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CANONICAL FORMULATION OF NONHOLONOMIC CONSTRAINED SYSTEMS*

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Based on the Ehresmann connection theory and symplectic geometry, the canonical formulation of nonholonomic constrained mechanical systems is described. Following the Lagrangian formulation of the constrained system, the Hamiltonian formulation is given by Legendre transformation. The Poisson bracket defined by an anti-symmetric tensor does not satisfy the Jacobi identity for the nonintegrability of nonholonomic constraints. The constraint manifold can admit symplectic submanifold for some cases, in which the Lie algebraic structure exists.

Keywords: nonholonomic constraints, canonical formulation, Ehresmann connection, symplectic submanifold
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I. INTRODUCTION

In the past forty years Dirac's theory of constrained Hamiltonian systems has been extensively applied to the fields of gravitation, particle physics and quantum fields, etc. Another kind of constrained systems, called nonholonomic constrained systems, realized by Hertz, Apell *et al.*, have attracted considerable attention in the last decade, which owes their applications to engineering issues and modern development of geometric mechanics.[1−6]

Canonical formulation of nonholonomic constrained systems appears frequently in the recent literature.[7−10] It is usually realized, beginning with Routh's equations and using Legendre transformation, by a kind of transformations to delete the Lagrange multipliers in Hamiltonian formulation of the equations. The systems do not admit Lie algebraic structure because the Poisson bracket does not satisfy the Jacobi identity unless the constraints are integrable. In this paper, we give a canonical formulation of nonholonomic mechanical systems based on Ehresmann connection theory and symplectic geometry, beginning with Chaplygin's equations and then using Legendre transformation to get Hamiltonian formulations for the system. The curvature makes coordinates and momenta of the system not conjugate canonically and the Poisson bracket not satisfy Jacobi identity. However, the structure of tangent bundle is related to the Lagrangian chosen and there exist alternative Lagrangians or Hamiltonians for the same dynamical system, which makes the equations of motion admit a genuine Euler-Lagrange representation or canonical one. The constraint manifold then may be of a symplectic submanifold, on which a Lie algebraic structure can be constructed. The research of symmetries of nonholonomic systems attracts much attention (see Refs.[11,12]).

Throughout this paper we have the Einstein's summation convention and Latin index: $i, j, s = 1, 2, \cdots, n$; Greek indices $\alpha, \beta, \gamma = 1, 2, \cdots, g < n$ and $\mu, \sigma = 1, 2, \cdots, \varepsilon = n - g$.

II. EHRESMANN CONNECTION ON A CONSTRAINT MANIFOLD

In this section, we review some results for the geometry of mechanical systems with linear nonholonomic constraints, making use of the formalism introducd in Refs.[4] and [13]. Consider a time-dependent mechanical system with n generalized coordinates. Let M be its smoothly extended configuration manifold of dimension $n+1$ with local coordinates $\{t, q^s\}$, which is a fibred manifold over R with canonical projection $\pi : M \to R$. TM denotes the tangent bundle to M and J_1M the 1-jet bundle. Suppose that the system is subject to g linear nonholonomic constraints

$$\dot{q}^{\beta} = B^{\beta}_{\sigma}(t, q^s)\, \dot{q}^{\sigma} + B^{\beta}(t, q^s). \tag{1}$$

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This construction distinguishes two lots of coordinates, $\{q^\sigma\}$ and $\{q^\beta\}$. Suppose that the configuration manifold M is of fibration structure over a manifold of dimension $n-g+1$ with local coordinates $\{t, q^\sigma\}$, denoted by M_0.

Let $\tau : M \to M_0$, $\pi_0 : M_0 \to R$, then $\pi = \pi_0 \cdot \tau$. J_1M_0 denotes a 1-jet manifold and it is a submanifold of TM_0. VM denotes the vertical subspace of TM with respect to the projection τ, i.e., the kernel of differential mapping $\mathrm{d}\tau$. And $\tau^*(J_1M_0)$ represents the pull back bundle over M, i.e., $\tau^*(J_1M_0) = M \times_{M_0} J_1M_0$. The vector fields along τ are sections: $M \to \tau^*(J_1M_0)$ which, in local coordinates, are made up of products of basic vector fields on M_0 and scalar functions $f \in C^\infty(M)$, i.e., a set of all these vector fields is the module over $C^\infty(M)$, denoted by $\chi(\tau)$.

