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THE EXACT SOLUTIONS OF THE BURGERS EQUATION AND HIGHER-ORDER BURGERS EQUATION IN (2+1) DIMENSIONS*

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Some exact solutions of the Burgers equation and higher-order Burgers equation in (2+1) dimensions are obtained by using the extended homogeneous balance method. In these solutions there are solitary wave solutions, close formal solutions for the initial value problems of the Burgers equation and higher-order Burgers equation, and also infinitely many rational function solutions. All of the solutions contain some arbitrary functions that may be related to the symmetry properties of the Burgers equation and the higher-order Burgers equation in (2+1) dimensions.

Keywords: (2+1)-dimensional Burgers equation, higher-order Burgers equation, extended homogeneous balance method, exact solutions

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I. INTRODUCTION

Nonlinear differential equations are known to describe a wide variety of phenomena not only in physics, in which applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions and ion acoustic waves in plasmas, to name just a few, but also in biology and chemistry, among several other fields. One of the important tasks in the study of nonlinear differential equations is to seek exact and explicit solutions. In the past few decades, both mathematicians and physicists have made many attempts in this direction.[1−21]

In this paper we consider the Burgers equation in (2+1) dimensions[22]

$$u_t = u_{xx} + 2u_x\partial_y^{-1}u_x, \tag{1}$$

and the higher-order Burgers equation in (2+1) dimensions[22]

$$\begin{aligned} u_t =& 4u_{xxx} + 12u_{xx}\partial_y^{-1}u_x + 12u_x\partial_y^{-1}u_{xx} \\ &+ 12u_x(\partial_y^{-1}u_x)^2, \end{aligned} \tag{2}$$

where $\partial_y^{-1} = \int \mathrm{d}y$. Their Painlevé properties have been checked by Chen and Lou[23] recently. We give some exact solutions of Eqs.(1) and (2) by using the extended homogeneous balance method and the chopping-off technique which we developed recently.[24] In these solutions there are solitary wave solutions, close formal solutions for the initial value problems of the Burgers equation and higher-order Burgers equation, and also infinitely many rational function solutions. All of the solutions contain some arbitrary functions that may be related to the symmetry properties of the Burgers equation and the higher-order Burgers equation in (2+1) dimensions.

In section II, we give equations that the quasi-solutions satisfy. In sections III and IV, we give exact solutions of Eqs.(1) and (2), respectively.

II. EQUATIONS WHICH THE QUASI-SOLUTIONS SATISFY

A. Differential equation which the quasi-solution of the Burgers equation in (2+1) dimensions satisfies

According to the extended homogeneous balance method, in order that the nonlinear term $2u_x\partial_y^{-1}u_x$ and the second-order derivative term u_{xx} in Eq.(1) can be partially balanced, we suppose that the solution of Eq.(1) is of the form

$$u(x,y,t) = p(x,y,t) + f'(\varphi)\varphi_x, \tag{3}$$

where $f(\varphi)$ and $\varphi(x,y,t)$ are functions to be determined, $\varphi(x,y,t)$ is called the quasi-solution and $p(x,y,t)$ is a known solution of Eq.(1), that is

$$p_t = p_{xx} + 2p_x\partial_y^{-1}p_x. \tag{4}$$

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Substituting Eq.(3) into Eq.(1) and using Eq.(4), we obtain

$$\begin{aligned} & u_t - u_{xx} - 2u_x\partial_y^{-1}u_x \\ =& -(f^{(3)} + 2f'f'')\varphi_x^2\varphi_y + (\varphi_t - \varphi_{xx} \\ & - 2\varphi_x\partial_y^{-1}p_x)\varphi_y f'' - 2(f'' + f'^2)\varphi_x\varphi_{xy} \\ & + (\varphi_t - \varphi_{xx} - 2\varphi_x\partial_y^{-1}p_x)_y f'. \end{aligned} \tag{5}$$

Putting the coefficient of $\varphi_x^2\varphi_y$ in Eq.(5) to zero, we obtain an ordinary differential equation of $f(\varphi)$

$$f^{(3)} + 2f'f'' = 0. \tag{6}$$

There is a special solution for Eq.(6) as follows

$$f(\varphi) = \ln(\varphi), \tag{7}$$

thereby

$$f'^2 = -f''. \tag{8}$$

Substituting Eq.(8) into Eq.(5) and setting the coefficients of f'' and f' to zero, we obtain an independent differential equation of $\varphi(x,y,t)$ under the condition of $\varphi_y \neq 0$,

$$\varphi_t - \varphi_{xx} - 2\varphi_x\partial_y^{-1}p_x = 0. \tag{9}$$

This is just the equation that the quasi-solution $\varphi(x,y,t)$ of the Burgers equation in (2+1) dimensions must satisfy.

