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KINETICS OF THE WAVETRAIN IN THE TWO-VARIABLE OREGONATOR MODEL*

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In this paper we first discuss the asymptotic behaviours of solitary impulses in reaction–diffusion equations of Oregonator under the equal, single and double diffusion conditions, respectively. Then we obtain the asymptotic equations of motion of wavetrains for this model by a superposition of solitary impulses, and finally we analyse the stability of wavetrains.

Keywords: excitable media, solitary impulse, wavetrain
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I. INTRODUCTION

In recent years, much effort has been focused on the understanding of patterns in excitable media (EM).[1−4] Perhaps the most famous example of EM is the Belousov–Zhabotinsky reaction since it leads to a variety of nonequilibrium spatial structures and serves as a paradigm for pattern formations in EM.[5] The simplest of such patterns is the travelling wave, in which regions of excited and quiescent reagents move uniformly through space. Understanding this structure would seem to be a necessary first step towards a complete picture of more complex structures such as the target,[6] the rotating spiral,[7] or in three dimensions, the scroll.[8,9]

In EM, the travelling-wave solutions usually consist of those that appear pulse-like or particle-like. Along with shocks (or fronts) these localized structures are among the simplest solutions found in nonlinear partial differential equations. It is worth exploring their usefulness as building blocks of more complex solutions. In this paper we study trains of such pulses.

Starting from the work of Keener and coworkers,[10,11] there has recently been considerable progress in finding some travelling-wave solutions in certain tractable models, such as the FitzHugh–Nagumo equation[12] and the piecewise-linear Oregonator.[13] On the other hand, there are many examples in which we usually do not know how to deal with such creation and destruction processes of solitary structures as a pulse tends to split into two and to develop important amplitude and velocity variations during strong interaction.[14]

A solution in the form of an individual pulse may be relatively easy to find numerically, but accurate solutions representing trains of many pulses require much effort. Of course, for some integrable systems, pulse trains can be obtained as solutions by the nonlinear superposition achieved in the inverse scattering transform.[15] But here we are interested in the reaction–diffusion equations of the two-variable Oregonator of EM.

The model can be described as follows

$$\frac{\partial U}{\partial t} = F(U) + D\frac{\partial^2 U}{\partial x^2}, \tag{1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, F(U) = \begin{pmatrix} f(U) \\ g(U) \end{pmatrix},$$

$$f(U) = \frac{1}{\varepsilon}\left[u(1-u) - fv\frac{u-\mu}{u+\mu}\right],$$

$$g(U) = u - v, D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

in which parameters satisfy: $0 < \varepsilon < 1$, $0 < \mu < 1$, $0 < f < +\infty$.

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Equation (1) can display various kinds of complicated behaviours.[3−5] The particular behaviour that we are going to study here comes from the interactions of many coherent structures or solitary waves. The asymptotics of this problem relies on the interactions among pulses which become quite weak when the interpulse spacing is large. When we take advantage of this weak coupling, we are able to replace Eq.(1) by a solution of ordinary differential equations (ODE) for the motion of the individual pulse (see below). Thus we reduce Eq.(1) to a simple N-body problem, whose particles are the solitary structures we shall classify in section II. Then in section III, we outline the reductions that lead to the equations of motion of these structures when we allow for nearest-neighbour interactions and neglect tidal distortions of pulses. In section IV, we discuss the stability of wavetrains.

II. SOLITARY IMPULSE

Let $U^* = \begin{pmatrix} u_+ \\ u_+ \end{pmatrix}$ be a solution of Eq.(1), representing the rest state of the medium,
where

$$u_+ = \frac{1 - f - \mu + \sqrt{(1 - f - \mu)^2 + 4\mu(1 + f)}}{2}.$$

It is assumed that Eq.(1) has a solitary-wave solution $S\left(t - \dfrac{x}{c_0}\right)$, propagating at a constant speed c_0. In Ref.[3], we have given an algebraic equation satisfied by c_0 in the thin boundary layer, that is

$$-\frac{18}{3125 D_1^2} c_0^6 + \frac{1 - 4 f v_0}{10 D_1} c_0^2 + \frac{5 \mu v_0}{3} = 0, \qquad (2)$$

where v_0 is the position of the wavefront.