The Ehresmann connection on M defines a horizontal lift $h : \chi(\tau) \to TM$, with $\mathrm{d}\tau \cdot h = \mathrm{id}_{\chi(\tau)}$. In local coordinates, we have

$$\begin{aligned} h_0 &= h(\partial/\partial t) = \partial/\partial t + B^\beta(t, q^s)\,\partial/\partial q^\beta, \\ h_\sigma &= h(\partial/\partial q^\sigma) = \partial/\partial q^\sigma + B^\beta_\sigma(t, q^s)\,\partial/\partial q^\beta, \end{aligned} \quad (2)$$

where $\{\partial/\partial q^\beta\}$ is a basis of vertical subspace of TM with respect to τ. And the set $\{h_0, h_\sigma\}$ expands the horizontal subspace of TM, i.e., the horizontal distributions on M, denoted by $h_\tau = h(\tau^*(J_1M_0))$ which is isomorphic to $\tau^*(J_1M_0)$ and referred to as the constrained submanifold.

By means of the basic vector fields $\{h_0, h_\sigma, \partial/\partial q^\beta\}$ and basic 1-forms $\{\mathrm{d}t, \mathrm{d}q^\sigma,\ \eta^\beta = \mathrm{d}q^\beta - B^\beta_\sigma \mathrm{d}q^\sigma - B^\beta \mathrm{d}t\}$ on M, the horizontal and vertical projection operators of vector fields on M with respect to τ will be

$$\begin{aligned} p_h &= h_0 \otimes \mathrm{d}t + h_\sigma \otimes \mathrm{d}q^\sigma, \\ p_v &= \partial/\partial q^\beta \otimes \eta^\beta. \end{aligned} \quad (3)$$

DEFINITION 1 A VM-valued connection 1-form ω corresponding to h is the canonical projection $p_v : TM \to T(M/M_0)$ whose VM-valued curvature 2-form r is the exterior covariant differentiation of p_v, i.e., $r = \nabla p_v = \mathrm{d}p_v \cdot p_h$.

By a new triad basis $\{\boldsymbol{T}, h_\mu, \partial/\partial q^\beta\}$ of vector fields on M, with

$$\boldsymbol{T} = \partial/\partial t + \dot{q}^\sigma\ \partial/\partial q^\sigma + \left(B^\beta_\sigma\ \dot{q}^\sigma + B^\beta\right)\partial/\partial q^\beta \quad (4)$$

and its dual $\{\mathrm{d}t, \theta^\mu, \eta^\beta\}$, with $\theta^\mu = \mathrm{d}q^\mu - \dot{q}^\mu\ \mathrm{d}t$, the curvature can be represented by

$$\begin{aligned} r = \Big\{ &- t^\beta_\sigma \mathrm{d}t \wedge \theta^\sigma + \frac{1}{2}\Big[h_\mu\left(B^\beta_\sigma\right) \\ &- h_\sigma\left(B^\beta_\mu\right)\Big]\theta^\sigma \wedge \theta^\mu\Big\} \otimes \partial/\partial q^\beta, \end{aligned} \quad (5)$$

where

$$\begin{aligned} t^\beta_\sigma &= \boldsymbol{T}\left(B^\beta_\sigma\right) - h_\sigma\left(B^\beta_\mu\ \dot{q}^\mu + B^\beta\right) \\ &= C^\beta_{\mu\sigma}\ \dot{q}^\mu + D^\beta_{0\sigma}, \end{aligned} \quad (6)$$

with

$$\begin{aligned} C^\beta_{\mu\sigma} = &\left(\frac{\partial B^\beta_\sigma}{\partial q^\mu} - \frac{\partial B^\beta_\mu}{\partial q^\sigma}\right) \\ &+ \left(B^\alpha_\mu \frac{\partial B^\beta_\sigma}{\partial q^\alpha} - B^\alpha_\sigma \frac{\partial B^\beta_\mu}{\partial q^\alpha}\right), \end{aligned} \quad (7a)$$

