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THERMODYNAMICS OF GLOBAL MONOPOLE ANTI-DE-SITTER BLACK HOLE IN GRAND CANONICAL ENSEMBLE*

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In this paper, we investigate the thermodynamics of the global monopole anti-de-Sitter black hole in the grand canonical ensemble following the York's formalism. The black hole is enclosed in a cavity with a finite radius where the temperature and potential are fixed. We have studied some thermodynamical properties, i.e. the reduced action, thermal energy and entropy. By investigating the stability of the solutions, we find stable solutions and instantons.

Keywords: reduced action, global monopole, grand canonical ensemble
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I. INTRODUCTION

In recent years some authors have concentrated on studying the thermodynamics of the black hole.[1−29] Hawking[14−17] studied the thermodynamical properties of black holes with micro-canonical ensemble using the path-integral approach. York *et al.*[18,21] further developed the canonical ensemble by defining the boundary conditions. Braden *et al.*[22] used this method to study the charged black hole in the grand canonical ensemble and the black hole in asymptotically anti-de-Sitter (AdS) space-times.[19−21] Hawking and Page[32,33] have investigated the thermodynamics of black holes for fixing the Hawking temperature at infinity, and York *et al.*[18,22,23] have done this by fixing local temperature at finite radius. Louko and Winters-Hilt[24] studied the thermodynamics of the Reissner–Nordström (R–N) AdS black hole by fixing a renormalized temperature at infinity. Peca and Lemos[21] have studied the thermodynamics of the R–N black hole following York's formalism.[18,22,23]

In the grand canonical ensemble, York *et al.*[18] assumed that a black hole is put in a finite-sized 'box', as Bose[30] did in general relativity. The boundary conditions are chosen according to the thermodynamical ensemble under study, and the thermodynamical action for the black hole can be calculated in a given thermodynamical ensemble. In the canonical ensemble the thermodynamical potential is the Helmholtz free energy, and in the grand canonical ensemble it is the grand canonical potential,[16,22] so we can investigate all the thermodynamical quantities from the thermodynamical action.[31]

In this paper, we follow York's formalism to investigate the global monopole AdS black hole in the grand canonical ensemble. The paper is organized as follows. In the next section, we introduce York's formalism. In section III, we investigate some thermodynamical properties of the global monopole AdS black hole in the grand canonical ensemble. In section IV, we analyse the numerical solutions under difference boundary conditions. In section V, we investigate the stability of solutions, and find stable solutions and instantons. In section VI, some special cases are given.

II. THE ACTION

The Euclidean action in general relativity[16] is given by

$$I = -\frac{1}{16\pi}\int_M \mathrm{d}^4x\sqrt{g}(R_2 - 2\Lambda) + \frac{1}{8\pi}\int_{\partial M}\mathrm{d}^3x\sqrt{h}\,K - \int \mathrm{d}^4x\sqrt{g}\,L_{\mathrm{m}} - I_{\mathrm{subtr}}, \quad (1)$$

where M is a compact region with boundary ∂M, R is the scalar curvature, Λ is the cosmological constant, g is the determinant of the Euclidean metrics, K is the trace of the extrinsic curvature of the boundary ∂M, h is the determinant of the Euclidean induced metrics on the boundary, L_{m} is the Euclidean Lagrangian density of the matter fields and I_{subtr} is an arbitrary

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term that can be used to define the zero of the energy to be seen later.

Barriola and Vilenkin[34,35] considered the gravitational field of a global monopole formed as a result of global $O(3)$ symmetry breaking. The simple mode of such a monopole is described by the Lagrangian

$$L_{\rm m} = \frac{1}{2}\left(\frac{\partial\,\Psi^a}{\partial x^b}\right)^2 - \frac{\lambda}{4}\left(\Psi^a\,\Psi^a - \frac{\eta^2}{8\pi}\right)^2, \qquad (2)$$

where Ψ^a is the iso-scalar triplet and a=1,2,3. The topologically nontrivial self-supporting Ansatz describing a monopole is

$$\Psi^a = -\frac{\eta}{\sqrt{8\pi}}\frac{x^a}{r}. \qquad (3)$$

The stress-energy tensor is given by

$$T_0^0 = T_1^1 = \frac{\eta^2}{8\pi r^2}, \qquad (4)$$

where $r^2 = x^a x^a$. It was shown that spherically symmetric space-time with the metric

$$\begin{aligned} {\rm d}s^2 =& -\left(1 - \frac{2M}{r} - \eta^2 - \frac{\Lambda}{3}r^2\right){\rm d}t^2 \\ &+ \left(1 - \frac{2M}{r} - \eta^2 - \frac{\Lambda}{3}r^2\right)^{-1}{\rm d}r^2 + r^2{\rm d}\Omega^2 \end{aligned} \qquad (5)$$

has the stress-energy tensor given by (4) ($T_2^2 = T_3^3$=0) and can approximately describe a global monopole with charge η and mass M. We consider the spherically symmetric gravitational collapse of the matter around the monopole. It can easily be seen that a black hole is formed. Now we can change the spherical symmetric static metric of the global monopole into the following Euclidean form

$${\rm d}s^2 = b^2{\rm d}\tau^2 + a^2{\rm d}y^2 + r^2{\rm d}\Omega^2, \qquad (6)$$

where a, b and r are functions of only the radial coordinate $y \in$[0,1]. The Euclidean time τ is chosen to have a period 2π, and the angles θ and φ are the usual coordinates of the unit sphere. By "black hole topology" we mean that the four geometries $(M,\ g)$ are regular with the product topology $R^2 \times S^2$, boundary $S^1 \times S^2$ at y=1, and Euler number χ=2.