A key feature in our analysis that follows is related to the asymptotic behaviour of $S(\tau)$ as $\tau = t - \dfrac{x}{c_0} \to \pm\infty$. Notice that a solitary-wave solution of Eq.(1) satisfies the ODE

$$V' = F(V) + c_0^{-2} D V'', \qquad (3)$$

where the prime denotes differentiation with respect to the argument τ and $F(V) \equiv F(V - U^*)$. More specifically, $S(\tau)$ describes an orbit of Eq.(3) which is bi-asymptotic to the rest state U^*, i.e. $S(\tau) \to 0$ as $\tau \to \pm\infty$. The asymptotic behaviour of $S(\tau)$ can therefore be obtained from a linear analysis of Eq.(3) around the rest state. For convenience, we rewrite Eq.(3) as follows

$$V' = AV + N(V) + c_0^{-2} D V'', \qquad (4)$$

where

$$A = \begin{pmatrix} a & b \\ 1 & -1 \end{pmatrix},\ a = \frac{1}{\varepsilon}\left[1 - 2u_+ - \frac{2 f \mu u_+}{(u_+ + \mu)^2}\right],$$

$b = -\dfrac{f}{\varepsilon} \cdot \dfrac{u_+ - \mu}{u_+ + \mu}$ and $N(V)$ are nonlinear parts of the excitable kinetics. Generally, solutions of Eq.(4) take the form of exponential functions. To see this, we let $V = \xi e^{\lambda\tau}$ (where ξ is assumed to be the eigenvector corresponding to the eigenvalue λ). Thus we have

$$(A + \lambda^2 c_0^{-2} D - \lambda I) e^{\lambda\tau} \xi + N(\xi e^{\lambda\tau}) = 0. \qquad (5)$$

To obtain the asymptotic behaviour of $S(\tau)$, we need only to consider the linear parts of Eq.(5). Notice that nontrivial solutions of $S(\tau)$ correspond to $\xi \neq 0$. Therefore λ has to satisfy

$$(c_0^{-2} D_1 \lambda^2 - \lambda)(c_0^{-2} D_2 \lambda^2 - \lambda) - P(c_0^{-2} D_2 \lambda^2 - \lambda) + c_0^{-2}(D_2 - D_1)\lambda^2 + Q = 0, \qquad (6)$$

where

$$P = 1 - a,$$
$$Q = -(a + b) = \frac{4 u_+}{\varepsilon}\left[1 + \frac{f\mu}{(u_+ + \mu)^2}\right] > 0.$$

To give the form of the solitary impulse in Eq.(1), we discuss the following three cases: equal, single and double diffusions.

For the equal diffusion, we have $D_1 = D_2 \equiv D_0$, and then Eq.(6) becomes

$$\eta^2 - P\eta + Q = 0, \qquad (7)$$

where $\eta = c_0^{-2} D_0 \lambda^2 - \lambda$. From Eq.(7), we obtain

$$\eta_\pm = \frac{P \pm \sqrt{P^2 - 4Q}}{2}. \qquad (8)$$

According to the nature of EM, an excitable medium returns to its initial state after propagation of an excitation wave. This implies that the rest state U^* is always stable. Therefore, $P \geq 0$.

(1) $P > 0$

We further distinguish three cases: (a) $P^2 = 4Q$; (b) $P^2 > 4Q$; (c) $P^2 < 4Q$.

For $P^2 = 4Q$, we have $c_0^{-2} D_0 \lambda^2 - \lambda - \dfrac{P}{2} = 0$, which gives

$$\lambda_\pm = \frac{1 \pm \sqrt{1 + 2 c_0^{-2} D_0 P}}{2 c_0^{-2} D_0}. \qquad (9)$$

With this, the solitary wave tail evolves according to

$$S(\tau) \sim U_R \mathrm{e}^{-\eta_R \tau}, \tag{10}$$

while its head develops according to

$$S(\tau) \sim U_L \mathrm{e}^{\eta_L \tau}, \tag{11}$$

as $\tau \to \infty$ or $x \to -\infty$, where

$$\eta_R = \frac{-1 + \sqrt{1 + 2c_0^{-2} D_0 P}}{2c_0^{-2} D_0},$$

$$\eta_L = \frac{1 + \sqrt{1 + 2c_0^{-2} D_0 P}}{2c_0^{-2} D_0}.$$

We call the solitary impulse a monotonic type, since the recovery to the rest state after excitation is monotonic (see Fig.1).