$$\begin{aligned} D^\beta_{0\sigma} = &\left(\frac{\partial B^\beta_\sigma}{\partial t} - \frac{\partial B^\beta}{\partial q^\sigma}\right) \\ &+ \left(B^\alpha \frac{\partial B^\beta_\sigma}{\partial q^\alpha} - B^\alpha_\sigma \frac{\partial B^\beta}{\partial q^\alpha}\right). \end{aligned} \quad (7b)$$

The constraints (1) are integral if and only if the curvature r vanishes. A creterion to determine if $r = 0$ can be given by the following.

THEOREM 1 The curvature r vanishes if and only if $t^\beta_\sigma = 0$.[4]

Ⅲ. LAGRANGE INVERSE PROBLEM

In this section we can briefly formulate the Lagrangian inverse problem. The equations of motion of constrained mechanical systems formulated in the above triad, called Generalized Chaplygin's equations, are given by

$$\boldsymbol{Z}\left(\frac{\partial L'}{\partial \dot{q}^\sigma}\right) - h_\sigma(L') - t^\beta_\sigma i^*\left(\frac{\partial L}{\partial \dot{q}^\beta}\right) = 0, \quad (8)$$

or globally by

$$i_{\boldsymbol{Z}}\Omega = 0, \quad i_{\boldsymbol{Z}}\mathrm{d}t = 0, \quad (9)$$

where the dynamical vector field $\boldsymbol{Z}$ and the fundamental 2-form Ω induced by the regular dynamical function $L' = i^*L \in C^2(N)$ $(i : N \to J_1M)$ and constraints are given by

$$\begin{aligned} \boldsymbol{Z} = &\partial/\partial t + \dot{q}^\sigma\ \partial/\partial q^\sigma \\ &+ \left(B^\beta_\sigma\ \dot{q}^\sigma + B^\beta\right)\partial/\partial q^\beta + f^\mu \partial/\partial \dot{q}^\mu, \end{aligned} \quad (10a)$$

$$\Omega = \mathrm{d}\left(\frac{\partial L'}{\partial \dot{q}^\sigma}\theta^\sigma\right) + \left[\mathrm{d}L' + t^\beta_\mu i^*\left(\frac{\partial L}{\partial \dot{q}^\beta}\right)\theta^\mu\right] \wedge \mathrm{d}t. \quad (10b)$$

In general, Eq.(8) for q^σ cannot be recast into the form of genuine Euler-Lagrange equations with a Lagrangian not dependent on the coordinates q^β, since the 2-form Ω is not closed usually, otherwise it is called *Lagrangian*.

In order to get a Lagrangian for the nonholonomic system, suppose that the dynamical equations decouple with the constraints. It is easy to verify the following.

THEOREM 2 If $\Psi = t^\beta_\mu i^*\left(\partial L/\partial \dot{q}^\beta\right)\theta^\mu$ satisfies the Helmholtz conditions,[14] then

$$\Psi = \left[-\boldsymbol{Z}\left(\frac{\partial \Phi}{\partial \dot{q}^\mu}\right) + \frac{\partial \Phi}{\partial q^\mu}\right]\theta^\mu \quad (11)$$

for some function $\Phi \in C^\infty(h_\tau)$ and the 2-form Ω is reduced to an exact 2-form $\Omega' = \mathrm{d}\Theta_{L'+\Phi}$, where

$$\Theta_{L'+\Phi} = S\left[\mathrm{d}\left(L'+\Phi\right)\right] + \left(L'+\Phi\right)\mathrm{d}t. \quad (12)$$

REMARK 1 This method can be extended to some cases that the Helmholtz conditions cannot be satisfied by Ψ. However, if there exists a 1-form Ψ' on h_τ as an annihilator of dynamical vector field Z such that the sum $\Psi + \Psi'$ satisfies the Helmholtz conditions, the system can still be recast into the Lagrangian one as above.[14]

Usually a Lagrangian system admits adjoint symmetries. If a constrained system admits some special adjoint symmetries, then a Lagrangian can be realized for this system.

