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EXACT AND ADIABATIC INVARIANTS OF FIRST-ORDER LAGRANGE SYSTEMS*

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A system of first-order differential equations is expressed in the form of first-order Lagrange equations. Based on the theory of symmetries and conserved quantities of first-order Lagrange systems, the perturbation to the symmetries and adiabatic invariants of first-order Lagrange systems are discussed. Firstly, the concept of higher-order adiabatic invariants of the first-order Lagrange system is proposed. Then, conditions for the existence of the exact and adiabatic invariants are proved, and their forms are given. Finally, an example is presented to illustrate these results.

Keywords: first-order Lagrange system, symmetry, exact invariant, adiabatic invariant
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I. INTRODUCTION

More and more attention has been paid to research on symmetries and conserved quantities of mechanical systems and many results have been reported in recent years. These results have enriched the knowledge of the essence of mechanical systems.[1−8] In 1917, Burgers first proposed adiabatic invariants which pointed to a special kind of Hamilton system.[9] The adiabatic invariants are almost unchanged when the parameter varies very slowly, and these play a very important role in research on the quasi-integrability of a mechanical system. In fact, the fact that the parameter varies very slowly is identical to the action of a small perturbation, so we have carried out research using the latter model in this paper.

The Lagrange systems in mechanics are usually of second order, i.e. the Lagrange functions include second-order terms of velocity. In biochemistry, molecular biology, physiology, ecology and social sciences, many systems can be described by first-order ordinary differential equations, and some of these can be expressed in Lagrange forms. So these Lagrange systems are of first order, i.e. the Lagrange functions are linear formulae of velocity. For example, the change rate of chemical concentration variables in the Lotka biochemistry oscillator[10] is just a first-order Lagrange system.

The study of adiabatic invariants has become a popular subject in mechanics,[11−13] atomic and molecular physics,[14] etc. In this paper, we discuss the adiabatic invariants of the first-order Lagrange systems from the viewpoint of symmetries and conserved quantities. Firstly, we construct the equations of the first-order Lagrange systems, and propose the concept of higher-order adiabatic invariants. Then, we prove the conditions for existence of the exact and adiabatic invariants, and give their forms. Finally, we present an example to illustrate these results.

II. DIFFERENTIAL EQUATIONS OF THE SYSTEM AND THEIR LAGRANGE REALIZATION

A general system of first-order differential equations can be expressed as

$$F_s(t, q_k, \dot{q}_k) = 0 \qquad (s, k = 1, \cdots, n). \tag{1}$$

Equations (1) can be expressed in the form of first-order Lagrange equations under certain conditions; that is, a function exists

$$L = \sum_{s=1}^{n} A_s(t, \boldsymbol{q})\dot{q}_s + B(t, \boldsymbol{q}) \tag{2}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = F_s \quad (s = 1, \cdots, n). \tag{3}$$

The conditions of the Lagrange realization for Eqs.(1)

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are their self-adjoint conditions as follows:

$$\frac{\partial F_s}{\partial \dot{q}_k} = -\frac{\partial F_k}{\partial \dot{q}_s},$$
$$\frac{\partial F_s}{\partial q_k} = \frac{\partial F_k}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F_k}{\partial q_s} \quad (s,k=1,\cdots,n). \tag{4}$$

The Lagrangian has the form[15]

$$L(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = -\sum_{s=1}^{n} q_s \int_0^1 F_s(t,\tau\boldsymbol{q},\tau\dot{\boldsymbol{q}})\mathrm{d}\tau, \tag{5}$$

where the Lagrangian is called the direct analytic expression.

If the equations are not self-adjoint, they can become self-adjoint after being multiplied by some multiplicators $h_{sk}(t,\boldsymbol{q})$. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_{k=1}^{n} h_{sk}(t,\boldsymbol{q})F_k(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (s=1,\cdots,n), \tag{6}$$

where the Lagrangian is called the indirect analytic expression.