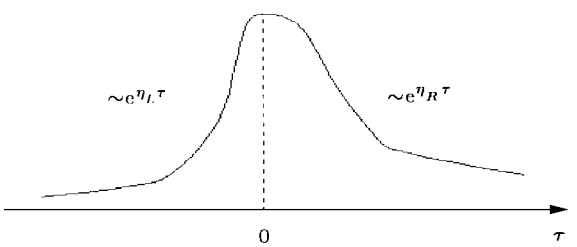


Fig.1. A possible profile of the monotonic solitary travelling impulse solution, $S(\tau)$. Asymptotic form at small and large τ values is displayed.

When $P^2 > 4Q$, analogously we may obtain

$$\lambda_1 = \frac{1 \pm \sqrt{1 + 2c_0^{-2} D_0 (P + \sqrt{P^2 - 4Q})}}{2c_0^{-2} D_0},$$

$$\lambda_2 = \frac{1 \pm \sqrt{1 + 2c_0^{-2} D_0 (P - \sqrt{P^2 - 4Q})}}{2c_0^{-2} D_0}. \tag{12}$$

The evolution of the tail and front of the corresponding solitary impulse takes a form similar to Eqs.(10) and (11), respectively, but we have

$$\eta_R = \frac{-1 + \sqrt{1 + 2c_0^{-2} D_0 (P - \sqrt{P^2 - 4Q})}}{2c_0^{-2} D_0},$$

$$\eta_L = \frac{1 + \sqrt{1 + 2c_0^{-2} D_0 (P - \sqrt{P^2 - 4Q})}}{2c_0^{-2} D_0}. \tag{13}$$

The solitary wave is also of monotonic type.

If $P^2 < 4Q$, then we have

$$\lambda_1 = \frac{1 \pm \sqrt{1 + 2c_0^{-2} D_0 (r + is)}}{2c_0^{-2} D_0},$$

$$\lambda_2 = \frac{1 \pm \sqrt{1 + 2c_0^{-2} D_0 (r - is)}}{2c_0^{-2} D_0}, \tag{14}$$

where $r = P$, $s = \sqrt{4Q - P^2}$, $i = \sqrt{-1}$. The tail of the corresponding solitary impulse evolves according to

$$S(\tau) \sim U_R \mathrm{e}^{-\eta_R \tau} \cos(v_R \tau + \phi_R). \tag{15}$$

While its front evolves according to

$$S(\tau) \sim U_L \mathrm{e}^{\eta_L \tau} \cos(v_L \tau + \phi_L), \tag{16}$$

as $\tau \to \infty$ or $x \to -\infty$, where

$$\eta_R = \frac{-1 + \sqrt{(1 + 2c_0^{-2} D_0 r)^2 + 4c_0^{-4} D_0^2 s^2} \cos\left(\arctan\left(\frac{2c_0^{-2} D_0 s}{1 + 2c_0^{-2} D_0 r}\right)\right)}{2c_0^{-2} D_0}, \tag{17}$$

$$\eta_L = \frac{1 + \sqrt{(1 + 2c_0^{-2} D_0 r)^2 + 4c_0^{-4} D_0^2 s^2} \cos\left(\arctan\left(\frac{2c_0^{-2} D_0 s}{1 + 2c_0^{-2} D_0 r}\right)\right)}{2c_0^{-2} D_0}. \tag{18}$$

We call the solitary impulse of this form a double-oscillatory type since its tail and head evolve with oscillations (see Fig.2).

(2) P=0

At the same time

$$c_0^{-2} D_0 \lambda^2 - \lambda = \pm \sqrt{Q}\, i, \tag{19}$$

which gives

$$\lambda_1 = \frac{1 \pm \sqrt{1 - 4c_0^{-2} D_0 \sqrt{Q} i}}{2c_0^{-2} D_0},$$

$$\lambda_2 = \frac{1 \pm \sqrt{1 + 4c_0^{-2} D_0 \sqrt{Q} i}}{2c_0^{-2} D_0}. \tag{20}$$

The tail of the corresponding solitary impulse takes

the form

$$S(\tau) \sim U_R \mathrm{e}^{-\eta_R \tau} \cos(v_R \tau + \phi_R)\,. \tag{21}$$

While its front evolves according to

$$S(\tau) \sim U_L \mathrm{e}^{\eta_L \tau} \cos(v_L \tau + \phi_L)\,, \tag{22}$$

as $\tau \to \infty$ or $x \to -\infty$, where

$$\eta_R = \frac{-1 + \sqrt{1 + 16 c_0^{-4} D_0^2 Q}\cos(\arctan(4 c_0^{-2} D_0))}{2 c_0^{-2} D_0}\,, \tag{23}$$

$$\eta_L = \frac{1 + \sqrt{1 + 16 c_0^{-4} D_0^2 Q}\cos(\arctan(4 c_0^{-2} D_0))}{2 c_0^{-2} D_0}\,. \tag{24}$$

The solitary impulse is also of the double-oscillatory type (see Fig.2).