B. Differential equation which the quasi-solution of the higher-order Burgers equation in (2+1) dimensions satisfies

Through a discussion similar to that for the case of the Burgers equation in (2+1) dimensions, we obtain

$$\begin{aligned} \varphi_t =& 4\varphi_{xxx} + 12\varphi_{xx}\partial_y^{-1}p_x \\ & + 12\varphi_x\partial_y^{-1}p_{xx} + 12\varphi_x(\partial_y^{-1}p_x)^2, \end{aligned} \tag{10}$$

where $\varphi(x,y,t)$ is the quasi-solution of the higher-order Burgers equation in (2+1) dimensions. $p(x,y,t)$ is a known solution of Eq.(2), that is,

$$\begin{aligned} p_t =& 4p_{xxx} + 12p_{xx}\partial_y^{-1}p_x \\ & + 12p_x\partial_y^{-1}p_{xx} + 12p_x(\partial_y^{-1}p_x)^2, \end{aligned} \tag{11}$$

and the new solution of the Eq.(2) may be expressed as

$$u(x,y,t) = p(x,y,t) + \frac{\partial}{\partial y}\ln(\varphi). \tag{12}$$

Eq.(10) is just the equation that the quasi-solution $\varphi(x,y,t)$ of the higher-order Burgers equation in (2+1) dimensions must satisfy.

III. EXACT SOLUTIONS OF THE BURGERS EQUATION IN (2+1) DIMENSIONS

A. Solitary wave solution and the closed form of the solution for the initial value problem

It can be verified that there is a special solution for Eq.(4),

$$p(x,y,t) = f(y), \tag{13}$$

where $f(y)$ is an arbitrary function of y. In this case, $p_x = 0, p_t = 0$ and

$$\int p_x \mathrm{d}y = g(t), \tag{14}$$

where $g(t)$ is an arbitrary function of t.

Substituting Eq.(14) into Eq.(9), we obtain

$$\varphi_t - \varphi_{xx} - 2g(t)\varphi_x = 0. \tag{15}$$

The solution of Eq.(15) is direct and simple. The result reads

$$\varphi(x,y,t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi,y,0)\mathrm{e}^{\frac{[(x-\xi)+G(t)]^2}{4t}}\mathrm{d}\xi, \tag{16}$$

where

$$G(t) = \int g(t)\mathrm{d}t. \tag{17}$$

Starting from Eq.(3) and using Eqs.(7) and (13), we obtain

$$u(x,y,0) = f(y) + \frac{\partial}{\partial y}\ln[\varphi(x,y,0)], \tag{18}$$

thereby

$$\varphi(x,y,0) = \varphi(x,0,0)\mathrm{e}^{\int_0^y[u(x,\xi,0)-f(\xi)]\mathrm{d}\xi}, \tag{19}$$

where $\varphi(x,0,0)$ is an arbitrary function of x.

Substituting Eq.(19) into Eq.(16), we obtain

$$\begin{aligned} \varphi(x,y,t) =& \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi,0,0) \\ & \cdot \mathrm{e}^{\int_0^y[u(\xi,\eta,0)-f(\eta)]\mathrm{d}\eta - \frac{[(x-\xi)+G(t)]^2}{4t}}\mathrm{d}\xi. \end{aligned} \tag{20}$$

Substituting Eq.(20) into Eq.(3) and using Eqs.(7) and (13), we obtain the closed form of the solution for the initial value problem of the Burgers equation in (2+1) dimensions.

If we choose $f(y)$=c_0=const and $g(t)$=c_1=const in Eqs.(13) and (14), respectively, we can obtain

$$\varphi(x,y,t) = 1 + \mathrm{e}^{\alpha(y)\{x+[\alpha(y)+c_1]t\}+\beta(y)}, \tag{21}$$

where $\alpha(y)$ and $\beta(y)$ are two arbitrary functions of y.

