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EFFECTS OF CONSTRAINTS ON LIE SYMMETRIES AND CONSERVED QUANTITIES OF A BIRKHOFF SYSTEM*

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After a Birkhoff system is restricted by constraints, the determining equations, the Lie symmetries, the structure equation and the form of conserved quantities corresponding to the Lie symmetries will change. Some Lie symmetries will disappear and under certain conditions some Lie symmetries will still remain present. The condition under which Lie symmetries and conserved quantities of the system will remain is given.

Keywords: Birkhoff system, Lie symmetry, conserved quantity, constraint
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I. INTRODUCTION

Studies on symmetry and conserved quantities are not only of mathematical significance, but also of more importance in the physical field. Lie symmetry is one of those important symmetries. There has been significant progress in the research on Lie symmetry. Monographs[1−3] about Lie symmetry from the mathematical point of view have come forth. Since Lutzky and others introduced Lie's invariance theory of the differential equations under infinitesimal transformations into the studies on mechanical systems in the 1970's, researches on Lie symmetries of mechanical system have made great progress.[4−7]

The dynamics of Birkhoff system[8] is an important branch of classical mechanics in modern time. Santilli made primary research on Birkhoff system and got some results.[9] Later, the frame of Birkhoff dynamics was constructed.[10] In recent years, the study on the Birkhoff dynamics has become a popular subject and many important results have been obtained.[11−13]

However, after constraints are imposed on a Birkhoff system, how Lie symmetries and conserved quantities of the system will change is not yet clear. So in this paper, we study the effects of constraints on Lie symmetries, structure equation and conserved quantities of Birkhoff systems. The condition under which Lie symmetries and conserved quantities of the system will remain is given.

II. EQUATIONS OF MOTION OF A BIRKHOFF SYSTEM

The equation of motion of a Birkhoff system can be expressed in the form[10]

$$\Omega_{\mu\nu}\dot{a}^{\nu}\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=0\quad(\mu=1,\cdots,2n),\tag{1}$$

where

$$\Omega_{\mu\nu}=\frac{\partial R_{\nu}}{\partial a^{\mu}}+\frac{\partial R_{\mu}}{\partial a^{\nu}}\tag{2}$$

is called Birkhoff's tensor, $B=B(t,\boldsymbol{a})$ and $R_{\mu}=R_{\mu}(t,\boldsymbol{a})$ are the Birkhoffian and the Birkhoff's functions respectively.

If variables $a^{\mu}(\mu=1,\cdots,2n)$ are not independent mutually, suppose that the system is subjected to g constrains

$$f_{\beta}(t,a^{\mu})=0\quad(\beta=1,\cdots,g),\tag{3}$$

then virtue displacements of the system (1) are restricted by

$$\frac{\partial f_{\beta}}{\partial a^{\mu}}\delta a^{\mu}=0\quad(\beta=1,\cdots,g).\tag{4}$$

By virtue of Pfaff-Birkhoff-D'Alembert principle[10] and Eq.(4), the equations with multipliers of constrained Birkhoff system can be easily acquired

$$\Omega_{\mu\nu}\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=\lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}}\quad(\mu=1,\cdots,2n).\tag{5}$$

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The dynamics of constrained Birkhoff system is determined by Eqs.(3) and (5).

Suppose that the system is not singular, i.e.,

$$\det(\Omega_{\mu\nu}) \neq 0, \tag{6}$$

Eq.(1) can be written in the form

$$\dot{a}^{\nu} = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right) \quad (\mu = 1, \cdots, 2n), \tag{7}$$

where

$$\Omega^{\mu\nu}\Omega_{\nu\tau} = \delta_{\mu\tau}\begin{cases} 1 & \mu = \tau, \\ 0 & \mu \neq \tau. \end{cases} \tag{8}$$

Similarly, Eq.(5) can also be expressed explicitly

$$\dot{a}^{\nu} = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t} + P_{\mu}\right) \quad (\mu = 1, \cdots, 2n), \tag{9}$$

where

$$P_{\mu} = P_{\mu}(t, \boldsymbol{a}) = \lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}}. \tag{10}$$

The multiplier λ_{β} can be solved as a function of t, $\boldsymbol{a}$ before integrating the equations of motion,

$$\lambda_{\beta} = -\frac{\Delta_{\gamma\beta}}{\Delta}\left\{\frac{\partial f_{\gamma}}{\partial a^{\rho}}\Omega^{\rho\mu}\left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right) + \frac{\partial f_{\gamma}}{\partial t}\right\} \quad (\beta = 1, \cdots, g), \tag{11}$$

where

$$\Delta = \det\left(\frac{\partial f_{\gamma}}{\partial a^{\rho}}\Omega^{\rho\mu}\frac{\partial f_{\beta}}{\partial a^{\mu}}\right), \tag{12}$$

$\Delta_{\gamma\beta}$ is the algebraic complement of the (l, s) element of Δ.[10] For λ_{β}, there are:

Proposition 1[7] If the number of constraints (3) is odd, there is no unique solution of λ_{β}.

