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A DIRECT PERTURBATION THEORY OF THE NONLINEAR SCHRÖDINGER EQUATION WITH CORRECTIONS*

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An exact direct perturbation theory of nonlinear Schrödinger equation with corrections is developed under the condition that the initial value of the perturbed solution is equal to the value of an exact multisoliton solution at a particular time. After showing the squared Jost functions are the eigenfunctions of the linearized operator with a vanishing eigenvalue, suitable definitions of adjoint functions and inner product are introduced. Orthogonal relations are derived and the expansion of the unity in terms of the squared Jost functions is naturally implied. The completeness of the squared Jost functions is shown by the generalized Marchenko equation. As an example, the evolution of a Raman loss compensated soliton in an optical fiber is treated.

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I. INTRODUCTION

Pertubation theory of the nearly integrable system (say, the perturbed nonlinear Schrödinger (NLS) equation) is of relevant importance for treating the practical physics problem.[1] The dircet perturbation theory[2] (or the Green function method) was developed in a time earlier than the perturbation theory based on the inverse scattering transform (IST).[3-6] Kaup did the pioneer work[7] to point out the connection between the direct perturbation method and the IST. Based on the result of IST, Kaup showed that the squared Jost functions are the eigenfunctions of the linearized operator for the perturbed NLS equation with vanishing eigenvalue. The adjoint operator of the linearized operator and the adjoint functions of the squared Jost functions were introduced, the orthogonal relations of the squared Jost functions were thus derived. By employing the hypothesis of compact support, Kaup showed the completeness of the squared Jost functions which is the corner stone for the direct perturbation theory. Nevertheless, the derivation of the completeness still has some problems. The first is the introduction of the adjoint operator. It was introduced by integration by parts. Since the contribution of Jost functions when $x \to \pm\infty$ will appear in the equation as the integration by parts was done, one extra term appeared. When the adjoint function of squared Jost function was introduced by the defined adjoint operator, this extra term was omitted without reason. Kaup employed the compact support to show the completeness of the squared Jost functions, while the compact support still needs to be verified. Besides, one still needs to omit some extra terms in Kaup's derivation in order to

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show the completeness. Since Kaup's work, various direct perturbation theories[8-14] have been developed to treat practical physics problems, while the derivatin of the completeness has not been improved. Kaup re-addressed this problem in a recent work.[13] However, the employed gauge transformation depends on x, thus the Jost functions after the gauge transformation did not satisfy the original first Lax equation. The problem has not yet been settled.

The aim of the present work is to give an exact formalism of the direct perturbation theory of the NLS equation under the condition that the initial value of a perturbed solution is equal to the value of an exact multi-soliton solution at a definite time. The theory is constructed with the help of IST. After showing that the squared Jost functions are eigenfunctions of the linearized operator with vanishing eigenvalue, suitable definitions of adjoint functions and inner product are introduced. The orthogonal relations are exactly evaluated and the expansion of the unity in terms of the squared Jost functions is naturally implied. The completeness of the squared Jost functions is then proven by the generalized Marchenko equation. As an example, the optical soliton transmission with damping and Raman loss compensation is also discussed.

II. LINEARIZED OPERATOR AND THE SQUARED JOST FUNCTIONS

The perturbed NLS equation can be written in the form

$$iv_t + v_{xx} + 2|v|^2 v = \epsilon r[v] \,, \tag{1}$$

where $r[v]$ is a functional of v and ϵ is a small parameter. The initial condition is assumed to be $v(x,0) = u(x,0)$, where $u(x,t)$ is an exact soliton solution. In the first order approximation, the solution of Eq.(1) under the initial condition can be expressed as

$$u(x,t) = u^a(x,t) + \epsilon q(x,t) \,, \tag{2}$$

where $u^a(x,t)$ is the so-called adiabatic solution, which has the same functional form as that of the exact soliton solution but the parameters involved may depend on t of the order of ϵ and ϵq represents the remaining term which is also of order of ϵ.