So the system of first-order equations (1) can be expressed as the following first-order Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s=1,\cdots,n). \tag{7}$$

III. INFINITESIMAL TRANSFORMATIONS AND EXACT INVARIANTS

We introduce the infinitesimal transformations of time and variables

$$t^* = t + \varepsilon\tau^0(t,\boldsymbol{q}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s^0(t,\boldsymbol{q}), \tag{8}$$

where ε is an infinitesimal parameter, and τ^0 and ξ_s^0 are infinitesimal generators. Now we introduce an infinitesimal generator vector

$$X^{(0)} = \tau^0\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s^0\frac{\partial}{\partial q_s} \tag{9}$$

and its first extension

$$X^{(1)} = X^{(0)} + \sum_{s=1}^{n}(\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0)\frac{\partial}{\partial \dot{q}_s}. \tag{10}$$

The invariance of Eqs.(1) under the infinitesimal transformations (8) leads to the satisfaction of the following determining equations

$$\frac{\partial F_s}{\partial t}\tau^0 + \sum_{k=1}^{n}\frac{\partial F_s}{\partial q_k}\xi_k^0 + \sum_{k=1}^{n}\frac{\partial F_s}{\partial \dot{q}_k}(\dot{\xi}_k^0 - \dot{q}_k\dot{\tau}^0)\big|_{F_k=0} = 0. \tag{11}$$

Definition 1 If the generators τ^0 and ξ_s^0 satisfy the determining equations (11), then the corresponding transformations are called the Lie symmetrical transformations, and the corresponding symmetries are called the Lie symmetries.

The Lie symmetries do not always imply exact invariants. The following theorem gives the conditions of the existence of exact invariants and their forms.

Theorem 1 For the infinitesimal generators τ^0 and ξ_s^0 satisfying the determining equations (11), if a gauge function $G^0 = G^0(t,\boldsymbol{q})$ exists satisfying the following structure equation

$$L\dot{\tau}^0 + X^{(1)}(L) + \dot{G}^0 = 0, \tag{12}$$

then the system (7) possesses the following exact invariants

$$I_0 = L\tau^0 + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\xi_s^0 - \dot{q}_s\tau^0) + G^0 = \mathrm{const.} \tag{13}$$

Proof

Differentiating I_0 with respect to time t, and considering Eqs.(12) and (7), we have

$$\begin{aligned}\frac{\mathrm{d}I_0}{\mathrm{d}t} =& \dot{L}\tau^0 + L\dot{\tau}^0 + \sum_{s=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(\xi_s^0 - \dot{q}_s\tau^0) \\ &+ \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s^0 - \ddot{q}_s\tau^0 - \dot{q}_s\dot{\tau}^0) \\ &- L\dot{\tau}^0 - \frac{\partial L}{\partial t}\tau^0 - \sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\xi_s^0 \\ &- \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0) \\ =& \sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s}\right)(\xi_s^0 - \dot{q}_s\tau^0) = 0.\end{aligned}$$

IV. PERTURBATION TO THE SYMMETRIES AND ADIABATIC INVARIANTS

Firstly we give the concept of higher-order adiabatic invariants.

Definite 2 If $I_z(t,\boldsymbol{q},\varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth-order adiabatic invariant of the mechanical system.

Suppose the first-order Lagrange system corresponding to Eq.(7) is perturbed by small quantities

εW_s, then the equations of motion of the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=\varepsilon W_s \quad (s=1,\cdots,n). \tag{14}$$

Due to the action of εW_s, the primary symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau(t,\boldsymbol{q})$ and $\xi_s(t,\boldsymbol{q})$ denote the generators of time and space respectively after being perturbed, then

$$\begin{aligned}\tau&=\tau^0+\varepsilon\tau^1+\varepsilon^2\tau^2+\cdots,\\ \xi_s&=\xi_s^0+\varepsilon\xi_s^1+\varepsilon^2\xi_s^2+\cdots,\end{aligned} \tag{15}$$

and they satisfy

$$L\dot{\tau}+X^{(1)'}(L)+\varepsilon\sum_{s=1}^{n}W_s(\xi_s-\dot{q}_s\tau)+\dot{G}=0; \tag{16}$$

where

$$\begin{aligned}X^{(1)'}=&X^{(0)'}+\sum_{s=1}^{n}(\dot{\xi}_s-\dot{q}_s\dot{\tau})\frac{\partial}{\partial\dot{q}_s}\\ =&\tau\frac{\partial}{\partial t}+\sum_{s=1}^{n}\xi_s\frac{\partial}{\partial q_s}+\sum_{s=1}^{n}(\dot{\xi}_s-\dot{q}_s\dot{\tau})\frac{\partial}{\partial\dot{q}_s}\end{aligned}$$

with G in Eq.(16) being a gauge function. We let

$$G=G^0+\varepsilon G^1+\varepsilon^2G^2+\cdots, \tag{17}$$

$$X^{(1)^*}=\tau^k\frac{\partial}{\partial t}+\sum_{s=1}^{n}\xi_s^k\frac{\partial}{\partial q_s}+\sum_{s=1}^{n}(\dot{\xi}_s^k-\dot{q}_s\dot{\tau}^k)\frac{\partial}{\partial\dot{q}_s}. \tag{18}$$