DEFINITION 2 The adjoint symmetries of $\boldsymbol{Z}$ are invariant 1-forms $\beta \in \wedge(\eta)$, i.e., $i_{\boldsymbol{Z}}\beta = 0$, $L_{\boldsymbol{Z}}\beta = 0$.

Without proof we give the following theorem for the existence of alternative Lagrangian of a nonholonomic mechanical system.

THEOREM 3 Surppose $\beta = \delta_\mu\omega^\mu + \tau_\alpha\xi^\alpha + \lambda_\mu\theta^\mu$ is an adjoint symmetry of $\boldsymbol{Z}$ and $\mathrm{d}\beta$ is of maximum rank ε. If there is a function $F \in C^\infty(h_\tau)$ such that

$$\delta_\sigma = \frac{\partial F}{\partial \dot{q}^\sigma}, \quad \tau_\alpha = \frac{\partial F}{\partial q^\alpha}, \quad (13)$$

then Eq.(8) is Lagrangian with regular $L^* = \boldsymbol{Z}(F)$.

IV. NONHOLONOMIC HAMILTONIAN SYSTEMS

In order to illustrate the algebraic structure of a system with affine nonholonomic constraints we begin with the unclosed 2-form Ω (see Eq.(10b)) on the manifold M. As an annihilator of the horizontal distribution h on M the constraints on M can be defined by a codistribution $h^0 \subset T^*M$, i. e.,

$$h^0 = \mathrm{Span}\{\omega^\beta \in T^*M | \omega^\beta = \mathrm{d}q^\beta - B^\beta_\sigma(t,q)\mathrm{d}q^\sigma - B^\beta(t,q)\mathrm{d}t\}. \quad (14)$$

The constraint phase space is $\mathcal{M} = FL(h) \subset T^*M$. The Legendre transformation is used to define the Hamiltonian $H' : \mathcal{M} \to R : H' = p_\sigma\dot{q}^\sigma - L'$, where the momentum is $p_\sigma = FL(\dot{q}^\sigma)$. Therefore, the fundamental 2-form on $\mathcal{M}$ is

$$\Omega_{H'} = FL(\Omega) = \mathrm{d}p_\sigma \wedge \mathrm{d}q^\sigma + \left(t^\beta_\mu p_\beta \mathrm{d}q^\mu - \mathrm{d}H'\right) \wedge \mathrm{d}t. \quad (15)$$

Here p_β is understood to be restricted to $\mathcal{M}$, as functions of t, q^s, p_μ. The dynamical vector field on $\mathcal{M}$ can be defined by

$$i_{\boldsymbol{Z}_{H'}}\Omega_{H'} = 0, \quad i_{\boldsymbol{Z}_{H'}}\mathrm{d}t = 0. \quad (16)$$

In local coordinates,

$$\boldsymbol{Z}_{H'} = \partial/\partial t + \dot{q}^\sigma\partial/\partial q^\sigma + \left(B^\beta_\mu\,\dot{q}^\mu + B^\beta\right)\partial/\partial q^\beta + \dot{p}_\sigma\partial/\partial p_\sigma. \quad (17)$$

The local representation of Eq.(16) gives the Hamiltonian formulation of equations of nonholonomic systems and the energy evolvement:

$$\dot{q}^\sigma = \frac{\partial H'}{\partial p_\sigma}, \quad (18a)$$

$$\dot{p}_\sigma = -\frac{\partial H'}{\partial q^\sigma} - B^\beta_\sigma\frac{\partial H'}{\partial q^\beta} + C^\beta_{\mu\sigma}p_\beta\frac{\partial H'}{\partial p_\mu} + D^\beta_{0\sigma}p_\beta, \quad (18b)$$

$$\dot{H}' = B^\beta\frac{\partial H'}{\partial q^\beta} + D^\beta_{0\mu}p_\beta\frac{\partial H'}{\partial p_\mu} + \frac{\partial H'}{\partial t}. \quad (18c)$$

