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EFFECTS OF COLOURED NOISE IN A TWO-MODE LASER SYSTEM*

Luo Xiao-qin(罗晓琴), Zhu Shi-qun(朱士群), and Gao Wei-jian(高伟建)

Department of Physics, College of Sciences, Suzhou University, Suzhou 215006, China

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A homogeneously broadened two-mode laser system with multiplicative coloured noise is investigated when the mode coupling constant ξ=2. An analytical result is obtained when two-dimensional decoupling theory is applied to the system. The intensity auto-correlation function and effective eigenvalue of the laser system are calculated when the difference of pump parameters, multiplicative noise strength and noise correlation time are varied. It is shown that the multiplicative noise can enhance the fluctuations while the noise colour can reduce the fluctuations in the laser system.

Keywords: coloured noise, two-mode laser, decoupling theory, mode coupling constant
PACC: 0540, 4225

I. INTRODUCTION

In recent years, much attention has been paid to the effects of coloured noise in nonlinear dynamical systems.[1−15] Phenomena such as noise-induced phase transitions,[1−5] stochastic resonance,[6−8] mean first-passage time,[9] non-equilibrium systems[10−13] and laser systems[14−22] are investigated when coloured noise is included in these systems. In most of these studies, only one-dimensional systems are considered. However, coloured noise usually appears not only in one-dimensional but also in two-dimensional (2D) systems.

In this paper, a two-mode laser system with multiplicative coloured noise is investigated theoretically. In section II, a 2D decoupling theory is applied to a two-mode laser system when both multiplicative coloured noise and additive white noise are included in the system. A mean field theory is employed to find analytical solutions. The approximate result is compared to the exact solution when the mode coupling constant ξ=1 in the white noise limit. In section III, the intensity auto-correlation function $\lambda_{11}(0)$ and the effective eigenvalue $\lambda_{\rm eff}^{11}$ of the homogeneously broadened two-mode laser system with mode coupling constant ξ=2 are calculated. The effects of different values of pump parameters, multiplicative noise strength and noise correlation time τ are studied. A discussion of the theory concludes the paper.

II. THEORETICAL ANALYSIS

A. Equations of motion

The dimensionless amplitudes $r_1(t)$ and $r_2(t)$ of a two-mode laser system with both coloured and white noises follow the Langevin equations[14]

$$\begin{aligned}\begin{bmatrix}\dfrac{{\rm d}r_1}{{\rm d}t}\\[2ex]\dfrac{{\rm d}r_2}{{\rm d}t}\end{bmatrix} &= \begin{bmatrix}a_1r_1 - Ar_1^3 - \xi r_2^2 r_1 + \dfrac{P}{2r_1}\\[2ex] a_2r_2 - Ar_2^3 - \xi r_1^2 r_2 + \dfrac{P}{2r_2}\end{bmatrix}\\ &\quad + \begin{bmatrix}r_1 & 0\\ 0 & r_2\end{bmatrix}\begin{bmatrix}p\\ p\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}\\ &= \begin{bmatrix}h_1(\boldsymbol{r})\\ h_2(\boldsymbol{r})\end{bmatrix} + \begin{bmatrix}g_{11}(\boldsymbol{r})\\ g_{22}(\boldsymbol{r})\end{bmatrix}\begin{bmatrix}p(t)\\ p(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}G_{11}(\boldsymbol{r})\\ G_{22}(\boldsymbol{r})\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}. \end{aligned} \tag{1}$$

Here the vector $\boldsymbol{r} = \boldsymbol{e}_x r_1 + \boldsymbol{e}_y r_2$ represents the amplitude of the two-mode laser field, a_1 and a_2 are the pump parameters, and ξ is the mode coupling constant for the two-mode laser that takes the values ξ=2 and ξ=1 for homogeneously and inhomogeneously broadened lasers, respectively. The multiplicative coloured noise $p(t)$ and additive white noise $q_1(t)$ and $q_2(t)$ are statistically independent. Their mean and variance

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are given by

$$\begin{aligned}\langle p(t)\rangle =&\langle q_i(t)\rangle = 0,\\ \langle p(t)p(t')\rangle =&\frac{P'}{2\tau}\exp\left[-\frac{|t-t'|}{\tau}\right]\xrightarrow[\tau\to 0]{}\\ &P'\delta(t-t'),\ \ (i,j=1,2)\\ \langle q_i(t)q_j(t')\rangle =&P\delta_{ij}\delta(t-t'),\end{aligned}\tag{2}$$

where P' and P are the strengths of coloured and white noises $p(t)$ and $q_i(t)$ respectively, and τ is the noise correlation time.