Substituting Eqs.(15) and (17) into Eq.(16), we obtain

$$\begin{aligned}&L\dot{\tau}^k+X^{(1)^*}(L)+\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\tau^{k-1})+\dot{G}^k=0\\ &(k=0,1,2,\cdots),\end{aligned} \tag{19}$$

when k=0, the condition W_s=0 holds.

Theorem 2 For a first-order Lagrange system perturbed by small quantities εW_s, if the generators $\tau^k(t,\boldsymbol{q})$ and $\xi_s^k(t,\boldsymbol{q})$ under infinitesimal transformations satisfy the structure equation (19), and $G^k(t,\boldsymbol{q})$ is the gauge function, then

$$I_z=\sum_{k=0}^{z}\varepsilon^k\left[L\tau^k+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s^k-\dot{q}_s\tau^k)+G^k\right]$$

is a zth-order adiabatic invariant of the mechanical system.

Proof

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\sum_{k=0}^{z}\varepsilon^k\Bigg[\dot{L}\tau^k+L\dot{\tau}^k\\ &+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\dot{\xi}_s^k-\ddot{q}_s\tau^k-\dot{q}_s\dot{\tau}^k)\\ &+\sum_{s=1}^{n}(\xi_s^k-\dot{q}_s\tau^k)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-L\dot{\tau}^k-X^{(1)^*}(L)\\ &-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Bigg]\\ =&\sum_{k=0}^{z}\varepsilon^k\Bigg[\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}\right)(\xi_s^k-\dot{q}_s\tau^k)\\ &-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Bigg].\end{aligned}$$

Using Eq.(14), we have

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\sum_{k=0}^{z}\varepsilon^k\Bigg[\sum_{s=1}^{n}\varepsilon W_s(\xi_s^k-\dot{q}_s\tau^k)\\ &-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Bigg].\end{aligned}$$

Expanding the above formula and making a summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^{z+1}\sum_{s=1}^{n}W_s(\xi_s^z-\dot{q}_s\tau^z).$$

It shows that $\mathrm{d}I_z/\mathrm{d}t$ is directly proportional to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the mechanical system.

V. AN ILLUSTRATIVE EXAMPLE

For a concrete system of first-order differential equations, firstly, we verify whether it satisfies the self-adjoint conditions, and if not, whether it satisfies them after being multiplied by some multiplicators. Secondly, we construct the Lagrangian by Eq.(5). Thirdly, we construct the determining equations. Lastly, we construct the structure equation. If we can find a gauge function from the structure equation, then we can solve for the exact and adiabatic invariants by theorems 1 and 2, respectively.

The differential equations of the Lotka biochemistry oscillator are[10]

$$\dot{q}_1=\alpha_1+\beta_1\mathrm{e}^{q_2},\quad \dot{q}_2=\alpha_2+\beta_2\mathrm{e}^{q_1}, \tag{20}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2$ are constants. Now we try to study the exact and adiabatic invariants of the system.

Firstly, the system (20) is expressed in the Lagrange form, and Eqs.(20) can be rewritten as

$$F_1 = \dot{q}_2 - \alpha_2 - \beta_2 \mathrm{e}^{q_1} = 0, \quad F_2 = -\dot{q}_1 + \alpha_1 + \beta_1 \mathrm{e}^{q_2} = 0, \tag{21}$$

which satisfy the self-adjoint conditions (4). The Lagrangian (5) gives

$$\begin{aligned} L =& -q_1 \int_0^1 (\tau \dot{q}_2 - \alpha_2 - \beta_2 \mathrm{e}^{q_1 \tau}) \mathrm{d}\tau \\ & - q_2 \int_0^1 (-\tau \dot{q}_1 + \alpha_1 + \beta_1 \mathrm{e}^{q_2 \tau}) \mathrm{d}\tau \\ =& -\frac{1}{2}(q_1 \dot{q}_2 - q_2 \dot{q}_1) + (\alpha_2 q_1 - \beta_1 \mathrm{e}^{q_2}) \\ & - (\alpha_1 q_2 - \beta_2 \mathrm{e}^{q_1}) - \beta_2 + \beta_1. \end{aligned}$$