The boundary at y=1 has a standard round two-sphere S^2 with area $4\pi r_{\rm B}^2$, where $r_{\rm B} = r(1)$. This sphere may be thought of as a cavity wall at $r_{\rm B}$ enclosing a spherical system with a black hole at the centre and through which heat can flow in either direction to maintain a temperature of the wall at $T = T(r_{\rm B}) \equiv \beta^{-1}$. The inverse temperature is related to the proper length of the round S^1 of the boundary by

$$\beta \equiv T^{-1} = \int_0^{2\pi} b(1){\rm d}\tau = 2\pi b(1). \qquad (7)$$

(We use units with $G = \hbar = k_{\rm B} = c = 1$ unless otherwise noted.) Of course there must be radiant energy between the surface of the black hole at $r_+ = r(0)$ and the cavity wall $r_{\rm B}$ even in order to speak of a temperature β^{-1} describing the system. The explicit effect of this radiation (black reaction) will be neglected in this paper,[36] but this will in no way effect the statement of boundary conditions, the method of analysis, and the qualitative conclusions. We see that these boundary conditions specify the three geometries of the boundary $S^1 \times S^2$ and furnish the thermodynamical data β and $r_{\rm B}$ that are needed for a canonical or grand canonical description.

The centre y=0, $r_+ = r(0)$, which represents the horizon of the black hole, is not a component of the three-dimensional boundary of M. Rather, it is a regular two-sphere with nonzero area $4\pi r_+^2$, and we can think of y=0 as a degenerate leaf of a foliation of M by three-dimensional hyper-surfaces y=const. These leaves have the topology $S^1 \times S^2$. As $y \to 0$, S^1 shrinks smoothly to zero to give S^2 at the centre. Thus we have

$$b(0) = 0. \qquad (8)$$

This description in the context of a smooth geometry requires that the $y - \tau$ plane near y=0 be isometric to a flat disk. The corresponding regularity condition is

$$\left.\frac{b'}{a}\right|_{y=0} = 1, \qquad (9)$$

where a prime denotes ∂_y. Combing Eqs.(8) and (9) shows that for a point $y = \overline{y}$ very near $y = 0$, the proper radial distance is

$$\int_0^{\overline{y}} a{\rm d}y \cong b(\overline{y}). \qquad (10)$$

The regularity conditions Eqs.(8) and (9) will not be varied in the action principles.

From the Gauss–Bonnet–Chern formula applied to the metric Eq.(6), the Euler number of M is deduced to be

$$\chi = 2\left\{\left(\frac{b'}{a}\right)\left[1 - \frac{(r')^2}{a^2}\right]\right\}\Bigg|_{y=0} = 2\left[1 - \frac{(r')^2}{a^2}\right]\Bigg|_{y=0}, \qquad (11)$$

where Eq.(8) was used to obtain the second form of Eq.(11). For χ=2, so

$$\left.\frac{(r')^2}{a^2}\right|_{y=0} = 0, \qquad (12)$$

for the metric Eq.(6), we can show that

$$R = -\frac{2}{abr^2}\left(\frac{r^2 b'}{a}\right)' - 2G^\tau_\tau, \tag{13}$$

where

$$G^\tau_\tau = \frac{1}{r^2 r'}\left[r\left(\frac{r'^2}{a^2} - 1\right)\right]'. \tag{14}$$

The other non-vanishing components of the Einstein tensor are

$$G^y_y = \frac{2b'r'}{a^2 br} + \left(\frac{r'^2}{ar}\right)^2 - \frac{1}{r^2}, \tag{15}$$

$$G^\theta_\theta = G^\varphi_\varphi = \frac{1}{abr^2}\left(\frac{r^2 b'}{a}\right)' + \frac{b}{a}\left(\frac{r'}{abr}\right)' + \left(\frac{r'}{ar}\right)^2. \tag{16}$$

The trace of the extrinsic curvature of a y=const hyper-surface is

$$K = -\frac{1}{abr^2}(br^2)'. \tag{17}$$

Evaluating the gravitational part $I_{\rm g}$ of the action (1) using the conditions (3)–(6), we have

$$\begin{aligned} I_{\rm g} =& -\frac{1}{16\pi}\int_M {\rm d}^4x\sqrt{g}(R - 2\Lambda) \\ &+ \frac{1}{8\pi}\int_{\partial M} {\rm d}^3x\sqrt{h}\ K - I_{\rm subtr} \\ =& \frac{1}{2}\int_0^{2\pi} {\rm d}\tau \int {\rm d}y\left(-\frac{2rb'r'}{a} - \frac{br'}{a} - ab + \Lambda abr^2\right) \\ &- \frac{1}{2}\int {\rm d}\tau\left(\frac{br^2}{a}\right)'\Big|_{y=0} - I_{\rm subtr}. \end{aligned} \tag{18}$$

The thermodynamical quantities and relations are obtained from the "classical action" $\tilde{I}$ (defined as the reduced action evaluated at its locally stable stationary points) using the well-known relation between the classical action and the thermodynamical potential

$$\tilde{I} = \beta F, \tag{19}$$

where F is the grand canonical potential since we are considering the grand canonical ensemble. All the thermodynamical quantities can be obtained from F using the classical relations.[31]

III. THE GLOBAL MONOPOLE ANTI-DE-SITTER BLACK HOLE

The global monopole AdS black hole in the grand ensemble is obtained using a negative cosmological constant Λ and the boundary condition $T \equiv T(r_{\rm B})$ and $\psi = \psi(r_{\rm B})$, where $r_{\rm B}$ is the boundary radius of the spherical cavity, T is the temperature at the boundary and ψ is the monopole potential difference between the horizon and the boundary. Instead of T, we can also use its inverse β.

