Vol 10 No 1, January 2001
1009-1963/2001/10(01)/0007-05

Chinese Physics

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CUMULANTS OF STOCHASTIC RESPONSE FOR A CLASS OF SPECIAL NONHOLONOMIC SYSTEMS*

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(Received 18 May 2000; revised manuscript received 26 June 2000)

This paper studies the response cumulants for a kind of special nonholonomic systems under non-Gaussian, delta-correlated excitations. We present a new methodology for formulating the equations governing the evolution of the response cumulants of the stochastic dynamic systems. The response cumulant differential equations (CDEs) derived can be used to calculate the response cumulants for both linear and nonlinear systems. One example is given to illustrate how to use the CDEs for calculating response cumulants.

Keywords: nonholonomic systems, delta-correlated processes, response cumulants
PACC: 0320

I. INTRODUCTION

The analysis of nonlinear mechanical systems under random environments is usually performed by means of the Fokker-Plank equation approach or by means of moment or cumulant equation approach. The Fokker-Plank equation approach[1,2] gives the differential equation for the evolution of the probability density function; however, the numerical solution is complicated, especially for multi-degree-of-freedom systems. The moment equation approach is relatively simple and the most used by scientists working in the random vibration field. For nonlinear systems the response in terms of moments is governed by an infinite hierarchy, and closure schemes have to be adopted for the numerical solution of the given systems. Stochastic cumulants often provide a more convenient method for presenting the information given by stochastic moment.[3] Relatively simple and reliable techniques exist by which one can approximate the probability distribution of the non-Gaussian process from a knowledge of the values of these higher-order cumulants. For stochastic response of linear systems subjected to delta-correlated stochastic process, a formulation has been presented for directed finding of the cumulants, instead of the moments, of any order.[4] Since the problem is directly formulated in terms of cumulants rather than the moments of the response, closure techniques based on cumulants can be directly and efficiently used to truncate the infinite hierarchy of the system of cumulant equations, and the use of complex relationship between moments and cumulants can be avoided.

II. DELTA-CORRELATED PROCESSES

Let the equation of motion be cast in the form

$$\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}, t) + \boldsymbol{G}(t)\boldsymbol{\eta}(t), \tag{1}$$

where $\boldsymbol{X}$ is state variables of dimension n, $\boldsymbol{f}$ is drift coefficients of dimension n, matrix $\boldsymbol{G}$ is diffusion coefficients of dimension $n \times M$, $\boldsymbol{\eta}(t)$ is delta correlated excitation of dimension M, $\eta_l(t)$ are components of $\boldsymbol{\eta}$(t). By definition the nth cumulant function of the process $\eta_l(t)$ is a multiplication of $(n-1)$ Dirac delta functions

$$\kappa_n[\eta_l(t_1), \cdots, \eta_l(t_n)] = I_{n,l}(t)\delta(t_2 - t_1)\cdots\delta(t_n - t_1), \tag{2}$$

where $I_{n,l}(t)$ is the nth cumulant intensity of $\eta_l(t)$. If we say $\eta_l(t)$ is a non-normal delta correlated process up to the sth order, that means

$$\kappa_r[\eta_l(t); t_1, t_2, \cdots, t_r] = I_{r,l}(t)\delta(t_2 - t_1)\cdots\delta(t_r - t_1),$$
$$r = 2, \cdots, s,$$
$$\kappa_{r+p}[\eta_l(t); t_1, t_2, \cdots, t_r] = 0, \quad p = 1, 2, \cdots$$

Introducing the increment of process $C_l(t)$ related to $\eta_l(t)$ as

$$\mathrm{d}C_l(t) = \eta_l(t)\mathrm{d}t, \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 19572028) and the Doctoral Program Foundation of Institution of Higher Education of China.

one obtains that the nth cumulant of $\mathrm{d}C_l(t)$ is proportional to $\mathrm{d}t$ as

$$\kappa_n[\mathrm{d}C_l(t)] = I_{n,l}(t)\mathrm{d}t.$$

The coefficient $I_{n,l}(t)$ becomes independent of time if $\eta_l(t)$ is assumed stationary. A special class of the stationary, delta correlated process is the Gaussian white-noise process in which cumulant functions of any order higher than 2 vanish, i.e., $I_{3,l} = I_{4,l} = \cdots = 0$. In this case the corresponding $C_l(t)$ becomes a Brownian motion process.