Here, q^σ and p_σ are not canonical in general, unless the fundamental 2-form is closed. The above equations and the constraint equations (1) can be represented by

$$\begin{pmatrix} \dot{q}^{\beta} \\ \dot{q}^{\sigma} \\ \dot{p}_{\sigma} \\ \dot{H}' \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{g\times g} & \mathbf{0}_{g\times\varepsilon} & \left(B_{\mu}^{\beta}\right)_{g\times\varepsilon} & 0 \\ \mathbf{0}_{\varepsilon\times g} & \mathbf{0}_{\varepsilon\times\varepsilon} & \left(\delta_{\mu}^{\sigma}\right)_{\varepsilon\times\varepsilon} & 0 \\ \left(-B_{\sigma}^{\beta}\right)_{\varepsilon\times g}^{\mathrm{T}} & \left(-\delta_{\sigma}^{\mu}\right)_{\varepsilon\times\varepsilon} & \left(C_{\mu\sigma}^{\beta}p_{\beta}\right)_{\varepsilon\times\varepsilon} & 0 \\ \left(B^{\beta}\right)_{1\times g} & \mathbf{0}_{1\times\varepsilon} & \left(D_{0\mu}^{\beta}p_{\beta}\right)_{1\times\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H'}{\partial q^{\beta}} \\ \dfrac{\partial H'}{\partial q^{\mu}} \\ \dfrac{\partial H'}{\partial p_{\mu}} \\ \dfrac{\partial H'}{\partial t} \end{pmatrix} + \begin{pmatrix} B^{\beta} \\ \mathbf{0} \\ D_{0\sigma}^{\beta}p_{\beta} \\ 0 \end{pmatrix}. \tag{19}$$

If the constraints are linear, i. e., $B^{\beta}=0$, $D_{0\mu}^{\beta}=0$, then the last equation is homogeneous and the transformation matrix becomes anti-symmetric:

$$J_{\mathcal{M}} = \begin{pmatrix} \mathbf{0}_{g\times g} & \mathbf{0}_{g\times\varepsilon} & \left(B_{\mu}^{\beta}\right)_{g\times\varepsilon} & 0 \\ \mathbf{0}_{\varepsilon\times g} & \mathbf{0}_{\varepsilon\times\varepsilon} & \left(\delta_{\mu}^{\sigma}\right)_{\varepsilon\times\varepsilon} & 0 \\ \left(-B_{\sigma}^{\beta}\right)_{\varepsilon\times g}^{\mathrm{T}} & \left(-\delta_{\sigma}^{\mu}\right)_{\varepsilon\times\varepsilon} & \left(C_{\mu\sigma}^{\beta}p_{\beta}\right)_{\varepsilon\times\varepsilon} & 0 \\ \mathbf{0}_{1\times g} & \mathbf{0}_{1\times\varepsilon} & \mathbf{0}_{1\times\varepsilon} & 1 \end{pmatrix} = \begin{pmatrix} j_{\mathcal{M}} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}. \tag{20}$$

In this case the Poisson bracket $\{\}_{\mathcal{M}}$ on the manifold $\mathcal{M}$ can then be defined by the reduced matrix $j_{\mathcal{M}}$ as

$$\{F_{\mathcal{M}}, G_{\mathcal{M}}\}_{\mathcal{M}}(t, q^{s}, p_{\sigma}) = \left(\frac{\partial F_{\mathcal{M}}^{\mathrm{T}}}{\partial q^{s}}, \frac{\partial F_{\mathcal{M}}^{\mathrm{T}}}{\partial p^{\sigma}}\right) j_{\mathcal{M}} \begin{pmatrix} \dfrac{\partial G_{\mathcal{M}}^{\mathrm{T}}}{\partial q^{s}} \\ \\ \dfrac{\partial G_{\mathcal{M}}^{\mathrm{T}}}{\partial p^{\sigma}} \end{pmatrix} \tag{21}$$

for any two smooth functions $F_{\mathcal{M}}$ and $G_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$ and the equations of motion can then be represented by

$$\dot{q}^{s} = \{q^{s}, H'\}, \quad \dot{p}_{\mu} = \{p_{\mu}, H'\}. \tag{22}$$

However, this bracket does not satisfy the Jacobi identity since the matrix $j_{\mathcal{M}}$ is not symplectic.