The reduced action of the global monopole AdS black hole is obtained from the Euclidean action I given in Eq.(1). In section II, we split the action into two terms

$$I = I_{\rm g} + I_{\rm m}, \tag{20}$$

where $I_{\rm g}$ is the gravitational term given in Eq.(18) and $I_{\rm m}$ is the matter field term.

Now, we evaluate $I_{\rm m}$ in the spherical coordinate system

$$I_{\rm m} = \int_M {\rm d}^4x\sqrt{g}\ L_{\rm m}. \tag{21}$$

We use Eq.(2)

$$I_{\rm m} = -\beta\eta\psi/2, \tag{22}$$

where η is the charge of the black hole and ψ is the difference of the potential between y=0 and y=1.

To evaluate the gravitational term $I_{\rm g}$ of Eq.(18), we use the constraint

$$G^\tau_\tau + \Lambda g^\tau_\tau = 8\pi T^\tau_\tau, \tag{23}$$

where the stress-energy tensor component T^τ_τ is given by

$$T^\tau_\tau = -\frac{\eta^2}{8\pi r^2}. \tag{24}$$

The component of the Einstein tensor G^τ_τ for the metric (6) is

$$G^\tau_\tau = \frac{r'^2}{a^2r^2} - \frac{1}{r^2} + \frac{2r''}{a^2 r} - \frac{2a'r'}{a^3 r}. \tag{25}$$

Substituting Eqs.(24) and (25) into Eq.(23), we obtain

$$\Lambda = -\frac{\eta^2}{r^2} - \frac{r'^2}{a^2r^2} + \frac{1}{r^2} - \frac{2r''}{a^2 r} + \frac{2a'r'}{a^3 r}. \tag{26}$$

Rearranging the terms in Eq.(26)

$$\frac{\Lambda}{3}(r^3)' + \eta^2 r' + \left[\left(\left(\frac{r'}{a}\right)^2 - 1\right)r\right]' = 0 \tag{27}$$

and integrating the equation, we have

$$\frac{\Lambda}{3}(r^3) + \eta^2 r' + \left(\left(\frac{r'}{a}\right)^2 - 1\right)r = -2M, \tag{28}$$

where $-2M$ is an integrative constant. We rearrange the above equation as

$$\frac{r'^2}{a^2} = 1 - \frac{\Lambda}{3}r^2 - \eta^2 - \frac{2M}{r}. \tag{29}$$

If given $a^2 = -\Lambda/3$, we obtain

$$\frac{r'^2}{a^2} = 1 + a^2r^2 - \eta^2 - \frac{2M}{r}, \tag{30}$$

then we use Eq.(12), so

$$2M = r_+ + a^2 r_+^3 - \eta^2 r_+. \tag{31}$$

This is the well-known relation between the ADM mass of the global monopole AdS black hole and its event horizon radius.

Substituting Eq.(26) into Eq.(18) and using Eqs.(9), (12), (17) and (31),

$$\begin{aligned} I_{\rm g}^* =& -\beta r_{\rm B}\sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}} \\ & - \frac{1}{2}\beta\eta\psi - \pi r_+^2 - I_{\rm subtr}. \end{aligned} \tag{32}$$

Adding Eq.(26) into Eq.(32), we get the reduced action

$$\begin{aligned} I^* =& -\beta r_{\rm B}\sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}} \\ & - \beta\eta\psi - \pi r_+^2 - I_{\rm subtr}. \end{aligned} \tag{33}$$

The term $I_{\rm subtr}$ is of the form $\beta E_{\rm subtr}$, where $E_{\rm subtr}$ is a constant that does not depend on β or ψ, since $I_{\rm subtr}$ is an arbitrary term that can be used to fix the zero of the energy, but does not affect other thermodynamical variables.[14] For convenience, we use for the zero of the energy

$$E_{\rm ADS} = E(r_+ = 0, \eta = 0) = 0, \tag{34}$$

where $E_{\rm ADS}$ is the thermal energy of AdS space-time.

To evaluate $I_{\rm subtr}$, we compute the thermal energy of the global monopole AdS black hole for Eq.(33) and use the condition Eq.(34). The thermal energy is given by[31]

$$\begin{aligned} E =& F + \beta\left(\frac{\partial F}{\partial\beta}\right)_{\psi,r_{\rm B}} - \left(\frac{\partial F}{\partial\psi}\right)_{\beta,r_{\rm B}} \\ =& \left(\frac{\partial\tilde{I}}{\partial\beta}\right)_{\psi,r_{\rm B}} - \frac{\psi}{\beta}\left(\frac{\partial\tilde{I}}{\partial\psi}\right)_{\beta,r_{\rm B}} \\ =& -r_{\rm B}\sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r}{r_{\rm B}}} \\ & - E_{\rm subtr}. \end{aligned} \tag{35}$$