The corresponding Fokker–Planck equation of the probability density $Q(\boldsymbol{r};t)$ of the variables $r_1(t)$ and $r_2(t)$ can be written as[23]

$$\begin{aligned}\frac{\partial Q((\boldsymbol{r},t)}{\partial t} =&-\sum_{i=1}^{2}\frac{\partial}{\partial r_i}\Big\{h_i(\boldsymbol{r})Q(\boldsymbol{r};t)\\ &-\frac{1}{2}g_{ii}(\boldsymbol{r})\langle p(t)\delta[\boldsymbol{r}(t)-\boldsymbol{r}]\rangle\\ &-\frac{P}{2}G_{ii}(\boldsymbol{r})\frac{\partial}{\partial r_i}[G_{ii}(\boldsymbol{r})Q(\boldsymbol{r};t)]\Big\}.\end{aligned}\tag{3}$$

Using the functional analysis, one has[16–19]

$$\langle p(t)\delta[\boldsymbol{r}(t)-\boldsymbol{r}]\rangle \approx P'_{\rm eff}\frac{\partial}{\partial r_i}[g_{ii}(\boldsymbol{r})Q(\boldsymbol{r};t)].\tag{4}$$

Here the effective noise strength[16,17] is given by

$$\begin{aligned}P'_{\rm eff} =&\frac{P'}{1-\tau\sum_{j=1}^{2}\left\{\left\langle\frac{\partial h_j(\boldsymbol{r})}{\partial r_j}\right\rangle-\left\langle h_j(\boldsymbol{r})\frac{\partial}{\partial r_j}\ln[g_{11}(\boldsymbol{r})g_{22}(\boldsymbol{r})]\right\rangle\right\}}\\ =&\frac{\langle I_1\rangle\langle I_2\rangle P'}{\langle I_1\rangle\langle I_2\rangle+\tau(2A\langle I_1\rangle\langle I_2\rangle+P)(\langle I_1\rangle+\langle I_2\rangle)}.\end{aligned}\tag{5}$$

where $\langle I_1\rangle=\langle r_1^2\rangle$, and $\langle I_2\rangle=\langle r_2^2\rangle$.

If the variables of amplitudes r_1 and r_2 are transformed to intensities I_1 and I_2 by the relations $I_1=r_1^2$ and $I_2=r_2^2$, the probability function $Q(I_1,I_2;t)$ corresponding to Eq.(3) can be written as

$$\begin{aligned}&\frac{\partial Q(I_1,I_2,t)}{\partial t}\\ =&-\frac{\partial}{\partial I_1}\Big\{F_1(I_1,I_2)Q-\frac{\partial}{\partial I_1}[D_{11}(I_1,I_2)Q]\Big\}\\ &-\frac{\partial}{\partial I_2}\Big\{F_2(I_1,I_2)Q-\frac{\partial}{\partial I_2}[D_{22}(I_1,I_2)Q]\Big\},\end{aligned}\tag{6}$$

where

$$\begin{aligned}F_1(I_1,I_2) =&2(a_1I_1-AI_1^2-\xi I_1I_2+P+P'_{\rm eff}I_1),\\ F_2(I_1,I_2) =&2(a_1I_2-AI_2^2-\xi I_2I_1+P+P'_{\rm eff}I_2),\\ D_{11}(I_1,I_2) =&2(PI_1+P'_{\rm eff}I_1^2),\\ D_{22}(I_1,I_2) =&2(PI_2+P'_{\rm eff}I_2^2),\end{aligned}\tag{7}$$

and Eqs.(4) and (5) are used to replace the corresponding terms in Eq.(3).