THEOREM 4 Surppose the constraints acting on the system are linear, there exists Lie algebra on the constraint manifold $\mathcal{M}$ defined by $j_{\mathcal{M}}$ if and only if the constraints are holonomic.

There exists the Lie algebra on the constraint manifold $\mathcal{M}$ if and only if the anti-symmetric bracket $\{\}_{\mathcal{M}}$ satisfies Jacobi identity, or $j_{\mathcal{M}}$ has a $\varepsilon\times\varepsilon$ reduced symplectic sub-matrix, which requires $C_{\mu\sigma}^{\beta}=0$ and $D_{0\mu}^{\beta}=0$ (or $B^{\beta}=0$), i.e., $t_{\sigma}^{\beta}=0$. We have shown that the curvature as the failure of integrability of the constraints vanishes if and only if $t_{\sigma}^{\beta}=0$ in Ref.[7], which is equivalent to say that the differential constraints are holonomic.

REMARK 2 Poisson structure, closely related with the symplectic structure of the manifold, does not need the constraints to be holonomic if a nonholonomic mechanical system admits an alternative Lagrangian representation.

THEOREM 5 Consider the 1-form $\kappa = t_{\mu}^{\beta}p_{\beta}\mathrm{d}q^{\mu} = \kappa_{\mu}\mathrm{d}q^{\mu}$ on constraint manifold $\mathcal{M}$, where t_{μ}^{β} is the curvature of the nonholonomic connection. Let the constraints be homogeneous and decoupled with the dynamical equations, i. e., $B^{\beta}=0$, $\partial H'/\partial q^{\beta}=0$. If $FL^{-1}(\kappa)$ satisfies the Helmholtz conditions on the Lagrangian side, then the constraint manifold admits presymplectic submanifold.

PROOF Take inverse Legendre tranformation of 1-form κ. If $\Psi = FL^{-1}(\kappa)$ satisfies the Helmholtz conditions on the Lagrangian side, then there exists a regular function Φ on manifold $\tau^*(J_1M_0)$ satisfying[14]

$$\Psi = \left[-\boldsymbol{Z}\left(\frac{\partial \Phi}{\partial \dot{q}^\mu}\right) + \frac{\partial \Phi}{\partial q^\mu}\right]\theta^\mu, \tag{23}$$

where Φ is independent of coordinates q^β since the dynamical equations are decoupled with the constraints. Take Legendre transformation $FL_{\psi+L'}: \tau^*(J_1M_0) \to \mathcal{M}$ to redefine $\mathcal{H} = p_\mu \dot{q}^\mu - (\psi + L')_{\mathcal{M}}$, where $p_\mu = \dfrac{\partial(\Phi + L')}{\partial \dot{q}^\mu}$. Then $\mathcal{M}$ admits a presymplectic submanifold $\mathcal{M}_0 \subset \mathcal{M}$, i. e.,

$$\begin{aligned}\mathcal{M}_0 = \{(t, q^\mu, p_\mu) \in \mathcal{M} | \Omega_{\mathcal{M}_0} \\ = \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t\}.\end{aligned} \tag{24}$$

V. APPLICATION

Consider a homogeneous ball with mass m and radius a, freely rolling on a rough horizontal plane. Appropriate generalized coordinates are: the coordinates (x, y) of the centre of mass of the ball, the three Euler angles ψ, θ, φ. The condition of rolling without slipping gives rise to non-holonomic constraints of the form

$$\begin{aligned}\dot{x} &= -a\left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right), \\ \dot{y} &= -a\left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right).\end{aligned}$$

The dimensions of the manifolds M_0 and M are 3 and 5, respectively, with notational identifications: $q^\beta = (x, y)$, $q^\sigma = (\psi, \theta, \varphi)$. The Lagrangian L' pull back to N is

$$\begin{aligned}L' =& \frac{1}{2}ma^2\Big[\frac{7}{5}\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta \\ &+ \frac{2}{5}\left(\dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta\right)\Big].\end{aligned}$$