Proposition 2[7] If the given constraints (3) are integrals of free Birkhoff system (1), then $\lambda_{\beta} = 0$ $(\beta = 1, \cdots, g)$.

Equations (7) and (9) can be simply written as

$$\dot{a}^{s} = g_{s}(t, \boldsymbol{a}) \qquad (s = 1, \cdots, 2n), \tag{13}$$

$$\dot{a}^{s} = h_{s}(t, \boldsymbol{a}) \qquad (s = 1, \cdots, 2n). \tag{14}$$

Equation (14) is called equations of motion of the corresponding free Birkhoff system of constrained Birkhoff system expressed by Eqs.(3) and (5). Apparently, we have

$$h_{s} = g_{s} + \Omega^{\mu\nu}P_{\mu}. \tag{15}$$

Ⅲ. INFINITESIMAL TRANSFORMATIONS AND DETERMINING EQUATIONS

Choosing the infinitesimal transformations of time and coordinates as

$$t^{*} = t + \Delta t,$$

$$a^{\mu^{*}}(t^{*}) = a^{\mu}(t) + \Delta a^{\mu} \quad (\mu = 1, \cdots, 2n), \tag{16}$$

or its extended form

$$t^{*} = t + \varepsilon\xi_{0}(t, \boldsymbol{a}),$$

$$a^{\mu^{*}} = a^{\mu} + \varepsilon\xi_{\mu}(t, \boldsymbol{a}) \quad (\mu = 1, \cdots, 2n), \tag{17}$$

where ε is the infinitesimal parameter, and ξ_{0}, ξ_{μ} are the generators of infinitesimal transformations. Then we introduce the generator vector of infinitesimal transformations

$$X^{(0)} = \xi_{0}\frac{\partial}{\partial t} + \xi_{\mu}\frac{\partial}{\partial a^{\mu}}, \tag{18}$$

and its first extended form

$$X^{(1)} = X^{(0)} + (\dot{\xi}_{\mu} - \dot{a}_{\mu}\dot{\xi}_{0})\frac{\partial}{\partial \dot{a}^{\mu}}. \tag{19}$$

According to invariance of ordinary differential equations under infinitesimal transformations in Lie's theory, the invariances of Eqs.(13) and (14) under the infinitesimal transformations (17) are respectively led up to the satisfaction of the following determining equations

$$\dot{\xi}_{\mu} - g_{s}\dot{\xi}_{0} = X^{(0)}[g_{s}], \tag{20}$$

$$\dot{\xi}_{\mu} - h_{s}\dot{\xi}_{0} = X^{(0)}[h_{s}]. \tag{21}$$

The invariance of constraint equation (3) under the infinitesimal transformation (15) can be expressed by the following restriction equation

$$X^{(0)}[f_{\beta}(t, \boldsymbol{a})]|_{f_{\beta}=0} = 0 \quad (\beta = 1, \cdots, g). \tag{22}$$

We hereinafter give the definitions of Lie symmetry.

Definition 1 If generators ξ_{0}, ξ_{μ} satisfy the determining equations (20), then the corresponding symmetries are called Lie symmetries of free Birkhoff system (1).

Definition 2 If generators ξ_{0}, ξ_{μ} satisfy the determining equations (21), then the corresponding symmetries are called Lie symmetries of corresponding free Birkhoff system of the constrained Birkhoff system expressed by Eqs.(3) and (5).

Definition 3 If generators ξ_{0}, ξ_{μ} satisfy the determining equations (21) and restriction equations (22), then the corresponding symmetries are called weak Lie symmetries of the constrained Birkhoff system expressed by Eqs.(3) and (5).

Because we obtain Eqs.(5) by virtue of Eq.(4), the generators ξ_0, ξ_μ are restricted by

$$\frac{\partial f_\beta}{\partial a^\mu}(\dot{\xi}_\mu - \dot{a}_\mu \dot{\xi}_0) = 0 \quad (\beta = 1, \cdots, g). \tag{23}$$

Then we have:

Definition 4 If generators ξ_0, ξ_μ satisfy the determining equations (21), restriction equations (22) and additional restriction equations (23), then the corresponding symmetries are called strong Lie symmetries of the constrained Birkhoff system expressed by Eqs.(3) and (5).