The author Hu[3] has presented the following relationships between probabilistic independence and the joint cumulants of random variables:

(1) If any one (or more) random variable among $X_1, \cdots, X_n$ is independent of all the other random variables, then

$$\kappa_n[X_1, \cdots, X_n] = 0. \tag{4}$$

(2) For $Y = \sum_{j=1}^{n} \alpha_j X_j$, if $X_j (j = 1, \cdots, n)$ are mutually independent, then

$$\kappa_m[Y] = \sum_{j=1}^{n} \alpha_j^m \kappa_m[X_j]. \tag{5}$$

(3) For $Z = \sum_{j=1}^{n} h_j(X_j)$, there exists

$$\kappa_m[Z] = \sum_{j=1}^{n} \kappa_m[h_j(X_j)]. \tag{6}$$

III. THE STOCHASTIC DIFFERENTIAL EQUATIONS OF NONHOLONOMIC SYSTEM

Assume that the configuration of a dynamical system is determined by n generalized coordinates $q_s (s = 1, \cdots, n)$ and the system is subjected to g ideal, 1st order, linear nonholonomic constraints

$$\dot{q}_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon} b_{\varepsilon+\beta,\sigma}(t)\dot{q}_\sigma + b_{\varepsilon+\beta}(q_s, t)$$
$$(\beta = 1, \cdots, g; s = 1, \cdots, n; \varepsilon = n - g). \tag{7}$$

The equations of motion can be written in the Voronets'[6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}}{\partial \dot{q}_\sigma} - \frac{\partial \tilde{T}}{\partial q_\sigma} = \overline{Q}_\sigma + \sum_{\beta=1}^{g} \frac{\partial T}{\partial \dot{q}_{\varepsilon+\beta}} A_\sigma^{\varepsilon+\beta}$$
$$+ \sum_{\beta=1}^{g} \left(\frac{\partial \tilde{T}}{\partial q_{\varepsilon+\beta}} + \overline{Q}_{\varepsilon+\beta} \right) b_{\varepsilon+\beta,\sigma} \quad (\sigma = 1, \cdots, \varepsilon), \tag{8}$$

where

$$A_\sigma^{\varepsilon+\beta} = \frac{\partial b_{\varepsilon+\beta,\sigma}}{\partial t} - \left(\frac{\partial b_{\varepsilon+\beta}}{\partial q_\sigma} + \sum_{\gamma=1}^{g} b_{\varepsilon+\gamma,\sigma} \frac{\partial b_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}} \right),$$

$\overline{Q}_s = Q_s + \xi_s (s = 1, \cdots, n)$, $Q_s = \dfrac{\partial U}{\partial q_s} - \dfrac{\partial F}{\partial \dot{q}_s}$, and $U = U(\boldsymbol{q})$ is the potential force function; $F = \dfrac{1}{2}\sum_{s,k=1}^{n} f_{sk}(\boldsymbol{q})\dot{q}_s\dot{q}_k$ is a dissipative function; $\xi_s = \sum_{i=1}^{M} C_{si}(t)\eta_i(t)$ is the stochastic forces; η_i is delta correlated process.

Suppose that the system is stationary, such that the kinetic energy T is in the quadratic form of $\dot{q}_s$, namely, $T = T_2 = \dfrac{1}{2}\sum_{s=1}^{n}\sum_{k=1}^{n} a_{ks}\dot{q}_s\dot{q}_k$. We further suppose that $\boldsymbol{r}_i$ is the linear function of q_s, then we have $a_{sk} = a_{ks} = \sum_{i=1}^{N} m_i \dfrac{\partial \boldsymbol{r}_i}{\partial q_s} \cdot \dfrac{\partial \boldsymbol{r}_i}{\partial q_k}$=const. And $\tilde{T} = \dfrac{1}{2}\sum_{\sigma,\nu=1}^{\varepsilon} \tau_{\sigma\nu}(t)\dot{q}_\sigma\dot{q}_\nu$,

$$\tau_{\sigma\nu} = a_{\sigma\nu} + \sum_{\beta=1}^{g}(a_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + a_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) + \sum_{\beta,\gamma=1}^{g} a_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\gamma,\nu}.$$