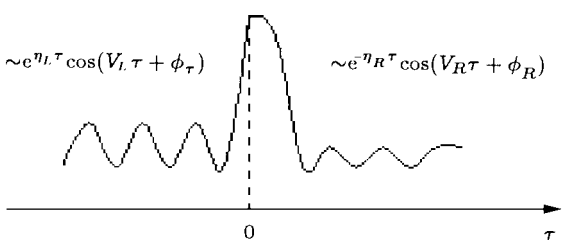


Fig.2. A possible profile of the solitary travelling impulse solution of double-oscillatory type, $S(\tau)$. Asymptotic form at small and large τ is displayed.

In the single diffusion case, for instance $D_1 > 0$ and $D_2 = 0$, it follows from Eq.(6) that we have

$$-c_0^{-2} D_1 \lambda^3 + (1 - c_0^{-2} D_1)\lambda^2 + P\lambda + Q = 0. \tag{25}$$

Notice that, in determining the asymptotic behaviour of $S(\tau)$, only those eigenvalues corresponding to slowest growth from $\tau = -\infty$ and slowest decay as $\tau \to \infty$ (i.e. smallest positive real part and least negative real part, respectively) should be considered, as all other eigenvalues contribute exponentially to smaller terms. Therefore, out of all possible patterns of eigenvalues the two that appear applicable to Eq.(6) are either $\eta_L, -\eta_R$ or $\eta_L, -\eta_R \pm i v_R$, where η_L and η_R are positive. Thus, in the single diffusion case, possibly only two kinds of solitary impulses appear: one corresponding to a solitary impulse with a monotonic tail, $S(\tau) \sim U_R \mathrm{e}^{-\eta_R \tau}$ as $\tau \to \infty$ or $x \to -\infty$; the other pertaining to the case of an oscillatory tail, $S(\tau) \sim U_R \mathrm{e}^{-\eta_R \tau} \cos(v_R \tau + \phi_R)$. In each case the head of the solitary impulse is monotonic, $S(\tau) \sim U_L \mathrm{e}^{-\eta_L \tau}$ as $\tau \to -\infty$ or $x \to \infty$ (see Fig.3). The system undergoes a succession of supernormal periods until a complete recovery is attained.

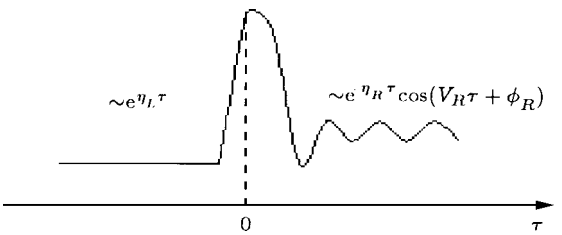


Fig.3. A possible profile of the solitary travelling impulse solution of oscillatory type, $S(\tau)$. Asymptotic form at small and large τ is shown.

Once we compare with the known Fitzhugh–Nagumo equation,[16] we have

$$\begin{cases} \partial_t u = -v + 3u - u^3 + D\partial_x^2 u\,, \\ \partial_t v = \sigma(u - \xi)\,, \end{cases} \tag{26}$$

where σ represents the ratio between the timescales of the two variables u and v, while ξ stands for the threshold of excitation; we find that here $1 + P$ corresponds to the parameter ξ, and Q corresponds to the parameter σ. Therefore, if we study only the asymptotic behaviour of a solitary impulse in the Oregonator model in the case of single diffusion, we construct the following system

$$\begin{cases} \partial_t u = -v + u - \dfrac{1}{3(1+P)} u^3 + D_0 \partial_x^2 u\,, \\ \partial_t v = Q(u - 1 - P) + D_0 \partial_x^2 v\,. \end{cases} \tag{27}$$

One advantage of doing this is that it can help us see clearly the threshold of excitation of Eq.(1).