From definitions 3 and 4, we have:

Proposition 3[7] Weak Lie symmetries of a constrained Birkhoff system must be its strong Lie symmetries.

When constraints are exerted on a free Birkhoff systems, some Lie symmetries will still remain present. We have:

Proposition 4 If infinitesimal generators ξ_0, ξ_μ and P_ν satisfy

$$-\Omega^{\mu\nu} P_\nu \dot{\xi}_0 = X^{(0)}(\Omega^{\mu\nu} P_\nu), \tag{24}$$

then Lie symmetries of free Birkhoff system (13) are also Lie symmetries of system (14).

Proof Substituting Eq.(13) into the determining equations (20) and taking note of relation (24), we obtain Eq.(21).

Proposition 5 If infinitesimal generators ξ_0, ξ_μ are corresponding to the Lie symmetries of free Birkhoff system (13) and satisfy Eqs.(22) and (24), then these symmetries are also Lie symmetries of the constrained Birkhoff system expressed by Eqs.(3) and (5).

IV. STRUCTURE EQUATION AND CONSERVED QUANTITY

Lie symmetries do not always generate conserved quantities. The following propositions 6–9 give the condition under which Lie symmetries generate conserved quantities and the form of conserved quantities.

Proposition 6[7] For those infinitesimal generators ξ_0, ξ_μ satisfying the determining equation (20), if there is a gauge function $G = G(t, \boldsymbol{a})$ satisfying the following structure equation

$$(R_\mu \dot{a}^\mu - B)\dot{\xi}_0 + X^{(1)}(R_\mu \dot{a}^\mu - B) + \dot{G} = 0, \tag{25}$$

then the free Birkhoff system (1) has conserved quantities as follows

$$I = R_\mu \xi_\mu - B\xi_0 + G = \text{const.} \tag{26}$$

Proposition 7[7] For those infinitesimal generators ξ_0, ξ_μ satisfying the determining equation (21), if there is a gauge function $G = G(t, \boldsymbol{a})$ satisfying the following structure equation

$$\begin{aligned}&(R_\mu \dot{a}^\mu - B)\dot{\xi}_0 + X^{(1)}(R_\mu \dot{a}^\mu - B)\\ &- P_\mu(\xi_\mu - \dot{a}^\mu \xi_0) + \dot{G} = 0,\end{aligned} \tag{27}$$

then the corresponding free Birkhoff system (14) of the constrained Birkhoff system expressed by Eqs.(3) and (5) has conserved quantities as Eq.(26).

Proposition 8[7] For those infinitesimal generators ξ_0, ξ_μ satisfying the determining equation (21) and restriction equation (22), if there is a gauge function $G = G(t, \boldsymbol{a})$ satisfying the structure equation (27), then the constrained Birkhoff system expressed by Eqs.(3) and (5) has conserved quantities as Eq.(26).

Proposition 9 If free Birkhoff system (13) and the corresponding free Birkhoff system (14) of the constrained Birkhoff system expressed by Eqs.(3) and (5) have the same Lie symmetries, and the condition

$$P_\mu(\xi_\mu - \dot{a}^\mu \xi_0) = 0 \tag{28}$$

holds, then both the free Birkhoff system (1) and the constrained Birkhoff system expressed by Eqs.(3) and (5) have the same conserved quantities.

V. ILLUSTRATIVE EXAMPLES

Suppose a free Birkhoff system is given by

$$\begin{aligned}&R_1 = a^3, \quad R_2 = a^4, \quad R_3 = R_4 = 0,\\ &B = \frac{1}{2}\{(a^1)^2 + (a^3)^2 + (a^4)^2\},\end{aligned} \tag{29}$$

and the following constraints are exerted on the system

$$f_1 = a^1 a^3 - c_1^2 = 0, \quad f_2 = a^1 + a^4 - c_2 = 0. \tag{30}$$

In light of Eqs.(2), we obtain

$$(\Omega_{\mu\nu}) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$(\Omega^{\mu\nu}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{31}$$

The equations of motion of the free Birkhoff system (29) are

$$\dot{a}^1 = a^3, \quad \dot{a}^2 = a^4,$$
$$\dot{a}^3 = -a^1, \quad \dot{a}^4 = 0, \tag{32}$$

and the equations of motion of the constrained Birkhoff system expressed by Eqs.(29) and (30)are

$$\begin{aligned}
\dot{a}^1 &= a^3 + \lambda_1 a^1, \\
\dot{a}^2 &= a^4 + \lambda_2, \\
\dot{a}^3 &= -a^1 - \lambda_1 a^3 - \lambda_2, \\
\dot{a}^4 &= 0.
\end{aligned} \tag{33}$$