B. Comparison of steady-state solutions

In the time limit $t\to\infty, \partial Q(I_1,I_2;t)/\partial t=0$, the laser system reaches a steady state. The stationary probability distribution $Q(I_1,I_2)$ can be obtained if the mean field theory is used with $\xi I_1I_2\approx\xi\langle I_2\rangle I_1$ in $F_1(I_1,I_2)$ and $\xi I_2I_1\approx\xi\langle I_1\rangle I_2$ in $F_2(I_1,I_2)$ in Eq.(7). Then one has[24]

$$\begin{aligned}Q(I_1,I_2) =&N(P+P'_{\rm eff}I_1)^{\beta_1}(P+P'_{\rm eff}I_2)^{\beta_2}\\ &\cdot\exp[-\alpha_1(I_1+I_2)].\end{aligned}\tag{8}$$

Here N is the normalization constant, and

$$\begin{aligned}\alpha_1 =&A/P'_{\rm eff},\\ \beta_1 =&(a_1-\xi\langle I_2\rangle+AP/P'_{\rm eff})/P'_{\rm eff}-1,\\ \beta_2 =&(a_2-\xi\langle I_1\rangle+AP/P'_{\rm eff})/P'_{\rm eff}-1.\end{aligned}\tag{9}$$

In order to see the validity of the 2D decoupling theory, it is necessary to compare the steady-state solution Eq.(8) with the known exact results. In Ref.[14], the exact analytical result of the steady-state intensity distribution function in an inhomogeneously broadened two-mode laser system has been derived for multiplicative white noise when the mode coupling constant ξ=1 with equal pump parameters, i.e. $a_1=a_2=a$. The exact steady-state distribution function is of the form[10]

$$Q_{\rm c}(I_1,I_2)=N_1[P+P'(I_1+I_2)]^{\beta}\times\exp[-\alpha(I_1+I_2)].\tag{10}$$

Here N_1 is the normalization constant and

$$\alpha=\frac{1}{P'},\qquad \beta=\frac{a}{P'}+\frac{P}{(P')^2}-2.\tag{11}$$

The steady-state intensity distribution functions from Eqs.(8) and (10) are plotted in Fig.1 for multiplicative white noise when the mode coupling constant ξ is set to one with equal pump parameters $a_1=a_2=a$ =–5 and 5 respectively. It is seen that

the shapes of the distribution function $Q(I_1, I_2)$ and $Q_c(I_1, I_2)$ are almost the same. There are small deviations between the two curves when the value of I_1 is small. For a relatively large value of I_1, the two curves are almost indistinguishable. Therefore, the 2D decoupling theory can be extended to a two-mode laser system for multiplicative coloured noise, especially when the mode coupling constant ξ is greater than 1 and the pump parameters a_1 and a_2 are different. In the following calculations, only Eqs.(8) and (9) are employed.

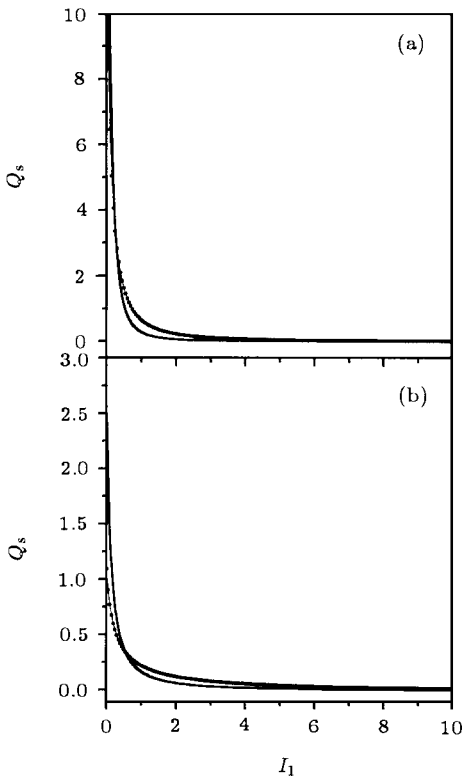


Fig.1. The steady-state intensity distribution $Q(I_1, I_2)$ of the two-mode laser is plotted as a function of the intensity I_1 when the multiplicative noise is white. The parameters are dimensionless and are chosen as: $P' = 10$, $P = 2$, $a_1 = a_2 = a$, $I_2 = 0$. —●—: decoupling theory with mean field approximation; ——: exact solution. (a) a=−5; (b) a=5.