The nonzero components of curvature are given by

$$\begin{aligned}C^x_{\theta\psi} &= -a\cos\psi, \\ C^x_{\varphi\psi} &= -a\sin\theta\sin\psi, \\ C^x_{\varphi\theta} &= a\cos\theta\cos\psi, \\ C^y_{\theta\psi} &= a\sin\psi, \\ C^y_{\varphi\theta} &= a\cos\theta\sin\psi, \\ C^y_{\varphi\psi} &= a\sin\theta\cos\psi.\end{aligned}$$

The system is obviously a Chaplygin's system. The 8×8 anti-symmetric matrix $j_{\mathcal{M}}$ determining the Hamiltonian equations reads

$$j_{\mathcal{M}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & a\sin\theta & -a\sin\theta\cos\psi \\
0 & 0 & 0 & 0 & 0 & 0 & -a\cos\psi & -a\sin\theta\sin\psi \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & ma^2(\dot{\varphi}\sin\theta\cos 2\psi - \dot{\theta}\sin 2\psi) & -ma^2\dot{\theta}\sin\theta \\
-a\sin\theta & a\cos\psi & 0 & -1 & 0 & -ma^2(\dot{\varphi}\sin\theta\cos 2\psi - \dot{\theta}\sin 2\psi) & 0 & -\frac{1}{2}ma^2\dot{\varphi}\sin 2\theta \\
a\sin\theta\cos\psi & a\sin\theta\sin\psi & 0 & 0 & -1 & ma^2\dot{\theta}\sin\theta & \frac{1}{2}ma^2\dot{\varphi}\sin 2\theta & 0
\end{pmatrix},$$

where $\dot{\psi}, \dot{\theta}$ and $\dot{\varphi}$ are understood to be represented by the corresponding coordinates and momenta. Under the given momentum variables and dynamical functions L' and H', $j_{\mathcal{M}}$ is not symplectic because of the non-integrability of the constraints given above. Thus the Poisson bracket $\{\}_{\mathcal{M}}$ defined by $j_{\mathcal{M}}$ does not satisfy the Jacobi identity.

However, making use of the theorem in section IV and the remark in section III, we can find an alternative Lagrangian as a sum of L' and $\Phi = \frac{1}{2}ma^2\left(\dot{\psi}+\dot{\varphi}\cos\theta\right)^2$, and the Hamiltonian

$$\tilde{L} = \frac{1}{2}\cdot\frac{7}{5}ma^2\left(\dot{\psi}^2+\dot{\theta}^2+\dot{\varphi}^2+2\dot{\varphi}\dot{\psi}\cos\theta\right),$$

$$\tilde{H} = \frac{5}{14ma^2\sin^2\theta}\left(p_\psi^2+p_\theta^2\sin^2\theta+p_\varphi^2-2p_\psi p_\varphi\cos\theta\right),$$

where the new momenta $(p_\psi, p_\theta, p_\varphi) = FL\left(\dot{\psi},\dot{\theta},\dot{\varphi}\right)$. By using the notational identifications: $a^1=\psi, a^2=\theta, a^3=\varphi, a^4=p_\psi, a^5=p_\theta, a^6=p_\varphi$, the canonical equations of motion read

$$\dot{a}^\mu = \sum_{\nu=1}^{6}\omega_0^{\mu\nu}\frac{\partial\tilde{H}}{\partial\dot{a}^\nu} \qquad (\mu=1,2,\cdots,6),$$

where $\omega_0^{\mu\nu}$ is a simple symplectic tensor and determines a Poisson structure of the symplectic submanifold $\mathcal{M}_0 = \{\psi,\theta,\varphi,p_\psi,p_\theta,p_\varphi|\omega_0 = \mathrm{d}p_\psi\wedge\mathrm{d}\psi + \mathrm{d}p_\theta\wedge\mathrm{d}\theta+\mathrm{d}p_\varphi\wedge\mathrm{d}\varphi\}$ as a conventional Hamiltonian system.

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