From Eqs.(30) and (33), we obtain

$$\begin{aligned}
P_1 &= \lambda_1 \frac{\partial f_1}{\partial a^1} + \lambda_2 \frac{\partial f_2}{\partial a^1} = -a^1, \\
P_2 &= 0, \\
P_3 &= -a^3, \\
P_4 &= -a^1 + \frac{(a^3)^2}{a^1}.
\end{aligned} \tag{34}$$

The condition (24) gives

$$\begin{aligned}
a^3 \dot{\xi}_0 &= X^{(0)}(-a^3), \\
-\left(-a^1 + \frac{(a^3)^2}{a^1}\right) \dot{\xi}_0 &= X^{(0)} \left(-a^1 + \frac{(a^3)^2}{a^1}\right), \\
-a^1 \dot{\xi}_0 &= X^{(0)}(a^1).
\end{aligned} \tag{35}$$

The determining equation (20) gives

$$\begin{aligned}
\dot{\xi}_1 &= a^3 \dot{\xi}_0 + \xi_3, \\
\dot{\xi}_2 &= a^4 \dot{\xi}_0 + \xi_4, \\
\dot{\xi}_3 &= -a^1 \dot{\xi}_0 - \xi_1, \\
\dot{\xi}_4 &= 0.
\end{aligned} \tag{36}$$

The determining equation (21) gives

$$\begin{aligned}
&\dot{\xi}_1 = 0, \\
&\dot{\xi}_2 - \left[a^4 - a_1 + \frac{(a^3)^2}{a^1}\right] \dot{\xi}_0 \\
&= \xi_4 - \xi_1 - \frac{(a^3)^2}{(a^1)^2} \xi_1 + \frac{2a^3}{a^1} \xi_3, \\
&\dot{\xi}_3 = 0, \\
&\dot{\xi}_4 = 0.
\end{aligned} \tag{37}$$

Equations (36) have the following solutions

$$\xi_0 = 1, \ \xi_2 = t, \ \xi_4 = 1, \ \xi_1 = \xi_3 = 0, \tag{38}$$

$$\xi_0 = 1, \ \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0, \tag{39}$$

$$\xi_0 = 0, \ \xi_2 = 1, \ \xi_1 = \xi_3 = \xi_4 = 0, \tag{40}$$

$$\xi_0 = 0, \ \xi_2 = t, \ \xi_4 = 1, \ \xi_1 = \xi_3 = 0, \tag{41}$$

$$\xi_0 = 1, \ \xi_2 = t, \ \xi_4 = 1, \ \xi_1 = \sin t, \ \xi_3 = \cos t, \tag{42}$$

$$\xi_0 = 0, \ \xi_2 = t, \ \xi_4 = 1, \ \xi_1 = \sin t, \ \xi_3 = \cos t. \tag{43}$$

Equations(38)–(43) are corresponding to Lie symmetries of the free Birkhoff system (29); Eqs.(38)–(41) also satisfy condition (35). By proposition 4, they are also Lie symmetries of the corresponding free Birkhoff system of the constrained Birkhoff system expressed by Eqs. (29) and (30).

The restriction equation (22) gives

$$a^3 \xi_1 + a^1 \xi_3 = 0, \quad \xi_1 + \xi_4 = 0. \tag{44}$$

Equations (39) and (40) satisfy the condition (44), so they are also corresponding to Lie symmetries of the constrained Birkhoff system expressed by Eqs. (29) and (30).

The condition (28) gives

$$\begin{aligned}
&- a^1(\xi_1 - \dot{a}^1 \xi_0) - a^3(\xi_3 - \dot{a}^3 \xi_0) \\
&+ \left[-a^1 + \frac{(a^3)^2}{a^1}\right] (\xi_4 - \dot{a}^4 \xi_0) = 0.
\end{aligned} \tag{45}$$

The solutions (39) and (40) also satisfy the condition (45), so by proposition 9 when they are chosen as generators of infinitesimal transformations, both the free Birkhoff system (29) and the constrained Birkhoff system expressed by Eqs. (29) and (30) have the same conserved quantities. From structure equations (25) and (27), we can obtain $G = 0$, then the same conserved quantities of the two systems are

$$I_1 = R_\mu \xi_\mu - B\xi_0 + G = -B = \text{const.} \tag{46}$$
$$I_2 = R_\mu \xi_\mu - B\xi_0 + G = a^4 = \text{const.} \tag{47}$$

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