Ⅲ. AUTO-CORRELATION FUNCTION AND EFFECTIVE EIGENVALUE

The mean values of the intensities $\langle I_1 \rangle$ and $\langle I_2 \rangle$, auto-correlation function $\lambda_{11}(0)$ and effective eigenvalue $\lambda_{\rm eff}^{11}$ of the laser system can be easily calculated from Eqs.(8) and (9), and are given by[15]

$$\langle I_i^n \rangle = \int_0^\infty I_i^n Q(I_1, I_2) \mathrm{d}I_1 \mathrm{d}I_2, \quad (i = 1, 2)$$

$$\lambda_{11}(0) = \frac{\langle I_1^2 \rangle}{\langle I_1 \rangle^2} - 1, \text{ and } \lambda_{\rm eff}^{11} = \frac{2P}{\lambda_{11}(0)\langle I_1 \rangle}. \quad (12)$$

The auto-correlation function $\lambda_{11}(0)$ and the effective eigenvalues $\lambda_{\rm eff}^{11}$ are plotted in Fig.2 as functions of the mean laser intensity $\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle$ for different values of the multiplicative coloured noise strength P', when other parameters are fixed constants. The mode coupling constant is ξ=2 and the difference of the pump parameter $\Delta a = a_1 - a_2 \neq 0$. It is seen that there is a peak in the curve of $\lambda_{11}(0)$. The height of the peak increases with the increasing value of multiplicative noise strength P', but the position of the peak is almost unchanged (see Figs.2(a) and 2(b)). It is also seen that the fluctuation of the weak mode shows the property of the thermal light since the value of $\lambda_{11}(0)$ is always close to one but never tends to zero as $\langle I \rangle$ becomes very large (see Fig.2(a)).

For the effective eigenvalue $\lambda_{\rm eff}^{11}$ of the two-mode laser system, it is seen that the value of $\lambda_{\rm eff}^{11}$ decreases with the increasing value of multiplicative noise strength P' (Figs.2(c) and 2(d)). However, a large difference exists between the weak mode with $\Delta a = -10$ (Fig.2(c)) and the strong mode with $\Delta a = 10$ (Fig.2(d)). For the weak mode, the curve of $\lambda_{\rm eff}^{11}$ is "U"-shaped. The value of the effective eigenvalue $\lambda_{\rm eff}^{11}$ is almost the same as P' increases for very small and very large values of $\langle I \rangle$ (Fig.2(c)). For the strong mode, the curve of $\lambda_{\rm eff}^{11}$ is "U"-shaped when P'=0 while it is "L"-shaped when P' is greater than zero. The value of the effective eigenvalue $\lambda_{\rm eff}^{11}$ is almost the same as P' increases for very small values of $\langle I \rangle$ but it decreases very fast as P' increases for very large values of $\langle I \rangle$.

The auto-correlation function $\lambda_{11}(0)$ and the effective eigenvalue $\lambda_{\rm eff}^{11}$ are plotted in Fig.3 against the mean laser intensity $\langle I \rangle$ for different values of noise correlation time τ when other parameters are fixed constants. It is seen that there is a peak in the curve of $\lambda_{11}(0)$. The height of the peak decreases as the value of the noise correlation time τ increases (Figs.3(a) and 3(b)). The position of the peak is shifted oppositely for weak and strong modes as τ increases. For the weak mode with $\Delta a = -10$, the position of the peak is shifted to a small value of $\langle I \rangle$ as the value of τ increases (Fig.3(a)). For the strong mode with $\Delta a = 10$, the position of the peak is shifted to a large value of $\langle I \rangle$ as the value of τ increases (Fig.3(b)). The fluctuation of the weak mode also shows the thermal light property.