Secondly, we construct the determining equations and solve them. The determining equations (11) give

$$\begin{aligned} \dot{\xi}_1^0 - (\alpha_1 + \beta_1 \mathrm{e}^{q_2}) \dot{\tau}^0 &= \beta_1 \mathrm{e}^{q_2} \xi_2^0, \\ \dot{\xi}_2^0 - (\alpha_2 + \beta_2 \mathrm{e}^{q_1}) \dot{\tau}^0 &= \beta_2 \mathrm{e}^{q_1} \xi_1^0, \end{aligned}$$

and they have solutions

$$\tau^0 = 1, \quad \xi_1^0 = \xi_2^0 = 0.$$

They correspond to the Lie symmetries of system (20).

Thirdly, we construct the structure equation. Eq.(12) gives

$$\begin{aligned} & L\dot{\tau}^0 + \xi_1^0 \left(-\frac{1}{2}\dot{q}_2 + \alpha_2 + \beta_2 \mathrm{e}^{q_1} \right) \\ & + \xi_2^0 \left(\frac{1}{2}\dot{q}_1 - \alpha_1 - \beta_1 \mathrm{e}^{q_2} \right) \\ & + \frac{1}{2} q_2 (\dot{\xi}_1^0 - \dot{q}_1 \dot{\tau}^0) - \frac{1}{2} q_1 (\dot{\xi}_2^0 - \dot{q}_2 \dot{\tau}^0) + \dot{G}^0 = 0 \end{aligned} \tag{22}$$

which has a solution

$$G^0 = 0. \tag{23}$$

So we can obtain the following exact invariant by theorem 1

$$\begin{aligned} I_0 =& L - \sum_{s=1}^{2} \frac{\partial L}{\partial \dot{q}_s} \dot{q}_s \\ =& \alpha_2 q_1 + \beta_2 \mathrm{e}^{q_1} - \alpha_1 q_2 - \beta_1 \mathrm{e}^{q_2} - \beta_2 + \beta_1 \\ =& \text{const.} \end{aligned} \tag{24}$$

Lastly, we study the adiabatic invariants of the system. We suppose the system is perturbed by small quantities as follows

$$\varepsilon W_1 = \varepsilon q_1, \quad \varepsilon W_2 = \varepsilon q_2. \tag{25}$$

The structure equation (19) gives

$$\begin{aligned} & L\dot{\tau}^1 + \xi_1^1 \left(-\frac{1}{2}\dot{q}_2 + \alpha_2 + \beta_2 \mathrm{e}^{q_1} \right) \\ & + \xi_2^1 \left(\frac{1}{2}\dot{q}_1 - \alpha_1 - \beta_1 \mathrm{e}^{q_2} \right) \\ & + \frac{1}{2} q_2 (\dot{\xi}_1^1 - \dot{q}_1 \dot{\tau}^1) - \frac{1}{2} q_1 (\dot{\xi}_2^1 - \dot{q}_2 \dot{\tau}^1) \\ & + W_1 (\xi_1^0 - \dot{q}_1 \tau^0) + W_2 (\xi_2^0 - \dot{q}_2 \tau^0) + \dot{G}^1 = 0 \end{aligned} \tag{26}$$

which has solutions as

$$\tau^1 = 1, \quad \xi_1^1 = \xi_2^1 = 0, \quad G^1 = \frac{1}{2}(q_1^2 + q_2^2). \tag{27}$$

So we can obtain the following first-order adiabatic invariants by theorem 2

$$\begin{aligned} I_1 =& \alpha_2 q_1 + \beta_2 \mathrm{e}^{q_1} - \alpha_1 q_2 - \beta_1 \mathrm{e}^{q_2} - \beta_2 + \beta_1 \\ & + \varepsilon \Big[\alpha_2 q_1 + \beta_2 \mathrm{e}^{q_1} - \alpha_1 q_2 - \beta_1 \mathrm{e}^{q_2} \\ & - \beta_2 + \beta_1 + \frac{1}{2}(q_1^2 + q_2^2) \Big]. \end{aligned}$$

Furthermore we can obtain more higher-order adiabatic invariants.

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