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LATTICE BOLTZMANN EQUATION MODEL IN THE CORIOLIS FIELD*

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In a large-scale field of rotational fluid, various unintelligible and surprising dynamic phenomena are produced due to the effect of the Coriolis force. The lattice Boltzmann equation (LBE) model in the Coriolis field is developed based on previous works.[1−4] Geophysical fluid dynamics equations are derived from the model. Numerical simulations have been made on an ideal atmospheric circulation of the Northern Hemisphere by using the model and they reproduce the Rossby wave motion well. Hence the applicability of the model is verified in both theory and experiment.

Keywords: distribution function, Coriolis force, geophysical hydrodynamic equations
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I. INTRODUCTION

The lattice Boltzmann equation (LBE) model is a new kind of computational method developed in recent years, using computer technology to solve the Boltzmann equation. The model gives excellent results in numerical simulation of flow fields, such as the fields of incompressible fluid, compressible fluid, thermal convection, multi-phase flow, electromagnetic flow and fluids with chemical reaction.[5−7] But the LBE model in the Coriolis force field has not been studied in detail previously.

It is well known that the Coriolis force in a large-scale field of rotational fluid is a very important force that cannot be neglected. More attention should be paid to the role of the Coriolis force, particularly in numerical simulations on atmospheric circulation, ocean general circulation, the mechanics of water-driven landscape erosion, astrophysics, space plasma dynamics, launch of astronautical rockets, and aerospace areas. We develop the LBE model in the Coriolis force field just to satisfy the requirements in research and application of several aspects mentioned above. It can be predicted that the model will also have applications in geodynamics. For example, by using the model we can simulate mantle convection and analyse the effect of transporting characters of mass and energy on crustal plate motion, earthquakes and so on.

II. LBE MODEL AND GEOPHYSICAL FLUID DYNAMICS EQUATION

The Boltzmann equation is the most fundamental equation governing the motion of particle populations. Being a complicated differential and integral equation, it is very difficult to solve directly. In general, the numerical computation uses the Bhatnagar–Gross–Krook (BGK) Boltzmann equation which simplifies the collision term.[8] In the Coriolis force field, the BGK equation is taken as follows

$$f^c_{ki}(r_\alpha, t) = f_{ki}(r_\alpha, t) - \Omega_{ki}, \tag{1}$$

$$f_{ki}(r_\alpha + c_{ki\alpha}\delta t, t+\delta t) = f^c_{ki}(r_\alpha, t) + F_{gki\alpha} - F_{cki\alpha}, \tag{2}$$

$$\Omega_{ki} = \frac{1}{\tau}[f_{ki}(r_\alpha, t) - f^{(0)}_{ki}(r_\alpha, t)],$$

$$F_{cki\alpha} = 2\delta t(\boldsymbol{\omega} \times \boldsymbol{V}) \cdot \boldsymbol{k}_\alpha \frac{f_{ki}}{c_{ki\alpha}},$$

$$F_{gki\alpha} = \delta t \eta \frac{f_{ki}}{c_{ki\alpha}} g;$$

$$i = 0, 1, 2, \cdots, b; \quad k = 1, 2, 3, \cdots, n.$$

Eq.(1) represents the collision between particles and Eq.(2) represents the movement of particles, where f_{ki} is the distribution function of particle velocity, and f^c_{ki} is the distribution function of particle velocity after collision. The subscript k represents the types of fluid, i is the number of moving directions of particles, and α represents the coordinates index. Ω_{ki} is

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the collision operator, and τ is the dimensionless relaxation time coefficient. $c_{ki\alpha}$ is the particle velocity which has the same modulus in all moving directions $|c_{ki\alpha}| = c_k$=const. c_k also represents the length of the discrete lattice, δt is the discrete time size, $F_{gki\alpha}$ represents the buoyancy effect, g is the gravitational acceleration, η is the buoyancy strength coefficient, $F_{cki\alpha}$ represents the Coriolis force effect, ω is the angular velocity of the rotating coordinate system, $\boldsymbol{v}$ is the fluid velocity relative to the rotating coordinate system, and $f_{ki}^{(0)}$ is the equilibrium velocity distribution function of local particles. To solve the BGK equation, one must obtain the equilibrium velocity distribution $f_{ki}^{(0)}$. Our study adopts the Maxwell equilibrium velocity distribution $f_{ki}^{(0)}$ with third-order accuracy[9,10]

$$\begin{aligned} f_{ki}^{(0)} =& W_i\rho_k(1 - 2E_k c_{ki\alpha}u_\alpha \\ &+ 2E_k^2 c_{ki\alpha}c_{ki\beta}u_\alpha u_\beta + E_k u^2 - 2E_k^2 c_{ki\alpha}u_\alpha u^2 \\ &- \frac{4}{3}E_k^3 c_{ki\alpha}c_{ki\beta}c_{ki\chi}u_\alpha u_\beta u_\chi). \end{aligned} \tag{3}$$

