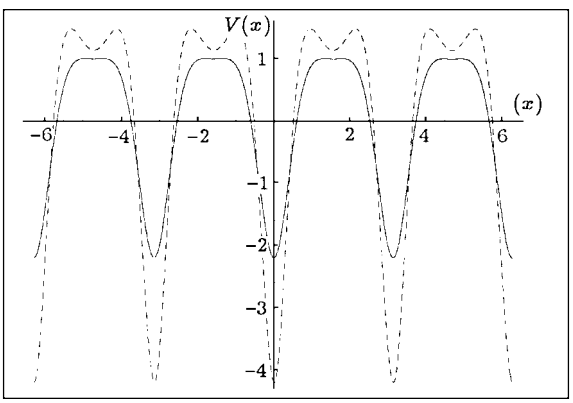


Fig.2. A new kind of periodic potential $V_+(x)$ in Eq.(12), with q=4/5 (solid line) and q=4/3 (dashed line).

The function $V_+(x)$ is a singularity-free function in the interval $(-\infty, \infty)$. The wavefunctions $\Psi_n^+(x)$ $(n = 0, 1, 2\ldots)$ can be calculated from Eqs.(6) and (7). To shorten this paper, we do not write them in detail here.

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