Equations (8) can be expressed in an explicit form

$$\sum_{\nu=1}^{\varepsilon} \tau_{\sigma\nu}(t)\ddot{q}_\nu + \sum_{\nu=1}^{\varepsilon} \dot{\tau}_{\sigma\nu}(t)\dot{q}_\nu = \frac{\partial U}{\partial q_\sigma} + \sum_{\beta=1}^{g} \frac{\partial U}{\partial q_{\varepsilon+\beta}} b_{\varepsilon+\beta,\sigma} - \sum_{\nu=1}^{\varepsilon} \varphi_{\sigma\nu}\dot{q}_\nu + \sum_{i=1}^{m} C_{\sigma i}(t)\eta_i + \sum_{\beta=1}^{g}\left[\sum_{i=1}^{M} b_{\varepsilon+\beta,\sigma}C_{\varepsilon+\beta,i}(t)\eta_i\right], \tag{9}$$

$$\varphi_{\sigma\nu} = f_{\sigma\nu} + \sum_{\beta=1}^{g}(f_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu} + f_{\varepsilon+\beta,\nu}b_{\varepsilon+\beta,\sigma}) + \sum_{\beta,\gamma=1}^{g} f_{\varepsilon+\beta,\varepsilon+\gamma}b_{\varepsilon+\beta,\sigma}b_{\varepsilon+\beta,\nu}.$$

Equations (9) can be written as

$$\ddot{q}_\nu = \sum_{\sigma=1}^{\varepsilon} \frac{\Delta_{\sigma\nu}}{\Delta} h_\sigma + \sum_{\sigma=1}^{\varepsilon}\sum_{i=1}^{M} \frac{\Delta_{\sigma\nu}}{\Delta} \cdot \left[C_{\sigma i} + \sum_{\beta=1}^{g} b_{\varepsilon+\beta,\sigma}C_{\varepsilon+\beta,i} \right] \eta_i \quad (\nu = 1, \cdots, \varepsilon), \tag{10}$$

where Δ is the determinant of $\tau_{\sigma\nu}(t)$, $\Delta_{\sigma\nu}$ is the algebraic complement of element (σ,ν) in Δ, and $h_\sigma = -\sum_{\nu=1}^{\varepsilon}\dot{\tau}_{\sigma\nu}\dot{q}_\nu + \frac{\partial U}{\partial q_\sigma} + \sum_{\beta=1}^{g}\frac{\partial U}{\partial q_{\varepsilon+\beta}}b_{\varepsilon+\beta,\sigma} - \sum_{\nu=1}^{\varepsilon}\varphi_{\sigma\nu}\dot{q}_\nu$. Let $X_s = q_s, X_{n+\nu} = \dot{q}_\nu (s = 1,\cdots,n;\nu = 1,\cdots,\varepsilon)$, then, from Eqs.(10) we have

$$\ddot{q}_\nu = f_\nu(t,q_s,\dot{q}_\sigma) + \sum_{i=1}^{M} g_{\nu i}(t)\eta_i(t) \quad (\nu = 1,\cdots,n). \tag{11}$$

So, the equations of motion of the nonholonomic system (7) and (8) can be expressed as

$$\begin{aligned}
\dot{X}_\nu &= X_{n+\nu},\\
\dot{X}_{\varepsilon+\beta} &= \sum_{\sigma=1}^{\varepsilon} b_{\varepsilon+\beta,n+\sigma}(t)X_{n+\sigma} + b_{\varepsilon+\beta}(x_s,t), \qquad (12)\\
\dot{X}_{n+\nu} &= f_\nu(t,X_s,X_{n+\sigma}) + \sum_{i=1}^{M} g_{\nu i}(t)\eta_i(t).
\end{aligned}$$

Or

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X},t) + \boldsymbol{G}(t)\boldsymbol{\eta}(t), \tag{13}$$

where

$$\boldsymbol{X} = (X_1, X_2, \cdots, X_{n+\varepsilon})^{\mathrm{T}},$$

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{X},t) = \Bigg\{ & X_1,\cdots,X_{n+\varepsilon},\left[\sum_{\sigma=1}^{\varepsilon} b_{\varepsilon+1,n+\sigma}X_{n+\sigma} + b_{\varepsilon+1}(X_s,t)\right],\cdots,\\
& \left[\sum_{\sigma=1}^{\varepsilon} b_{\varepsilon+1,n+\sigma}X_{n+\sigma} + b_{\varepsilon+1}(X_s,t)\right], f_1,\cdots,f_\varepsilon\Bigg\}^{\mathrm{T}},
\end{aligned}$$

$$\boldsymbol{G}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ g_{11} & g_{12} & \cdots & g_{1M}\\ \vdots & \vdots & \cdots & \vdots\\ g_{\varepsilon 1} & g_{\varepsilon 2} & \cdots & g_{\varepsilon M}\end{bmatrix}.$$