For the $D2Q9$ model, the weight W_i, the factor E_k and the relationship between the internal energy e_k and the particle velocity are found in Ref.[9]

$$W_0 = (18 - 5D)/18;\ \ W_1 = 2D/9b_1;$$

$$W_2 = D/18b_2;\ \ E_k = -D/4e_k;\ \ c_k^2 = 6e_k/D,$$

where D is the space dimension, $b_1 = b_2 = 4$ is the direction number of the lattice, and W_0, W_1 and W_2 are the weights of stationary and moving particles, respectively.

Using the Taylor and Chapman–Enskog expansions,[7] the following equation can be derived from Eq.(2)

$$\varepsilon : \frac{\partial f_{ki}^{(0)}}{\partial t} + c_{ki\alpha}\frac{\partial f_{ki}^{(0)}}{\partial r_\alpha} = \frac{F_{gki\alpha}}{\delta t} - \frac{F_{cki\alpha}}{\delta t} - \frac{1}{\tau\delta t}f_{ki}^{neq}, \tag{4}$$

$$\begin{aligned} \varepsilon^2 :& \frac{\partial f_{ki}^{(0)}}{\partial t} + c_{ki\alpha}\frac{\partial f_{ki}^{(0)}}{\partial r_\alpha} + \left(1 - \frac{1}{2\tau}\right) \\ &\left(\frac{\partial f_{ki}^{neq}}{\partial t} + c_{ki\alpha}\frac{\partial f_{ki}^{neq}}{\partial r_\alpha}\right) \\ =& \frac{F_{gki\alpha}}{\delta t} - \frac{F_{cki\alpha}}{\delta t} - \frac{1}{\tau\delta t}f_{ki}^{neq}. \end{aligned} \tag{5}$$

Here $f_{ki}^{neq} \to \varepsilon$ is a non-equilibrium component. From Eqs.(4), (5) and the corresponding conservation law,[1,3] the geophysical hydrodynamic equations can be derived

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\alpha)}{\partial r_\alpha} = 0. \tag{6}$$

$$\begin{aligned} &\rho\frac{\partial u_\alpha}{\partial t} + \rho u_\alpha\frac{\partial u_\beta}{\partial r_\alpha} \\ &= -\frac{\partial P}{\partial r_\alpha} - 2\rho(\boldsymbol{\omega}\times\boldsymbol{V}) + \delta\rho g\boldsymbol{k} + \frac{\partial F_\mu}{\partial r_\alpha}, \end{aligned} \tag{7}$$

$$\begin{aligned} P =& \frac{2}{D}\rho e; \Delta\rho = \eta\rho = \rho - \rho_0; \gamma = \frac{2e\delta t}{D}\left(\tau - \frac{1}{2}\right), \\ F_\mu =& \gamma\frac{\partial \rho u_\alpha}{\partial r_\beta} + \frac{2e\delta t}{D}\left(\frac{1}{2} - \tau\right)\frac{\partial \rho u_\alpha}{\partial r_\alpha} \\ &+ \frac{2\rho\delta t}{D}\left(\tau - \frac{1}{2}\right)\frac{\partial e}{\partial t}\delta_{\alpha\beta} + \delta t\left(\tau - \frac{1}{2}\right)\frac{\partial \rho u_\alpha u_\beta}{\partial t}, \\ &\rho\frac{\partial e}{\partial t} + \rho u_\alpha\frac{\partial e}{\partial r_\alpha} - \delta t\left(\tau - \frac{1}{2}\right)\frac{\partial}{\partial r_\alpha} \\ &\cdot\left(\frac{2}{D}\rho e\frac{\partial e}{\partial r_\beta} + \rho u_\alpha\frac{\partial e}{\partial t} + \rho u_\alpha u_\beta\frac{\partial e}{\partial r_\beta}\right) = 0, \end{aligned} \tag{8}$$

$$\kappa = \frac{2e\delta t}{D}\left(\tau - \frac{1}{2}\right);\ \ \mathrm{Pr} = \frac{\gamma}{\kappa} = 1,$$

where ρ_0 is the fluid density in the static equilibrium state, $\Delta\rho$ is the deviation of density, $\boldsymbol{k}$ is the vertical unit vector, γ is the kinematic viscosity coefficient, F_μ is the viscous stress, κ is the diffusion coefficient, and Pr is the Prandtl constant. The relationship between the deviation of density $\eta\rho$ and Grashof Gr can be derived[9]