We use Fig.4 to collect the above-obtained results for the possible profiles of solitary impulse in Eq.(1).

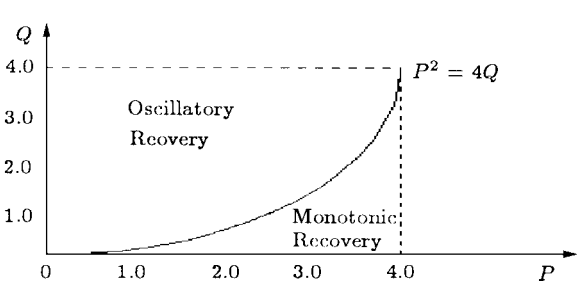


Fig.4. Phase diagram for the Oregonator model from Eq.(1), showing parameter regimes which give rise to a monotonic, oscillatory or double oscillatory tail of the propagating impulse.

If we consider the case of double diffusion, then from Eq.(6) we know that possibly another kind of solitary impulse appears rather than the three kinds of solitary impulse obtained above: its head and tail are not monotonic but non-oscillatory (i.e. the double root in Eq.(6) appears). The tail of the corresponding solitary impulse takes the form

$$S(\tau) \sim \sqrt{A_1 + B_1^2\tau^2}\ \mathrm{e}^{-\eta_R\tau}\cos(v_R\tau + \phi_R); \qquad (28)$$

while the head has the form

$$S(\tau) \sim \sqrt{A_2^2 + B_2^2\tau^2}\ \mathrm{e}^{\eta_L\tau}\cos(v_L\tau + \phi_L), \qquad (29)$$

as $\tau \to \infty$ or $x \to -\infty$ (see Fig.5).

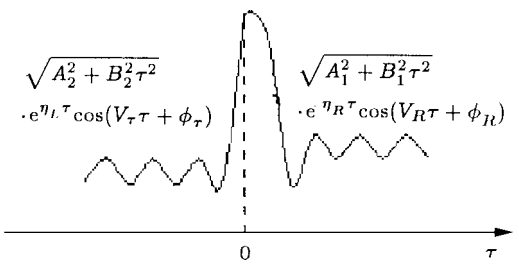


Fig.5. Asymptotic form of the solitary travelling-wave solution of super-double-oscillatory type.

We call the solitary impulse of this form a super-double-oscillatory type.

However, notice that, when recovering is not monotonic but non-oscillatory, decay from a transient overshoot of excitability can occur. This case can lead to the formation of burst-like patterns during propagation.[17]

Ⅲ. ASYMPTOTIC BEHAVIOUR OF WAVE-TRAINS

We let $t_i(x)$ denote the instant at which successive impulses pass through a given point x in the space. Then we have $t_i(x) = \dfrac{x}{c_0} + \tau_i$, where the relative phase τ_i depends on the impulse initiation time. We approximate a general wavetrain, in which interpulse times $t_j - t_{j-1}$ are large relative to the absolute recovery period, by a superposition of solitary impulses[18] of the form

$$U(x,t) = \sum_j S(\tau - \tau_j) + \gamma R, \qquad (30)$$

where R is a correction term, and γ is a small parameter which vanishes in the limit of infinite interpulse spacing.

If we use a regular perturbation scheme and employ Eq.(30) with τ_j being constant, then R would diverge. According to the singular perturbation theory, the standard way for correcting this is to let the constants depend weakly on x in a way which guarantees that R will remain small. Therefore we seek a solution of the form of Eq.(30) with $\tau_j = \tau_j(\gamma x)$ and $R = R(\tau, \gamma x)$. This representation thus allows for the transient interactions between impulses. They will not, in general, all travel at the same uniform speed

$$\left(\frac{\mathrm{d}t_i}{\mathrm{d}x}\right)^{-1} = c_0. \qquad (31)$$

We first assume that η_R is smaller than η_L but of comparable size. The small parameter γ is defined as $\gamma \equiv \mathrm{e}^{-\eta_R T}$, where T is a typical time interval between successive impulses. Substituting Eq.(30) into Eq.(4), where $\tau_j = \tau_j(\gamma x)$ and $R = R(\tau, \gamma x)$, we have

$$\partial_x^2 U = \sum_j c_0^{-2} S_j'' + \gamma\Big(2c_0^{-1}\sum_j S_j''\tau_j' + c_0^{-2}\partial_\tau^2 R\Big) + O(\gamma^2), \qquad (32)$$