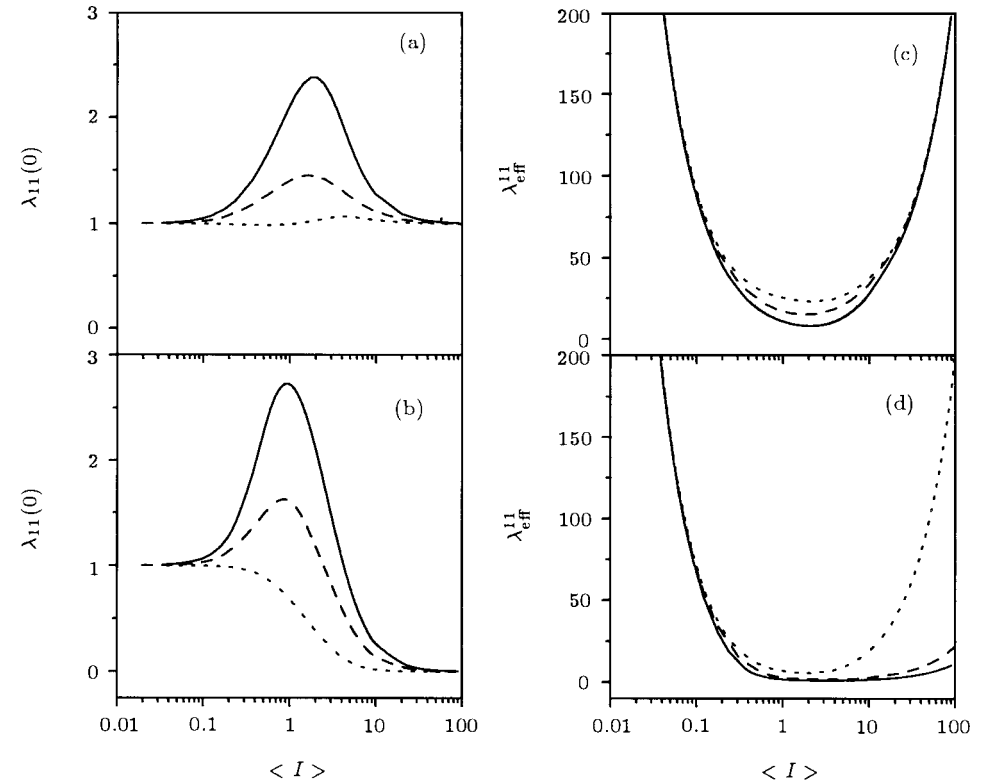


Fig.2. The auto-correlation function $\lambda_{11}(0)$ and effective eigenvalue $\lambda_{\rm eff}^{11}$ of the two-mode laser as functions of the average intensity $\langle I\rangle = \langle I_1\rangle + \langle I_2\rangle$ for different values of multiplicative coloured noise strength P'. The parameters are dimensionless and are chosen to be: $\xi = 2$, $P = 2$, $\tau = 0.2$. ····· P'=0; ------ P'=10; —— P'=20. (a) and (c) Δa=–10; (b) and (d) $\Delta a = 10$.