For Eqs.(28) and (35), we have

$$E_{\rm subtr} = -r_{\rm B}\sqrt{1+\alpha^2 r_{\rm B}^2}. \tag{36}$$

Substituting Eq.(36) into Eq.(33), we finally obtain the reduced action for the monopole AdS black hole

$$\begin{aligned} I^* =& \beta r_{\rm B}\left[\sqrt{1+\alpha^2 r_{\rm B}^2}\right. \\ & \left. - \sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}\right] \\ & - \beta\eta\psi - \pi r_+^2. \end{aligned} \tag{37}$$

Similarly, substituting Eq.(36) into Eq.(35), we obtain its thermal energy

$$\begin{aligned} E =& r_{\rm B}\left[\sqrt{1+\alpha^2 r_{\rm B}^2}\right. \\ & \left. - \sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}\right]. \end{aligned} \tag{38}$$

The entropy is obtained from

$$S = \beta^2\left(\frac{\partial F}{\partial\beta}\right)_{\psi,r_{\rm B}} = \beta\left(\frac{\partial\tilde{I}}{\partial\beta}\right)_{\psi,r_{\rm B}} - \tilde{I} = \pi r_+^2 = \frac{A_+}{4}, \tag{39}$$

where A_+ is the area of the event horizon. This is the usual Bekenstein–Hawking entropy.[37,38]

Once the values of β, ψ and $r_{\rm B}$ are fixed by the boundary conditions, then the reduced action is a function of only r_+ and η, i.e. $I^* = I^*(r_+, \eta)$. The local stability conditions are then: (1) $\delta I^* = 0$ and (2) the Hessian matrix is positive definite. The latter condition corresponds to a condition of dynamical as well as thermodynamical stability[22] and will be discussed in section V. We will start by investigating the first condition.

The condition of stationary $\delta I^* = 0$ gives

$$\begin{aligned} \frac{\partial I^*}{\partial\eta} =& \left[-\beta r_{\rm B}\left(1 - \frac{r_+}{r_{\rm B}}(-2\eta)\right]\right. \\ & \left.\Big/\left[2\sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}\right]\right. \\ & - \beta\psi = 0, \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{\partial I^*}{\partial r_+} =& \beta r_{\rm B}\left(\frac{1-\eta^2}{r_{\rm B}} - \frac{3\alpha^2 r_+^2}{r_{\rm B}}\right) \\ & \Big/\left[2\sqrt{1+\alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}\right] \\ & - 2\pi r_+ = 0. \end{aligned} \tag{41}$$

Solving Eqs.(40) and (41), we obtain

$$\psi = \beta(r_{\rm B} - r_+) \Big/ \sqrt{1 + \alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}, \quad (42)$$

$$\beta = \frac{4\pi r_+}{1 - \eta^2 + 3\alpha^2 r_+^2} \cdot \sqrt{1 + \alpha^2 r_{\rm B}^2 - \eta^2 - \frac{r_+}{r_{\rm B}} - \frac{\alpha^2 r_+^3}{r_{\rm B}} + \frac{\eta^2 r_+}{r_{\rm B}}}, \quad (43)$$

which are the differences of the global monopole potential between r_+ and $r_{\rm B}$ and the inverse Hawking temperature, respectively.

We define the more convenient variables

$$x \equiv \frac{r_+}{r_{\rm B}}, \ \overline{\beta} \equiv \frac{\beta}{4\pi r_{\rm B}}, \ \overline{\alpha}^2 \equiv \alpha^2 r_{\rm B}^2, \ \overline{\psi} \equiv \frac{\psi}{r_{\rm B}}. \quad (44)$$

In terms of these variables, Eqs.(42) and (41) are written as

$$\overline{\psi} = \eta(1-x)/\sqrt{(1-\eta^2)(1-x) + \overline{\alpha}^2 - \overline{\alpha}^2 x^3}, \quad (45)$$

$$\overline{\beta} = \frac{x}{1 - \eta^2 + 3\overline{\alpha}^2 x^2}\sqrt{(1-\eta^2)(1-x) + \overline{\alpha}^2 - \overline{\alpha}^2 x^3}. \quad (46)$$

The elimination of the variable η in Eqs.(44) and (45) yields the fifth degree equation

$$\begin{aligned} &\overline{\beta}^2(1 - \overline{\alpha}^2\overline{\psi}^2)^2 - 4\overline{\alpha}^2\overline{\beta}^2\overline{\psi}^2(1 - \overline{\alpha}^2\overline{\psi}^2)x \\ &- (1 + \overline{\alpha}^2 - 6\overline{\alpha}^2\overline{\beta}^2 + \overline{\psi}^2 + \overline{\alpha}^2\overline{\psi}^2 + 6\overline{\alpha}^2\overline{\beta}^2\overline{\psi}^2 \\ &- 4\overline{\alpha}^4\overline{\beta}^2\overline{\psi}^4)x^2 + (1 - \overline{\alpha}^2\overline{\psi}^2 - 12\overline{\alpha}^4\overline{\beta}^2\overline{\psi}^2)x^3 \\ &+ \overline{\alpha}^2(9\overline{\alpha}^2\overline{\beta}^2 - \overline{\psi}^2)x^4 + \overline{\alpha}^2 x^5 = 0. \end{aligned} \quad (47)$$

However, not every solution of this equation corresponds to a physical solution of black hole. This is because the radius of the event horizon of the global monopole AdS black hole must satisfy the following condition

$$1 - \eta^2 + 3\overline{\alpha}^2 x^2 \geq 0. \quad (48)$$

Comparing Eq.(48) with Eq.(42), we obtain

$$\psi^2 \leq (r_{\rm B} + 3\alpha^2 r_+^2 r_{\rm B})/[\alpha^2(2r_+ + r_{\rm B})]. \quad (49)$$

With the coordinate given in Eq.(44), the inequality (49) becomes

$$\overline{\psi}^2 \leq (1 + 3\overline{\alpha}^2 x)/[\overline{\alpha}^2(1 + 2x)]. \quad (50)$$

This is the condition that solutions of Eq.(47) must satisfy in order to correspond to physical black hole solutions.