Substituting Eq.(2) into Eq.(1), we obtain

$$\{\partial_t - L(u,t)\}q = R[u] \,, \tag{3}$$

where $L(u,x)$ is an operator known as the linearized operator defined by

$$L(u) = \begin{pmatrix} -\partial_x^2 - 4|u|^2 & -2u^2 \\ 2\bar{u}^2 & \partial_x^2 + 4|u|^2 \end{pmatrix} ; \tag{4}$$

while $R[u]$ is given by

$$R[u] = r[u] - s[u], \qquad s[u] = \frac{1}{\epsilon}\{iu_t + u_{xx} + 2|u|^2 u\} = \frac{\mathrm{d}}{\mathrm{d}\tau} u \,, \tag{5}$$

where $\tau = \epsilon t$. It should be noted that the u on the right hand side and in $r[u]$ is an exact soliton solution of the unperturbed NLS equation, while the u in $s[u]$ is the adiabatic solution (the superscript a has been dropped).

It is easy to find out the eigenfunctions of the linearized operator $L(u,x)$ with a zero eigenvalue by employing the Lax equations of the NLS equation.[15] Since the full Jost function $w(x,t,\lambda)$ satisfies the two Lax equations, we have

$$\{\partial_t - L(u,x)\}W(x,t,\lambda) = 0, \qquad W(x,t,\lambda) = w^2(x,t,\lambda)\,. \tag{6}$$

Here $W(x,t,\lambda)$ is the squared Jost function.

The usual Jost functions are defined as the solution of only the first Lax equation with suitable boundary behavior. By multiplying a factor determined by the second Lax equation, the full Jost functions are obtained. Examples are $h(t,\lambda)^{-1}\psi(x,t,\lambda), h(t,\lambda)\phi(x,t,\lambda)$, etc., where $h(t,\lambda) = \mathrm{e}^{-\mathrm{i}2\lambda^2 t}$. Hence $W(x,t,\lambda)$ can be expressed as $h(t,\lambda)^{-2}\Psi(x,t,\lambda), h(t,\lambda)^2\Phi(x,t,\lambda)$, etc., where $\Psi(x,t,\lambda) = \psi^2(x,t,\lambda)$, etc..

From Eq.(6) we have

$$\{i\partial_t - L(u)\}\Psi(x,t,\lambda) = 4\lambda^2\Psi(x,t,\lambda)\,, \tag{7}$$

and

$$\{i\partial_t - L(u)\}\Psi(x,t,\lambda_n) = 4\lambda_n^2\Psi(x,t,\lambda_n)\,, \tag{8}$$

where λ_n is a pole of the transmission coefficient $a(k)^{-1}$. From Eq.(7), we can also find

$$\{i\partial_t - L(u)\}\dot{\Psi}(x,t,\lambda_n) = 4\lambda_n^2\dot{\Psi}(x,t,\lambda_n) + 8\lambda_n\Psi(x,t,\lambda_n)\,, \tag{9}$$

where dot means

$$\dot{\Psi}(x,t,\lambda_n) = \frac{\partial}{\partial\lambda}\Psi(x,t,\lambda)|_{\lambda=\lambda_n}\,. \tag{10}$$

Other squared Jost functions, $\Psi(x,t,\lambda)$, $\widetilde{\Phi}(x,t,\lambda)$, etc., satisfy the same eigen-equations.

III. ORTHOGONAL RELATIONS

For convenience, the Dirac ket of the eigenfunctions of $L(u,x)$ will be denoted as $|\Psi(\lambda)\rangle$, etc.. The adjoint operator and the adjoint eigenfunctions are introduced by many authors using various ways for the 1-soliton case.[7-14] The essential point is that the inner product of the squared function and its adjoint is proportional to a δ function in the case of continuous spectrum. In reference to the previons work, we directly define the inner product as

$$\begin{aligned}&\langle\Psi(k')|\Psi(k)\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\,\Psi^A(x,k')\Psi(x,k)\,,\\ &\Phi(x,\lambda)^A = \Psi(x,\lambda)^T(i\sigma_2) = (\psi_2^2(x,\lambda)\ \psi_1^2(x,\lambda))\,.\end{aligned} \tag{11}$$

From the first Lax equation, one can find[15]

$$\langle\Psi(k')|\Psi(k)\rangle = \lim_{L\to\infty}\frac{1}{-i2(\lambda-\lambda')}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2\Big|_{-L}^{L} = -a(\lambda)^2\delta(\lambda-\lambda')\,, \tag{12}$$

where $W[\cdot,\cdot]$ is the Wronskian, and we take the Cauchy principal value to get the second equation.