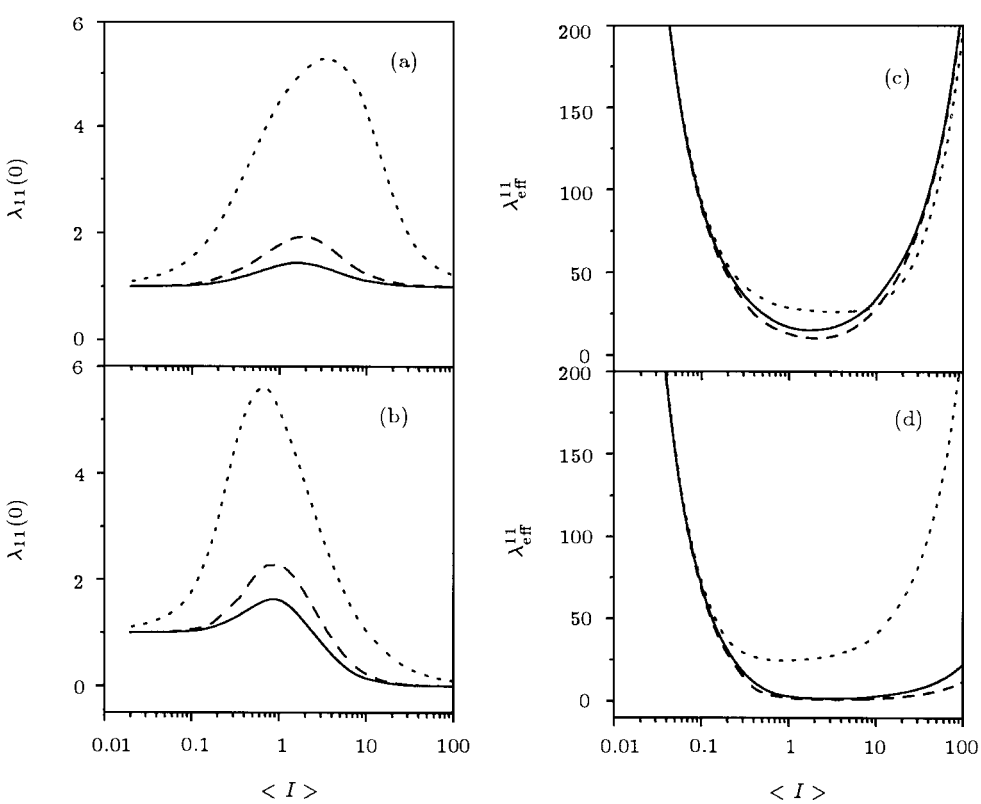


Fig.3. The auto-correlation function $\lambda_{11}(0)$ and effective eigenvalue $\lambda_{\rm eff}^{11}$ of the two-mode laser as functions of $\langle I\rangle$ for different values of the noise correlation time τ. The parameters are dimensionless and are chosen to be: $\xi = 2$, P=2, P'=10. ····· τ=0; ------ τ=0.1; —— τ=0.2. (a) and (c) $\Delta a = -10$; (b) and (d) $\Delta a = 10$.

The curve of the effective eigenvalue $\lambda_{\rm eff}^{11}$ also decreases as the value of the noise correlation time τ increases (Figs.3(a) and 3(b)). For the weak mode with $\Delta a = -10$, the curve of $\lambda_{\rm eff}^{11}$ is "U"-shaped. There is only a small deviation when the value of τ is changed. For the strong mode with $\Delta a = 10$, the curve of $\lambda_{\rm eff}^{11}$ is "U"-shaped for white multiplicative noise with τ=0, but it is "L"-shaped with a long tail when the value of τ is greater than zero.

The auto-correlation function $\lambda_{11}(0)$ and the effective eigenvalue $\lambda_{\rm eff}^{11}$ versus the mean laser intensity $\langle I \rangle$ are plotted in Fig.4 for different mode coupling constants ξ=1 and 2 while other parameters are fixed constants. The curve of $\lambda_{11}(0)$ decreases as the mode coupling constant ξ increases from 1 to 2. For the weak mode with $\Delta a = -10$, there is almost no difference between the curves of ξ=1 and ξ=2 when the two-mode laser system is below the threshold with a small value of $\langle I \rangle$. There are large deviations when the system is above the threshold with a large value of $\langle I \rangle$ (Fig.4(a)). For the strong mode with $\Delta a = 10$, there is only a small difference between the curves of ξ=1 and ξ=2 (Fig.4(b)).

From the curves of the effective eigenvalue $\lambda_{\rm eff}^{11}$ for the weak mode, it is very interesting to note that $\lambda_{\rm eff}^{11}$ is "U"-shaped for the homogeneously broadened two-mode laser system with the mode coupling constant ξ=2 while $\lambda_{\rm eff}^{11}$ is "L"-shaped for the inhomogeneously broadened laser system with mode coupling constant ξ=1 (Fig.4(c)). For the strong mode, the curve of $\lambda_{\rm eff}^{11}$ is "L"-shaped and there is almost no evident difference between the curves of ξ=1 and ξ=2.