The stochastic differential equation of the dynamic system can be written as

$$\mathrm{d}\boldsymbol{X} = \boldsymbol{F}(\boldsymbol{X},t)\mathrm{d}t + \boldsymbol{G}(t)\mathrm{d}\boldsymbol{C}(t). \tag{14}$$

Expressed in a component form, the sth equation in (14) is written as

$$\mathrm{d}X_s = F_s(\boldsymbol{X},t)\mathrm{d}t + \sum_{l=1}^{m} G_{s,l}(t)\mathrm{d}C_l(t). \tag{15}$$

IV. THE DIFFERENTIAL EQUATIONS OF RESPONSE CUMULANT

The evolution of the nth autocumulant of state variables X_s is defined as

$$\dot{\kappa}_n(X_s) = \frac{\kappa_n(X_s + \mathrm{d}X_s) - \kappa_n(X_s)}{\mathrm{d}t}. \tag{16}$$

Under the following assumptions:

(1) $\mathrm{d}\boldsymbol{C}(t)$ are independent of $\boldsymbol{X}(t)$,

(2) $\mathrm{d}C_l(t)(l = 1,\cdots,m)$ are mutually independent,

(3) Terms of order equal to or higher than $(\mathrm{d}t)^2$ are negligible,

We have the following general cumulant differential equations (CDEs) associated with Eq.(14).[5]

The nth autocumulant differential equation can be expressed as

$$\begin{aligned}
\dot{\kappa}_n(X_s) =& n\kappa_{(n-1)+1}[X_s, F_s(\boldsymbol{X},t)]\\
&+ \sum_{l=1}^{m}[G_{s,l}(t)]^n I_{n,l}(t),
\end{aligned} \tag{17}$$

where $\kappa_{(n-1)+1}[X_s, F_s(\boldsymbol{X},t)]$ is the $[(n-1+1]$th joint cumulant of X_s and $F_s(\boldsymbol{X},t)$.

For two and three variables, one obtains

$$\begin{aligned}
\dot{\kappa}_{n+m}(X_s,X_r) =& n\kappa_{(n-1)+1+m}[X_s, F_s(\boldsymbol{X},t), X_r]\\
&+ m\kappa_{n+(m-1)+1}[X_s, X_r, F_r(\boldsymbol{X},t)]\\
&+ \sum_{l=1}^{m}[G_{s,l}(t)]^n[G_{r,l}(t)]^m I_{(n+m),l}(t),
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\dot{\kappa}_{n+m+j}(X_s,X_r,X_q)\\
=& n\kappa_{(n-1)+1+m+j}[X_s, F_s(\boldsymbol{X},t), X_r, X_q]\\
&+ m\kappa_{n+(m-1)+1+j}[X_s, X_r, F_r(\boldsymbol{X},t), X_q]\\
&+ j\kappa_{n+m+(j-1)+1}[X_s, X_r, X_q, F_q(\boldsymbol{X},t)]\\
&+ \sum_{l=1}^{m}[G_{s,l}(t)]^n[G_{r,l}(t)]^m[G_{q,l}(t)]^j I_{(n+m+j),l}(t),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\dot{\kappa}_{n+m+j+i}(X_s,X_r,X_q,X_p)\\
=& n\kappa_{(n-1)+1+m+j+i}[X_s, f_s(\boldsymbol{X},t), X_r, X_q, X_p]\\
&+ m\kappa_{n+(m-1)+1+j+i}[X_s, X_r, f_r(\boldsymbol{X},t), X_q, X_p]\\
&+ j\kappa_{n+m+(j-1)+1+i}[X_s, X_r, X_q, f_q(\boldsymbol{X},t), X_p]\\
&+ i\kappa_{n+m+j+(i-1)+1}[X_s, X_r, X_q, X_p, f_p(\boldsymbol{X},t)]\\
&+ \sum_{l=1}^{M}[G_{s,l}(t)]^n[G_{r,l}(t)]^m\\
&\cdot [G_{q,l}(t)]^j[G_{p,l}(t)]^i I_{(n+m+j+i),l}(t).
\end{aligned} \tag{20}$$

CDEs involving fewer random variables can readily be obtained from CDEs involving more random variables. For the case of N state variables, when cumulants of order l are sought ,there are $S_{lN} = \dfrac{(l+N-1)!}{l!(N-1)!}$ simultaneous equations to be formed. The number S_{lN} represents unordered selections of repeatable l terms from N possibilities. In practice, when N is large, one seldom intends to calculate response cumulants of order l greater then 4.