Especially, if choosing $\alpha(y)$ as a constant and $\beta(y)$ as a linear function of y, we can obtain a special solution of Eq.(15) as follows

$$\varphi(x,y,t)=1+\mathrm{e}^{\alpha[x+(\alpha+c_1)t]+\beta y}, \tag{22}$$

where α and β are two arbitrary constants. Therefore, we obtain

$$\begin{aligned}&u(x,y,t)\\ =&c_0+\frac{1}{2}\beta\left[1+\tanh\left(\frac{1}{2}\{\alpha[x+(\alpha+c_1)t]+\beta y\}\right)\right].\end{aligned} \tag{23}$$

It is very interesting that the soliton wave moves at the vertical direction of beeline $\alpha x+\beta y$, and the velocity and amplitude of the soliton relate to the arbitrary constant β.

B. Rational function solutions

Supposing another special solution of Eq.(4) is of the form

$$p(x,y,t)=a(y,t)x^2+b(y,t)x+c(y,t), \tag{24}$$

where $a(y,t)$, $b(y,t)$ and $c(y,t)$ are functions to be determined. Therefore we have

$$\begin{aligned}&p_t-p_{xx}-2p_x\partial_y^{-1}p_x\\ =&\left[\frac{\partial a(y,t)}{\partial t}-8a(y,t)\int a(y,t)\mathrm{d}y\right]x^2\\ &+\left[\frac{\partial b(y,t)}{\partial t}-4a(y,t)\int b(y,t)\mathrm{d}y\right.\\ &\left.-4b(y,t)\int a(y,t)\mathrm{d}y\right]x\\ &+\left[\frac{\partial c(y,t)}{\partial t}-2a(y,t)-2b(y,t)\int b(y,t)\mathrm{d}y\right]\\ =&x^2\frac{\partial}{\partial y}\left[\frac{\partial A(y,t)}{\partial t}-4A^2(y,t)\right]\\ &+x\left[\frac{\partial B(y,t)}{\partial t}-4A(y,t)B(y,t)\right]\\ &+\frac{\partial}{\partial y}\left[\frac{\partial C(y,t)}{\partial t}-2A(y,t)-B^2(y,t)\right],\end{aligned} \tag{25}$$

where

$$\begin{aligned}A(y,t)=&\int a(y,t)\mathrm{d}y, B(y,t)\\ =&\int b(y,t)\mathrm{d}y, C(y,t)=\int c(y,t)\mathrm{d}y.\end{aligned} \tag{26}$$

Setting the coefficients of x^2, x and x^0 to zero, we obtain

$$\frac{\partial}{\partial y}\left[\frac{\partial A(y,t)}{\partial t}-4A^2(y,t)\right]=0, \tag{27}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial B(y,t)}{\partial t}-4A(y,t)B(y,t)\right]=0, \tag{28}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial C(y,t)}{\partial t}-2A(y,t)-B^2(y,t)\right]=0. \tag{29}$$

Solving Eqs.(27)–(29), we have

$$A(y,t)=-\frac{1}{4[t+A_0(y)]}, \tag{30}$$

$$B(y,t)=\frac{B_0 y}{t+A_0(y)}, \tag{31}$$

$$C(y,t)=C_0(y)-\frac{B_0^2(y)}{t+A_0(y)}-\frac{1}{2}\ln[t+A_0(y)], \tag{32}$$

where $A_0(y), B_0(y)$ and $C_0(y)$ are new arbitrary functions of y. Starting from Eq.(26), we obtain

$$a(y,t)=\frac{A_0'(y)}{4[t+A_0(y)]^2}, \tag{33}$$

$$b(y,t)=\frac{B_0'(y)}{t+A_0(y)}-\frac{B_0(y)A_0'(y)}{[t+A_0(y)]^2}, \tag{34}$$

$$\begin{aligned}c(y,t)=&C_0'(y)-\frac{4B_0(y)B_0'(y)+A_0'(y)}{2[t+A_0(y)]}\\ &+\frac{A_0'(y)}{[t+A_0(y)]^2}.\end{aligned} \tag{35}$$

Substituting $a(y,t)$, $b(y,t)$ and $c(y,t)$ in Eqs.(33)–(35), we obtain a special solution $p(x,y,t)$ of Eq.(4)

$$\begin{aligned}&p(x,y,t)\\ =&\frac{A_0'(y)}{4[t+A_0(y)]^2}x^2+\left\{\frac{B_0'(y)}{t+A_0(y)}-\frac{B_0(y)A_0'(y)}{[t+A_0(y)]^2}\right\}x\\ &+\left\{C_0'(y)-\frac{4B_0(y)B_0'(y)+A_0'(y)}{2[t+A_0(y)]}+\frac{A_0'(y)}{[t+A_0(y)]^2}\right\}.\end{aligned} \tag{36}$$