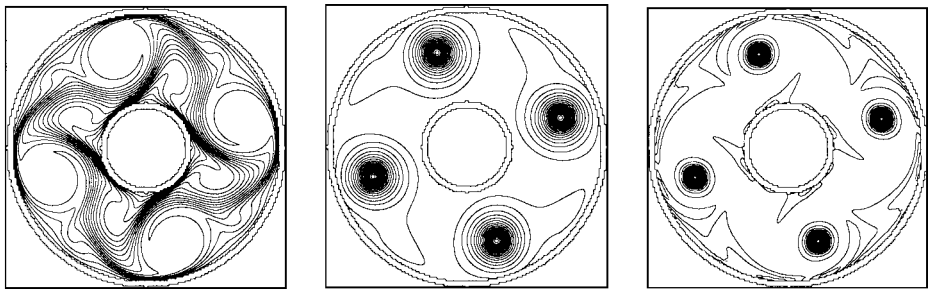
$$F_{gri_\alpha} = +\frac{Gr\rho_0 c_r(\tau - 0.5)^2 f_{ri}}{9\rho N^3 c_{ri\alpha}};$$

$$F_{gbi_\alpha} = -\frac{Gr\rho_0 c_b(\tau - 0.5)^2 f_{bi}}{9\rho N^3 c_{bi\alpha}},$$

where N is the discrete number of lattices.

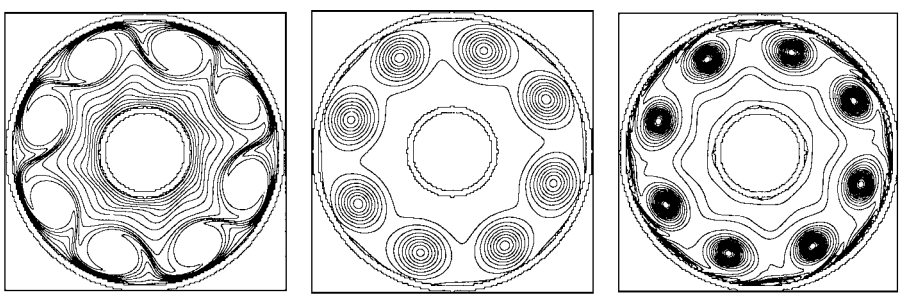
III. NUMERICAL SIMULATION

To simulate the rotational effect of geophysical fluid and verify the applicability of this LBE model, the numerical computation makes use of a typical example of an ideal atmospheric circulation in the Northern Hemisphere. The outer circle in the figure denotes the equator which is the high-temperature zone, and the inner circle similar to the Arctic is the low-temperature zone. Numerical results with wavenumbers four and eight are given respectively in Figs.1 and 2. The results show that the wavenumber increases with the rotation velocity in agreement with the Rossby wave analytical solution.



(a) internal energy contour (b) pressure contour (c) vortex contour

Fig.1. Integral T=5000 steps. Computing conditions: angular rotating rate ω = 0.003rad/sec; dimensionless relaxation time coefficient τ=0.55; Grashof number: Gr=1.0 × 10^5; Prandtl number Pr=1.0. Characteristic dimensions: inner diameter/outer diameter = 50/160 (number of discrete lattices).



(a) internal energy contour (b) pressure contour (c) vortex contour

Fig.2. Integral steps T = 5000. Computing conditions: angular rotating rate ω = 0.003red/sec; dimensionless relaxation time coefficient τ=0.55; Grashof number: Gr=1.0 × 10^5; Prandtl number Pr=1.0. Characteristic dimensions: inner diameter/outer diameter = 50/160 (number of discrete lattices).

IV. CONCLUSION

The Coriolis force is introduced into the BGK Boltzmann equation and the lattice Boltzmann equation model in rotational flow field is developed. Geophysical hydrodynamic equations are derived from the lattice Boltzmann model with the Coriolis force. It is explained in theory that the model can be applied in the simulation of thermal and dynamical fields in rotational systems. The numerical simulation provides flowing behaviour of large-scale ideal geostrophic flow, and the reliability and applicability of the lattice Boltzmann equation model with the Coriolis force are also verified in the numerical experiment.

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