V. ILLUSTRATED EXAMPLE

Assume that the configuration of a dynamical system is determined by generalized coordinates q_1, q_2; the kinetic energy of the system is $T = \dfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)$, the nonholonomic constraint imposed on the system is $f = \dot{q}_1 + at\dot{q}_2 - aq_2 + t = 0$, the potential force function is $U = -\dfrac{1}{2}(q_1^2 + q_2^2)$, and the stochastic forces $\eta_1(t), \eta_2(t)$ are delta correlated processes.

The stochastic differential equation of the system is

$$\ddot{q}_1 = -\frac{1}{1+a^2t^2} + \frac{a^2t^2}{1+a^2t^2}\eta_1(t) - \frac{at}{1+a^2t^2}\eta_2(t) + q_1,$$
$$\ddot{q}_2 = -\frac{at}{1+a^2t^2} - \frac{at}{1+a^2t^2}\eta_1(t) + \frac{1}{1+a^2t^2}\eta_2(t) + q_2.$$

By introducing state variables $X_1 = q_1, X_2 = q_2, X_3 = \dot{q}_1, X_4 = \dot{q}_2$, the preceding equation can be rewritten in a form similar to Eqs.(12),

$$\dot{X}_1 = X_3,$$
$$\dot{X}_2 = \frac{1}{t}X_2 - \frac{1}{at}X_3 - \frac{1}{a},$$
$$\dot{X}_3 = -\frac{1}{1+a^2t^2} + \frac{a^2t^2}{1+a^2t^2}\eta_1(t) - \frac{at}{1+a^2t^2}\eta_2(t) + X_1,$$
$$\dot{X}_4 = -\frac{1}{1+a^2t^2} - \frac{at}{1+a^2t^2}\eta_1(t) + \frac{1}{1+a^2t^2}\eta_2(t) + X_2,$$

in particular, we have

$$f_1 = X_3;\quad f_2 = \frac{1}{t}X_2 - \frac{1}{at}X_3 - \frac{1}{a};\quad f_3 = -\frac{1}{1+a^2t^2};$$
$$f_4 = -\frac{at}{1+a^2t^2};\quad G_{31} = \frac{a^2t^2}{1+a^2t^2};$$
$$G_{32} = -\frac{at}{1+a^2t^2} = G_{41};\quad G_{42} = \frac{1}{1+a^2t^2};$$
$$G_{11} = G_{12} = G_{21} = G_{22} = 0.$$

Substituting the corresponding $\boldsymbol{F}$ and $\boldsymbol{G}$ into Eqs.(20), one obtains the governing CDEs

$$\begin{aligned}&\dot{\kappa}_{n+m+j+i}(X_1, X_2, X_3, X_4)\\ =&n\kappa_{(n-1)+1+m+j+i}[X_1, f_1, X_2, X_3, X_4]\\ &+ m\kappa_{n+(m-1)+1+j+i}[X_1, X_2, f_2, X_3, X_4].\end{aligned}$$

Hereafter, $\kappa_{n+m+j+i}(X_1, X_2, X_3, X_4)$ will be abbreviated as $\kappa_{n+m+j+i}$, then we have the 2nd order CDEs as