In order to obtain new solutions of the Eq.(1) from $p(x,y,t)$ in Eq.(24), we suppose the solution of Eq.(9) is of the series form

$$\varphi(x,y,t)=\sum_{n=0}^{\infty}a_n(y,t)x^n. \tag{37}$$

Substituting $p(x,y,t)$ in Eq.(36) and $\varphi(x,y,t)$ in Eq.(37) into Eq.(9), we obtain

$$\begin{aligned}&\frac{\partial a_n(y,t)}{\partial t}-4nA(y,t)-2(n+1)B(y,t)a_{n+1}(y,t)\\ &-(n+2)(n+1)a_{n+2}(y,t)=0,\end{aligned} \tag{38}$$

where $n\geq 0$.

Using the chopping-off technique which we have developed[24] to solve Eq.(38), we may obtain $a_n(y,t) \equiv 0$ for $n \geq N+1$

$$\begin{aligned}
a_N(y,t) =& \frac{c_N(y)}{[t+A_0(y)]^N},\\
a_{N-1}(y,t) =& \frac{c_{N-1}(y)}{[t+A_0(y)]^{N-1}} - \frac{2Nc_N(y)B_0(y)}{[t+A_0(y)]^N},\\
a_{N-2}(y,t) =& \frac{c_{N-2}(y)}{[t+A_0(y)]^{N-2}}\\
&- \frac{2(N-1)c_{N-1}(y)B_0(y)}{[t+A_0(y)]^{N-1}}\\
&- \frac{N(N-1)c_N(y)}{[t+A_0(y)]^{N-1}}\\
&+ \frac{2N(N-1)c_N(y)B_0^2(y)}{[t+A_0(y)]^N},\\
&\cdots\cdots
\end{aligned} \tag{39}$$

where N is an arbitrary positive integer, $c_N(y)$, $c_{N-1}(y), \cdots, c_1(y)$, $c_0(y)$ are arbitrary functions of y.

Substituting $a_N(y,t)$, $a_{N-1}(y,t), \cdots$, $a_1(y,t)$, $a_0(y,t)$ in Eq.(39), we obtain $\varphi_N(x,y,t)$. Also, substituting $\varphi_N(x,y,t)$ obtained here and $p(x,y,t)$ in Eq.(36) into Eq.(3), we obtain a new solution $u_N(x,y,t)$ of Eq.(1). In principle, for any positive integer N we can obtain a $\varphi_N(x,y,t)$. Therefore we may obtain infinitely many solutions of Eq.(1) starting from $p(x,y,t)$ in Eq.(36), which are of the form of the rational functions.

IV. THE EXACT SOLUTIONS OF THE HIGHER-ORDER BURGERS EQUATION IN (2+1) DIMENSIONS

For the special solution $p(x,y,t) = p_0(y)$ of Eq.(11), where $p_0(y)$ is an arbitrary function of y, Eq.(10) may be written as

$$\varphi_t - 4\varphi_{xxx} - 12g_1(t)\varphi_{xx} - 12g_2(t)\varphi_x - 12g_1^2(t)\varphi_x = 0, \tag{40}$$

where $g_1(t) = \int p_x \mathrm{d}y = \int 0 \mathrm{d}y, g_2(t) = \int p_{xx}\mathrm{d}y$ are two arbitrary functions of t.

The solution of Eq.(40) is simple and direct. The result reads

$$\begin{aligned}
\varphi(x,y,t) =& \frac{1}{2\pi^3\sqrt{12t}} \int_{-\infty}^{+\infty} \varphi(\xi,y,0) Ai\left\{ \frac{144G_1(t)}{(12t)^{2/3}} - \frac{(x-\xi)+12[G_2(t)+G_3(t)]}{(12t)^{1/3}} \right\}\\
&\times \exp\left(\frac{72G_1^3(t)}{t} - \frac{\{(x-\xi)+12[G_2(t)+G_3(t)]\}G_1(t)}{(12t)^{2/3}} \right) \mathrm{d}\xi,
\end{aligned} \tag{41}$$

where $Ai(x)$ is the Airy function, and

$$\begin{aligned}
&G_1(t) = \int g_1(t)\mathrm{d}t, \quad G_2(t) = \int g_2(t)\mathrm{d}t,\\
&G_3(t) = \int g_1^2(t)\mathrm{d}t.
\end{aligned} \tag{42}$$

Using $p(x,y,t) = p_0(y)$ and starting from Eq.(12), we obtain

$$\varphi(x,y,0) = \varphi(x,0,0)\mathrm{e}^{\int_0^y [u(x,\xi,0) - f(\xi)]\mathrm{d}\xi}. \tag{43}$$

Substituting $\varphi(x,y,0)$ in Eq.(43) into Eq.(41) and substituting the result obtained here into Eq.(12), we obtain the closed form for the initial value problem of the higher-order Burgers equation (2).