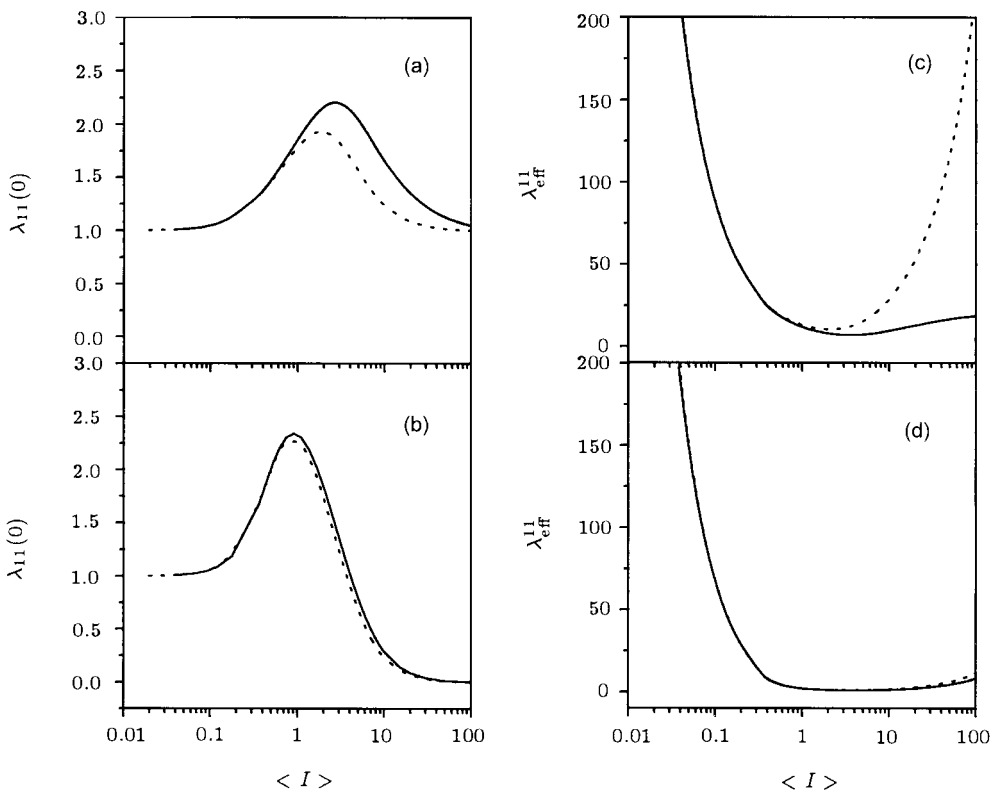


Fig.4. The auto-correlation function $\lambda_{11}(0)$ and effective eigenvalue $\lambda_{\rm eff}^{11}$ of the two-mode laser as functions of $\langle I \rangle$ for different values of the mode coupling constant ξ. The parameters are dimensionless and are chosen to be: $P = 2$, $P' = 10$, $\tau = 0.1$. —— ξ=1; ····· ξ=2. (a) and (c) $\Delta a = -10$; (b) and (d) $\Delta a = 10$.

IV. DISCUSSION

The 2D decoupling theory is applied to a two-mode laser system that includes both multiplicative coloured noise and additive white noise. The approximate result from the decoupling theory is compared to the exact solution from a two-mode laser theory with mode coupling constant ξ=1 in a multiplicative white noise limit. Excellent agreement is obtained. It is evident that the decoupling theory can be extended to the 2D coupled nonlinear system that includes multiplicative coloured noise.

Then the auto-correlation function $\lambda_{11}(0)$ and the effective eigenvalue $\lambda_{\rm eff}^{11}$ are calculated from the steady-state intensity distribution function $Q(I_1, I_2)$ with the mode coupling constant ξ=2 for different values of multiplicative coloured noise strength P' and noise correlation time τ when pump parameters are different with $\Delta a = -10$ or 10. It is shown that increasing the value of multiplicative noise strength P' can enhance the fluctuations in the laser system, while increasing the value of the noise correlation time τ can reduce the fluctuations in the system. When the mode coupling constant ξ is changed from 1 to 2 the value of the effective eigenvalue $\lambda_{\rm eff}^{11}$ is increased sharply as the mean laser intensity $\langle I \rangle$ is very large for the weak mode with $\Delta a = -10$. For the strong mode with $\Delta a = 10$, there is almost no difference between the curves of $\xi = 1$ and $\xi = 2$.

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