$$\begin{aligned}\dot{\kappa}_{n+m+j+i} &= n\kappa_{(n-1)+m+(j+1)+i} + m\frac{1}{t}\kappa_{n+m+j+i}\\ &\quad - m\frac{1}{at}\kappa_{n+(m-1)+(j+1)+i},\\ \dot{\kappa}_{2+0+0+0} &= 2\kappa_{1+0+1+0},\\ \dot{\kappa}_{1+1+0+0} &= \dot{\kappa}_{0+1+1+0} + \frac{1}{t}\dot{\kappa}_{1+1+0+0} - \frac{1}{at}\dot{\kappa}_{1+0+1+0},\\ \dot{\kappa}_{1+0+1+0} &= \kappa_{0+0+2+0},\\ \dot{\kappa}_{1+0+0+1} &= \kappa_{0+0+1+1},\\ \dot{\kappa}_{0+2+0+0} &= 2\frac{1}{t}\kappa_{0+2+0+0} - 2\frac{1}{at}\kappa_{0+1+1+0},\\ \dot{\kappa}_{0+1+1+0} &= \frac{1}{t}\kappa_{0+1+1+0} - \frac{1}{at}\kappa_{0+0+2+0},\\ \dot{\kappa}_{0+1+0+1} &= \frac{1}{t}\kappa_{0+1+0+1} - \frac{1}{at}\kappa_{0+0+1+1},\\ \dot{\kappa}_{0+0+2+0} &= 0,\\ \dot{\kappa}_{0+0+1+1} &= 0,\\ \dot{\kappa}_{0+0+0+2} &= 0.\end{aligned}$$

The simplicity of the present method allows easier derivation of analytical results for other higher-order response cumulants.

VI. CONCLUDING REMARKS

Cumulant differential equations that can be used to calculate response cumulants for linear or nonlinear dynamic systems under non-Gaussian delta-correlated excitations are derived. For linear systems (i.e., $\boldsymbol{F}$ are linear functions of $\boldsymbol{X}$), the corresponding CDEs have the superiority in simplicity over their counterparts moment equations. There are no clear advantages or disadvantages in calculating response statistics for nonlinear systems by using moment differential equations. When $\boldsymbol{G}$ is treated as a function of both $\boldsymbol{X}$ and t, the first and second terms on the right side of Eq.(13) are no longer statistically independent. In this case, using response moment equations to calculate response statistics might be preferred.

ACKNOWLEDGEMENT

We are indebted to our advisor, Mr. F. X. Mei, who gives us lots of help.

Appendix

Presented in the following is the proof for Eq.(17).

Adding X_s on both sides of Eq.(15) yields

$$X_s + \mathrm{d}X_s = X_s + F_s(\boldsymbol{X}, t)\mathrm{d}t + \sum_{l=1}^{m} G_{s,l}(t)\mathrm{d}C_l(t). \tag{A.1}$$

Under the assumption of the first condition from Eq.(A.1), one can immediately write

$$\kappa_n(X_s + \mathrm{d}X_s) = \kappa_n[X_s + F_s(\boldsymbol{X}, t)\mathrm{d}t] + \kappa_n\left[\sum_{l=1}^{m} G_{s,l}(t)\mathrm{d}C_l(t)\right]. \tag{A.2}$$

Furthermore, assuming the second condition, one then obtains

$$\kappa_n\left[\sum_{l=1}^{m} G_{s,l}(t)\mathrm{d}C_l(t)\right] = \sum_{l=1}^{m}[G_{s,l}(t)]^n \kappa_n[\mathrm{d}C_l(t)] = \sum_{l=1}^{m}[G_{s,l}(t)]^n I_{n,l}(t)\mathrm{d}t. \tag{A.3}$$

The first term on the right side of Eq.(A.2) can be decomposed into

$$\kappa_n[X_s + F_s(\boldsymbol{X}, t)\mathrm{d}t] = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\kappa_{j+(n-j)} \cdot [X_s, F_s(\boldsymbol{X}, t)]\mathrm{d}t^{(n-j)}. \tag{A.4}$$

According to the third condition, Eq.(A.4) becomes

$$\kappa_n[X_s + F_s(\boldsymbol{X}, t)\mathrm{d}t] = \kappa_n[X_s] + n\kappa_{(n-1)+1} \cdot [X_s, F_s(\boldsymbol{X}, t)]\mathrm{d}t. \tag{A.5}$$

Combining Eqs.(A.2), (A.3) and (A.5), one obtains

$$\kappa_n[X_s + \mathrm{d}X_s] = \kappa_n[X_s] + n\kappa_{(n-1)+1}[X_s, F_s(\boldsymbol{X}, t)]\mathrm{d}t + \sum_{l=1}^{m}[G_{s,l}(t)]^n I_{n,l}(t)\mathrm{d}t. \tag{A.6}$$

From Eqs.(16) and (A.6) one thus obtains Eq.(17).

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