If choosing $g_1(t)$=c_1=const and $g_2(t)$=c_2=const in Eq.(40), we can obtain a solution of Eq.(40) as follows

$$\begin{aligned}
\varphi(x,y,t) =& 1 + \exp(\alpha(y)\{x + 4[\alpha^2(y)\\
&+ 3c_1\alpha(y) + 3(c_2 + c_1^2)]t + \beta(y)]\}),
\end{aligned} \tag{44}$$

where $\alpha(y)$ and $\beta(y)$ are two arbitrary functions of y.

Especially if we choose $\alpha(y)$ as a constant, and $\beta(y)$ as a linear function of y, we obtain

$$\begin{aligned}
\varphi(x,y,t) =& 1 + \exp(\alpha\{x + 4[\alpha^2\\
&+ 3c_1\alpha + 3(c_2 + c_1^2)]t + \beta y]\}).
\end{aligned} \tag{45}$$

Therefore we obtain a solitary wave solution of the higher-order Burgers equation (2)

$$\begin{aligned}
u(x,y,t) =& c_0 + \frac{1}{2}\beta\left[1 + \tanh\left(\frac{1}{2}\alpha\{x + 4[\alpha^2\right.\right.\\
&\left.\left.+ 3c_1\alpha + 3(c_2 + c_1^2)]t + \beta y]\}\right)\right].
\end{aligned} \tag{46}$$

We also see that the soliton wave moves at the vertical direction of beeline $\alpha x + \beta y$, and the velocity and amplitude of the soliton relate to β.

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References

[1] Abdelkaser M A 1982 *J. Math. Anal. Appl.* **85** 287
[2] Dodd R and Fordy A 1982 *Phys. Lett.* A **89** 168
[3] Drazin P G and Johnson R S 1989 *Soliton: an introduction* (Cambridge: Cambridge University Press)
[4] Fan E G and Zhang H Q 1998 *Phys. Lett.* A **245** 389
[5] Fan E G and Zhang H Q 1998 *Phys. Lett.* A **246** 403
[6] Feng X 1994 *Math. Manus.* **84** 361
[7] Feng X *1996 Phys. Lett.* A **213** 167
[8] Hirota R and Satsuma J 1981 *Phys. Lett.* A **85** 407
[9] Hirota R and Satsuma J 1982 *J. Phys. Soc. Jpn* **51** 3390
[10] Lu B Q 1994 *Phys. Lett.* A **189** 25
[11] Lu B Q *et al* 1993 *Phys. Lett.* A **180** 61
[12] Murray J D 1989 *Mathematical Biology* (New York: Springer)
[13] Olver P L 1984 *Hamiltonian and Non-Hamiltonian Models for Water Waves, in Lecture Notes in Physics* (New York: Springer-Verlag) **195** 273
[14] Wang M L *et al* 1996 *Phys. Lett.* A **216** 67
[15] Wang M X *et al* 1994 *J. Math. Anal. Appl.* **182** 705
[16] Wilson G 1982 *Phys. Lett.* A **89** 332
[17] Yang S Q 1999 *Acta Phys. Sin. (overseas edition)* **8** 801
[18] Ruan H Y and Chen Y X 1998 *Acta Phys. Sin. (overseas edition)* **7** 241
[19] Ruan H Y and Chen Y X 1999 *Acta Phys. Sin. (overseas edition)* **8** 241
[20] Ruan H Y and Chen Y X 2000 *Acta Phys. Sin.* **49** 177 (in Chinese)
[21] Bai C L, Bai C J and Xu B I 2000 *Chin. Phys.* **9** 161
[22] Burgers J M 1958 *Adv. Appl. Mech.* **1** 171
[23] Chen L L and Lou S Y 1998 *Commun. Theor. Phys.* **29** 315
[24] Bai C L 2000 *Commun. Theor. Phys.* **34** 729