$$N(U) = N\Big(\sum_j S_j\Big) + \gamma\nabla N\Big(\sum_j S_j\Big)R + O(\gamma^2), \qquad (33)$$

$$(\partial_\tau - A)U = \sum_j (S_j' - AS_j) + \gamma(\partial_\tau - A)R + O(\gamma^2), \qquad (34)$$

$$\gamma\pounds R = -H(S_j) - 2c_0^{-1}D\sum_j S_j''\frac{\mathrm{d}\tau_j}{\mathrm{d}x} + O(\gamma^2), \qquad (35)$$

where

$$\pounds = c_0^{-2}D\partial_\tau^2 - \partial_\tau + A + \nabla N\Big(\sum_j S_j\Big), \quad S_j = S(\tau - \tau_j), \qquad (36)$$

$$H(S_j) = N\Big(\sum_j S_j\Big) - \sum_j N(S_j). \qquad (37)$$

Based on the fact that $\Big(\sum_j S_j)^n = \sum_j S_j^n + O(\gamma\Big)$, we now make some significant simplifications. For an analytic function F, we may write $F\Big(\sum_j S_j\Big) = \sum_j F(S_j) + O(\gamma)$. We assume that this relation holds, in particular for N and ∇N. Then we see that $H(S_j) = O(\gamma)$ and

$$\pounds = c_0^{-2}D\partial_\tau^2 - \partial_\tau + A + \sum_j \nabla N(S_j) + O(\gamma). \qquad (38)$$

Now we define an operator $\pounds_j$

$$\pounds_j S_j' = 0. \qquad (39)$$

Therefore

$$\pounds S_j' = O(\gamma). \qquad (40)$$

To continue, we define an inner product by

$$< U, V >= \int_{-\infty}^{+\infty} U(\tau) \cdot V(\tau) \mathrm{d}\tau, \tag{41}$$

where the dot denotes the usual scalar product in $\boldsymbol{R}^2$. Consider the adjoint $\pounds_j^+$ of $\pounds_j$ and assume its null space has a group of bases by G_j

$$\pounds_j^+ G_j = 0, \tag{42}$$

where $G_j \equiv G(\tau - \tau_j)$. Thus

$$\pounds^+ G_j = O(\gamma). \tag{43}$$

When we multiply Eq.(35) from the left with G_i, the left-hand side is $O(\gamma^2)$. Hence, the right-hand side must also be $O(\gamma^2)$. On using this condition, we obtain

$$\frac{\mathrm{d}\tau_j}{\mathrm{d}x} = \frac{c_0 < H, G_j >}{2 < DS'', G >}. \tag{44}$$

We now focus on evaluating the integral $< H, G_j >$ in Eq.(44). First we split the integration interval into small segments $\triangle_j$, each of which contains only one impulse peak. We then make use of the exponential decay of $S(\tau)$ as $\tau \to \pm\infty$. To neglect contributions of $N(S_j \neq k)(k \geq 1)$ and terms proportional to $S_j \neq l(l \geq 2)$ in the integral over $\triangle_j$. Thus we obtain the nearest-neighbour approximation

$$< H, G_j > \approx \sum_k \int_{\triangle_k} (S_{k-1} + S_{k+1}) \cdot \nabla N(S_k) \cdot G_j \mathrm{d}\tau. \tag{45}$$

According to our assumptions, the dominant contribution to the term on the right-hand side of Eq.(45) comes from the integral over $\triangle_k$. For clarity, now we change the integration variable from τ to $\theta \equiv \tau - \tau_k$, then we have

$$< H, G_j > \approx \int_{\triangle} [S(\theta + \tau_j - \tau_{j-1})] + S(\theta + \tau_j - \tau_{j+1})) \cdot \nabla N[S(\theta)] \cdot G(\theta) \mathrm{d}\tau, \tag{46}$$

where $\triangle$ is the small neighbourhood of $\theta = 0$ in which G and S are of significant size.