Equation (47) has no analytical solutions. However, it can be numerically computed and presented in graphics. This will be done in the following section.

IV. ANALYSIS OF THE BLACK HOLE SOLUTIONS

In this section we present in graphics an analysis of the solutions of the equation (47).

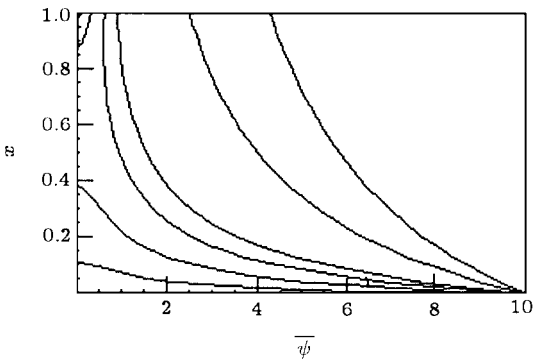


Fig.1. Solutions of Eq.(47) for $\overline{\alpha}$=0.1 as a function of the global monopole at the boundary for fixed values $\overline{\beta}$=0.1, 0.3, 0.6, 0.9, 3, 9 (from left to right); $\overline{\psi} \leq 10$, as imposed by condition (50). The stable solutions correspond to the upper branch of the curves. This means that for given values of $\overline{\beta}$ and $\overline{\psi}$, only the solution with a higher value of x is stable.

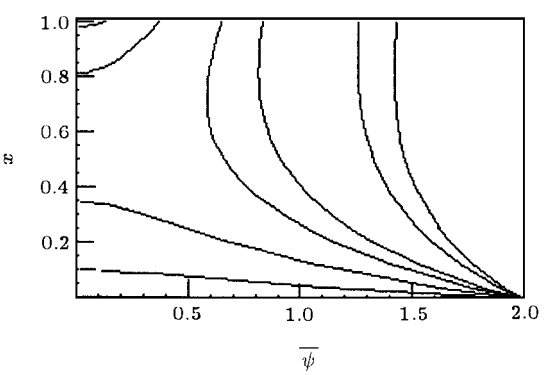


Fig.2. Solutions of Eq.(47) for $\overline{\alpha}$=0.2 as a function of the global monopole at the boundary for fixed values $\overline{\beta}$=0.1, 0.3, 0.6, 0.9, 3, 9 (from left to right); $\overline{\psi} \leq 5$, as imposed by condition (50). The stable solutions correspond to the upper branch of the curves. This means that for given values of $\overline{\beta}$ and $\overline{\psi}$, only the solution with a higher value of x is stable.

In the Figs.1–4, for fixed $\overline{\psi}$, there is a maximum of $\overline{\beta}_{\rm max}$. When $\overline{\beta} > \overline{\beta}_{\rm max}(\overline{\psi})$, there are no solutions. When $\overline{\beta} \leq \overline{\beta}_{\rm max}(\overline{\psi})$, there are two or one solutions, depending on the value of $\overline{\beta}$. In the limiting case $\overline{\beta} \to 0$ (i.e. $r_{\rm B}T \to \infty$), we find that there is a solution with $x = r_+/r_{\rm B} \to 1$ (we will see in section V that this is a stable solution). For $\overline{\psi}$=0 and $\overline{\beta} > \overline{\beta}_{\rm max}$ there are no solutions; for $\overline{\psi} \neq 0$, with fixed $\overline{\psi}$, we can see that x

increases as β has a higher value. Also, when we fix β, we can see that x increases as $\overline{\psi}$ has a higher value for a stable black hole.

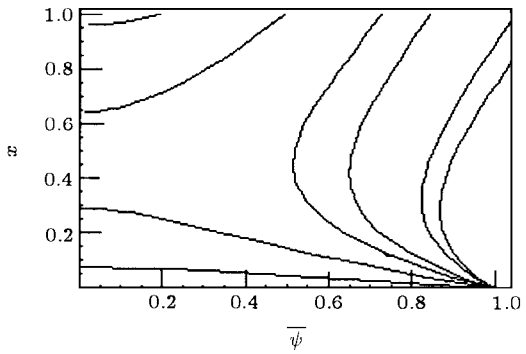


Fig.3. Solutions of Eq.(47) for $\overline{\alpha}$=1 as a function of the global monopole at the boundary for fixed values $\overline{\beta}$=0.1, 0.3, 0.6, 0.9, 3, and 9 (from left to right); $\overline{\psi} \leq 1$, as imposed by condition (50). The stable solutions correspond to the upper branch of the curves. This means that for given values of $\overline{\beta}$ and $\overline{\psi}$, only the solution with a higher value of x is stable.

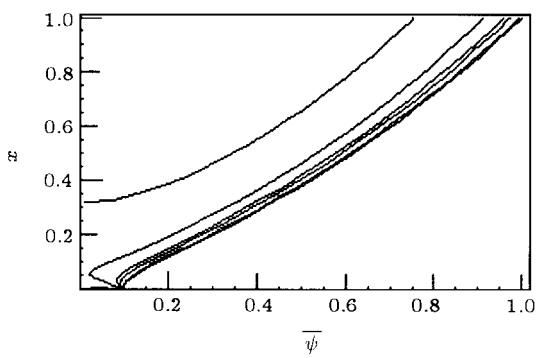


Fig.4. Solutions of Eq.(47) for $\overline{\alpha}$=10 as a function of the global monopole at the boundary for fixed values $\overline{\beta}$=0.1, 0.3, 0.6, 0.9, 3, and 9 (from left to right); $\overline{\psi} \leq 0.1$, as imposed by condition (50). The stable solutions correspond to the upper branch of the curves. This means that for given values of $\overline{\beta}$ and $\overline{\psi}$, only the solution with a higher value of x is stable.

The solutions at $T = 0$ (i.e. $\overline{\beta} = \infty$) correspond to extreme black holes. This can be analytically verified by replacing $\overline{\beta} = \infty$ in Eq.(47) and obtaining

$$1 - \overline{\alpha}^2\overline{\psi}^2 - 2\overline{\alpha}^2\overline{\psi}^2 x + 3\overline{\alpha}^2 x^2 = 0. \tag{51}$$

This corresponds to the condition of the extreme global monopole AdS black hole, which is in agreement with the known fact that only the extreme black hole has zero temperature.

Now, we investigate the solutions of Eq.(51). It has two solution if

$$\overline{\alpha}^2 > 2/3 \quad \overline{\psi}^2 > (1 + 3\overline{\alpha}^2)/3\overline{\alpha}^2, \tag{52}$$

or

$$\overline{\alpha}^2 < 2/3 \quad \overline{\psi}^2 < 1/\overline{\alpha}^2. \tag{53}$$

It has one solution if

$$\overline{\alpha}^2 > 2/3 \quad 1/\overline{\alpha}^2 < \overline{\psi}^2 < (1 + 3\overline{\alpha}^2)/3\overline{\alpha}^2, \tag{54}$$

or

$$\overline{\alpha}^2 < 2/3 \quad (1 + 3\overline{\alpha}^2)/3\overline{\alpha}^2 < \overline{\psi}^2 < 1/\overline{\alpha}^2. \tag{55}$$

And it has no solution if

$$\overline{\alpha}^2 > 2/3 \quad \overline{\psi}^2 < 1/\overline{\alpha}^2,$$

or

$$\overline{\alpha}^2 < 2/3 \quad \overline{\psi}^2 < (1 + 3\overline{\alpha}^2)/3\overline{\alpha}^2. \tag{56}$$

The solutions for $\overline{\alpha}$=0.5 are presented in Fig.2. We can see that the maximum value of $\overline{\beta}$, $\overline{\beta}_{\max}$ for which there are solutions is increasing, i.e. there are solutions for lower values of the temperature at $\overline{\alpha}$=0.5 rather than $\overline{\alpha}$=0.1.

Condition (50) implies for $\overline{\alpha}^2 < 2/3$

$$\overline{\psi} < 1/\overline{\alpha}. \tag{57}$$

This condition corresponds to the upper limit of the interval given in Eq.(53), which in this case is $\overline{\psi} < 10$. Notice that if in Eq.(47) we make x=0, we obtain precisely $\overline{\psi} = 1/\overline{\alpha}$ as we can see in Fig.1.

At the same time, for $\overline{\alpha}$=0.5, 1 and 10, Figs.2, 3 and 4 show similar properties as Fig.1. In these figures we find that $\overline{\beta}_{\max}(\overline{\psi})$ increase as $\overline{\alpha}$ increase, i.e. there are solutions for slightly lower values of the temperature where $\overline{\alpha}$ is increasing.

An important value of $\overline{\beta}$, first studied by York[14] in connection with the Schwarzschild black hole is $\beta = 8\pi r_{\rm B}/\sqrt{27}$. For higher values of β, hence lower temperature values, there are no black hole solutions. This is a quantum effect and, following York,[18] it can be understood as follows. One can associate a Compton-type wavelength λ to the energy $k_{\rm B}T$ of the thermal particles by $\lambda = \hbar c/(k_{\rm B}T)$, or in Planck units, if $\lambda > 8\pi r_{\rm B}/\sqrt{27} \approx 4.8 r_{\rm B}$. Then the thermal particles cannot be confined within the cavity and do not collapse to form a black hole. Using Eq.(46) (with $\overline{\psi}$=0, i.e. η=0), we can show that York's criterion for the absence of a black hole becomes

$$\lambda = \beta > \frac{8\pi r_{\rm B}}{\sqrt{27}}\left(1 - \frac{5}{18}\alpha^2 r_{\rm B}^2\right). \tag{58}$$

So, for a given temperature, it is more difficult to confine the thermal particles and harder to form a black hole. This accords with the idea that a negative cosmological constant shrinks space.