Since we consider the asymptotic behaviour of solitary waves, we arrive at the final expression after substituting the two typical asymptotic forms (i.e. solitary waves of monotonic and oscillatory) for $S(\tau)$

$$< H, G_j > \approx V_R \mathrm{e}^{-\eta_R (t_j - t_{j-1})} \cos[v_R (t_j - t_{j-1}) + \psi_R] + V_L \mathrm{e}^{-\eta_L (t_{j+1} - t_j)}, \tag{47}$$

where

$$V_R = \sqrt{\xi_1^2 + \xi_2^2},$$

$$V_L = \int_{\triangle} \mathrm{e}^{\eta_L \theta} U_L \cdot \nabla N[S(\theta)] \cdot G(\theta) \mathrm{d}\theta,$$

$$\psi_R = \arctan\left(\frac{\xi_2}{\xi_1}\right),$$

and

$$\xi_1 = \int_{\triangle} \mathrm{e}^{\eta_R \theta} \cos[v_R \theta + \phi_R] U_R \cdot \nabla N[S(\theta)] \cdot G(\theta) \mathrm{d}\theta,$$

$$\xi_2 = \int_{\triangle} \mathrm{e}^{\eta_R \theta} \sin[v_R \theta + \phi_R] U_R \cdot \nabla N[S(\theta)] \cdot G(\theta) \mathrm{d}\theta.$$

Letting

$$\alpha = -\frac{c_0 V_R}{2 < DS'', G >}, \quad \beta = -\frac{c_0 V_L}{2 < DS'', G >} \tag{48}$$

and using the relation $t_i(x) = \dfrac{x}{c_0} + \tau_i(x)$, then from Eq.(44) we finally obtain

$$\frac{\mathrm{d}t_j}{\mathrm{d}x} = c_0^{-1} + \alpha \mathrm{e}^{-\eta_R (t_j - t_{j-1})} \cos[v_R (t_j - t_{j-1}) + \psi_R] + \beta \mathrm{e}^{-\eta_L (t_{j+1} - t_j)}. \tag{49}$$

Equation (49) describes the asymptotic behaviour of motion of the wavetrain in Eq.(1).

IV. STABILITY OF WAVETRAINS

A steadily propagating general wavetrain, $t_j^0(x) = \dfrac{x}{c} + \sum_{k=1}^{j} T_k$ is stable if every perturbed train, $t_j(x) = t_j^0(x) + P_j(x)$, converges to the unperturbed one during propagation.[19] Let us begin with uniformly spaced, infinite trains of the form

$$t_j^0(x) = \frac{x}{c} + (j-1)T + T_1, \tag{50}$$

where T is the period of a train, c is speed, and T_1 is an arbitrary phase. Substituting Eq.(50) into Eq.(49), we find, to leading order in γ,

$$c = c_0 - c_0^2 \mathrm{e}^{-\eta_R T} \cos(v_R T + \psi_R). \tag{51}$$

Inserting the perturbed form $t_j(x) = t_j^0(x) + P_j(x)$ in Eq.(50), keeping only terms linear in P_j and using Eq.(51), we find

$$\frac{\mathrm{d}P_j}{\mathrm{d}x} = -c_0^{-2} (P_j - P_{j-1}) c'(T). \tag{52}$$

Equation (52) admits solutions in the form

$$P_j = A_\omega \mathrm{e}^{\sigma_\omega x + i\omega j} + \mathrm{const}, \tag{53}$$

where ω ranges over integral multiples of $\frac{2\pi}{N}$, and N is the number of impulses (N sufficiently large).

Once using the solutions of Eq.(53), we find that the perturbation grows spatially at rates

$$\mathrm{Re}\sigma_\omega = -c_0^{-2}(1-\cos\omega)c'(T). \tag{54}$$

Therefore, a uniformly spaced infinite wavetrain with period T is stable if and only if $c'(T)$ is positive.

Now we consider a non-uniformly spaced finite wavetrain and assume that it takes the form $t_j^0 = \frac{x}{c_0} + \sum_{k=1}^{j} T_k (j = 1, 2, \cdots, N)$. Proceeding as before, we find the linear equations to be

$$\frac{\mathrm{d}P_1}{\mathrm{d}x} = 0, \ \frac{\mathrm{d}P_j}{\mathrm{d}x} = -c_0^{-2}(P_j - P_{j-1})c'(T_j), \tag{55}$$

where $j = 2, 3, \cdots, N$. The characteristic equation becomes

$$\prod_{k=2}^{N}[c_0^{-2}c'(T_k) + \sigma] = 0\,. \tag{56}$$

Thus a non-uniform finite wavetrain is stable if and only if $c'(T_k) > 0$ for each k.

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