If we extend our previous analysis, we obtain

$$\lambda = \beta > \frac{8\pi r_{\rm B}}{\sqrt{27}}\left(1 - \frac{5}{18}\alpha^2 r_{\rm B}^2 + \frac{1}{2}\psi^2\right). \tag{59}$$

We can see that the global monopole potential has the opposite effect of the cosmological constant.

V. STABILITY

As mentioned previously, there is a second condition of local stability that has not yet been investigated. We will follow the same procedure as given in Ref.[22]. This is the condition that the Hessian matrix of reduced action be positively defined. For convenience in this analysis, we will use the variable $S = \pi r_+^2$ instead of r_+. The Hessian matrix of $I^*(S,\eta)$ is

$$I^*_{,ij} = \begin{pmatrix} I^*_{,\eta\eta} & I^*_{,\eta S} \\ I^*_{,\eta S} & I^*_{,SS} \end{pmatrix}. \tag{60}$$

The matrix is positively defined if its pivots are all positive.

The pivots of Eq.(60) are

$$I^*_{,\eta\eta} \tag{61}$$

and

$$\det(I^*_{,ij})/I^*_{,\eta\eta}. \tag{62}$$

From the first condition of local stability, $\nabla I^*=0$ yields

$$\left(\frac{\partial I^*}{\partial \eta}\right)_S = \beta\left(\frac{\partial H}{\partial \eta}\right)_S - \beta\psi = 0 \Rightarrow \psi = \left(\frac{\partial H}{\partial \eta}\right)_S, \tag{63}$$

$$\left(\frac{\partial I^*}{\partial S}\right)_\eta = \beta\left(\frac{\partial H}{\partial S}\right)_\eta - 1 = 0 \Rightarrow \beta^{-1} = \left(\frac{\partial H}{\partial S}\right)_\eta. \tag{64}$$

These are well-known thermodynamical relations.[31]

From Eq.(64), we compute the second derivative of I^* at the stationary points

$$\left.\frac{\partial^2 I^*}{\partial \eta^2}\right|_{\rm eq} = \beta\left.\frac{\partial^2 H}{\partial \eta^2}\right|_{\rm eq} = \beta\left(\frac{\partial \psi}{\partial \eta}\right)_S, \tag{65}$$

$$\left.\frac{\partial^2 I^*}{\partial \eta \partial S}\right|_{\rm eq} = \beta\left.\frac{\partial^2 H}{\partial \eta \partial S}\right|_{\rm eq} = \beta\left(\frac{\partial \psi}{\partial S}\right)_\eta = -\frac{1}{\beta}\left(\frac{\partial \beta}{\partial \eta}\right)_S, \tag{66}$$

$$\left.\frac{\partial^2 I^*}{\partial S^2}\right|_{\rm eq} = \beta\left.\frac{\partial^2 H}{\partial S^2}\right|_{\rm eq} = -\frac{1}{\beta}\left(\frac{\partial \beta}{\partial S}\right)_\eta, \tag{67}$$

where 'eq' represents quantities evaluated at equilibrium, i.e. the stationary points of the reduced action I^*.

The first pivot (61) is simply the first of these equations.

The second pivot (62) is easily obtained from Eqs.(65)–(67)

$$\begin{aligned}\frac{\det I^*_{,ij}}{I^*_{,\eta\eta}} &= -\frac{1}{\beta}\left(\frac{\partial \beta}{\partial S}\right)_\eta + \frac{1}{\beta}\left(\frac{\partial \beta}{\partial \eta}\right)_S\left(\frac{\partial \psi}{\partial S}\right)_\eta \Big/ \left(\frac{\partial \psi}{\partial \eta}\right)_S \\ &= -\frac{1}{\beta}\left[\left(\frac{\partial \beta}{\partial S}\right)_\eta + \left(\frac{\partial \beta}{\partial \eta}\right)_S\left(\frac{\partial \eta}{\partial S}\right)_\psi\right] \\ &= -\frac{1}{\beta}\left(\frac{\partial \beta}{\partial S}\right)_\psi = \frac{1}{C_{\psi,r_{\rm B}}},\end{aligned} \tag{68}$$

where $C_{\psi,r_{\rm B}}$ is the heat capacity at constant ψ and $r_{\rm B}$. Notice that in all these calculations we have implicitly held $r_{\rm B}$ constant, since in the grand canonical ensemble, the dimension of the system is held constant.

Imposing positive pivots for local stability yields

$$\left(\frac{\partial \psi}{\partial \eta}\right)_{S,r_{\rm B}} \geq 0, \tag{69}$$

$$C_{\psi,r_{\rm B}} \geq 0. \tag{70}$$

These conditions are identical to the classical thermodynamical stability conditions.[31] Therefore we can conclude that the dynamical stability conditions given in Eqs.(64), (69) and (70) are identical to the thermodynamical stability conditions.[22]

For the global monopole AdS black hole, the pivots obtained from Eqs.(65)–(68) are

$$\begin{aligned}\left(\frac{\partial \psi}{\partial \eta}\right)_{S,r_{\rm B}} =& (r_{\rm B} - r_+)\left[\left(1 - \frac{r_+}{r_{\rm B}}\right) - \left(1 - \frac{r_+}{r_{\rm B}}\right)\eta^2 \right. \\ &\left. + \alpha^2 r_{\rm B}^2 - \frac{\alpha^2 r_+^2}{r_{\rm B}}\right]^{-\frac{1}{2}} + \frac{\eta^2}{r_{\rm B}}(r_{\rm B} - r_+)^2 \\ &\cdot \left[\left(1 - \frac{r_+}{r_{\rm B}}\right) - \left(1 - \frac{r_+}{r_{\rm B}}\right)\eta^2 \right. \\ &\left. + \alpha^2 r_{\rm B}^2 - \frac{\alpha^2 r_+^2}{r_{\rm B}}\right]^{-\frac{3}{2}} > 0,\end{aligned} \tag{71}$$

which is positive, and

$$\begin{aligned} C_{\psi,r_{\rm B}} =& 4\pi[r_{\rm B}-r_{+}+\alpha^2(r_{\rm B}^3-r_{+}^3)] \\ & \cdot(2\alpha^2r_{+}^2\psi^2-\alpha^2r_{\rm B}^2\psi^2+r_{\rm B} \\ & +3\alpha^2r_{+}^2r_{\rm B}-\alpha^2\psi^2r_{+}r_{\rm B}) \\ & /[-3\alpha^4\psi^2r_{+}^4r_{\rm B}+\alpha^2(3r_{\rm B}+2\psi^2) \\ & +3r_{+}r_{\rm B}(1-\alpha^2r_{\rm B}\psi^2)+2r_{\rm B}^2(1+\alpha^2r_{\rm B}^2) \\ & \cdot(-1+\alpha^2r_{\rm B}^2\psi^2)+\alpha^2r_{+}^3(2r_{\rm B}-2\psi^2 \\ & -5\alpha^2r_{\rm B}\psi^2)+\alpha^2r_{+}^2r_{\rm B}(6r_{\rm B} \\ & +6\alpha^2r_{\rm B}^3+3\psi^2+4\alpha^2r_{\rm B}^2\psi^2)]. \end{aligned} \tag{72}$$

We have investigated Eq.(72). When $r_{\rm B}\to\infty$, $C_{\psi,r_{\rm B}}<0$, that is to say, the lower branches of the curves presented in Figs.1–4 correspond to the unstable solutions. However, $r_{\rm B}=ar_{+}$, $C_{\psi,r_{\rm B}}>0$ (a is a finite constant), i.e. the upper branches of the curves presented in Figs.1 to 4 correspond to the stable solutions. Therefore we can say that in general only the solutions with a higher value of x, i.e. with a larger event horizon radius, are stable.

In more detail, we can distinguish three cases: (1) for low $\overline{\alpha}$ (see Figs.1 and 2) and for values of $\overline{\beta}$ and $\overline{\psi}$, there is one solution which corresponds to a lower branch, therefore this solution is unstable; (2) for a higher value of $\overline{\alpha}$ (see Figs. 3 and 4), whenever there is only one solution, it corresponds to an upper branch and therefore this solution is stable; (3) whenever there are two solutions, for given values of $\overline{\beta}$, $\overline{\psi}$, $\overline{\alpha}$, the smaller is an instanton (unstable solution) and that with a larger event horizon radius is a stable black hole.

VI. COMMENTS ON SPECIAL CASES

Several black holes may be considered as the special cases of the global monopole AdS black hole.

(1) For ψ=0 and Λ=0, we obtain the Schwarzschild black hole studied in Ref.[18], so Eq.(47) becomes $\overline{\beta}^2-x^2+x^3=0$. Only the solution with a larger event horizon radius is stable.

(2) For Λ=0, we obtain the global monopole black hole. Equation (47) becomes $\overline{\beta}^2-(1+\overline{\psi}^2)x^2+x^3=0$. Under proper conditions, only the solution with a larger event horizon radius is stable. In the limit $r_{\rm B}\to\infty$, the heat capacity $C_{\psi,r_{\rm B}\to\infty}=-2\pi<0$ is negative and the solution is unstable.

(3) For $\psi=0$, we obtain the Schwarzschild AdS black hole. This black hole has been studied previously in Refs.[23] and [38]. Equation (47) becomes $\overline{\beta}^2-(1+\overline{\alpha}^2-6\overline{\alpha}^2\overline{\beta}^2)x^2+x^3+9\overline{\alpha}^4\overline{\beta}^2x^4+\overline{\alpha}^2x^5=0$. This black hole has two solutions for $\overline{\beta}<\overline{\beta}_{\rm max}$ and only the one with a larger horizon radius is stable. The horizon radius is $r_{+}=cr_{\rm B}$ and the temperature $T=\dfrac{4\pi r_{+}(1+6\overline{\alpha}^2r_{+}^2+9\overline{\alpha}^4r_{+}^4)^{\frac{1}{2}}}{\overline{\alpha}^2}\dfrac{1}{r_{\rm B}}$ goes to zero as $r_{\rm B}\to\infty$.

VII. CONCLUSIONS

We have studied the thermodynamics of the global monopole AdS black hole in York's formalism. In the grand canonical ensemble where the temperature and the global monopole potential are fixed at the boundary with a finite radius, we have found one or two black hole solutions, depending on the boundary conditions and the value of the cosmological constant. In general, when there are two solutions, only the black hole with a larger event horizon radius is stable; the other is unstable. For the fixed global monopole potential, the radius of the thermodynamical equilibrium region becomes smaller as the temperature of the black hole becomes higher. For the fixed temperature of the black hole, the radius of the thermodynamical equilibrium region becomes smaller as the global monopole potential becomes higher. We also found that the negative cosmological constant makes the temperature of a stable black hole higher, and the global monopole potential makes the temperature of a stable black hole lower.

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