From the first Lax equation, we also obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\frac{\mathrm{d}}{\mathrm{d}x}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2\Big|_{\lambda=\lambda'=\lambda_n} = -i4\Phi(x,\lambda_n)^A\dot{\Psi}(x,\lambda_n)\,. \tag{13}$$

Hence we have

$$\langle\Psi(\lambda_m)|\dot{\Psi}(\lambda_n)\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\,\Psi(x,\lambda_m)^A\dot{\Psi}(x,\lambda_n) = -i\frac{1}{2}\dot{a}(\lambda_n)^2\delta_{mn}\,. \tag{14}$$

Similarly, we have

$$\langle\dot{\Psi}(\lambda_m)|\Psi(\lambda_n)\rangle = -i\frac{1}{2}\dot{a}(\lambda_n)^2\delta_{mn}\,. \tag{15}$$

Applying the operator $\left\{\dfrac{\mathrm{d}^3}{\mathrm{d}\lambda^3}+3\dfrac{\mathrm{d}}{\mathrm{d}\lambda'}\dfrac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\right\}$ to Eq.(13), taking $\lambda=\lambda'=\lambda_n$, and then integrating, we obtain

$$\langle\dot{\Psi}(\lambda_m)|\dot{\Psi}(\lambda_n)\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\,\dot{\Psi}(x,\lambda_m)^A\dot{\Psi}(x,\lambda_n) = -i\frac{1}{2}\dot{a}(\lambda_n)\ddot{a}(\lambda_n)\delta_{mn}\,. \tag{16}$$

In the same way, one can find that

$$\langle\tilde{\Psi}(\lambda')|\tilde{\Psi}(\lambda)\rangle = \tilde{a}(\lambda)^2\pi\delta(\lambda-\lambda')\,, \tag{17}$$

$$\langle(\dot{\tilde{\Psi}}(\bar{\lambda}_m)|\tilde{\Psi}(\bar{\lambda}_n)\rangle\langle(\dot{\tilde{\Psi}}(\bar{\lambda}_m)|\tilde{\Psi}(\bar{\lambda}_n)\rangle = -i\frac{1}{2}\dot{\tilde{a}}(\bar{\lambda}_n)^2\delta_{mn}\,, \tag{18}$$

and

$$\langle(\dot{\tilde{\Psi}}(\bar{\lambda}_m)|\dot{\tilde{\Psi}}(\bar{\lambda}_n)\rangle = -i\frac{1}{2}\dot{\tilde{a}}(\bar{\lambda}_n)\dot{\tilde{a}}(\bar{\lambda}_n)\delta_{mn}\,. \tag{19}$$

We have got all the non-zero inner products.

IV. EXPANSION OF THE UNITY

If the above kets form a complete set, a state $|f\rangle$ can be expanded as

$$\begin{aligned}|f\rangle =& \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\{f(\lambda)|\Psi(\lambda)\rangle + \tilde{f}(\lambda)|\tilde{\Psi}(\lambda)\rangle\}\\ &+\sum_{n=1}^{N}\{f_n|\Psi(\lambda_n)\rangle + g_n|\dot{\Psi}(\lambda_n)\rangle\} + \sum_{n=1}^{N}\{\tilde{f}_n|\tilde{\Psi}(\bar{\lambda}_n)\rangle + \tilde{g}_n|\dot{\tilde{\Psi}}(\bar{\lambda}_n)\rangle\}\,.\end{aligned} \tag{20}$$

By using the orthogonal relations, one can obtain expressions of $f(\lambda),\tilde{f}(\lambda),f_n$, etc., in terms of $\langle\Psi(\lambda)|f\rangle,\langle\tilde{\Psi}(\lambda_n)|f\rangle$, etc.. Substituting these expressions into Eq.(20) we obtain

$$\begin{aligned}\delta(x-y) =& \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\frac{1}{a(\lambda)^2}\Phi(x,\lambda)\Phi(y,\lambda)^A + \sum_{n=1}^{N}i\frac{2\ddot{a}(\lambda_n)}{\dot{a}(\lambda_n)^3}\Phi(x,\lambda_n)\Phi(y,\lambda_n)^A\\ &-\sum_{n=1}^{N}i\frac{2}{\dot{a}(\lambda_n)^2}\{\dot{\Phi}(x,\lambda_n)\Phi(y,\lambda_n)^A+\Phi(x,\lambda_n)\dot{\Phi}(y,\lambda_n)^A\}\\ &-\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\frac{1}{\tilde{a}(\lambda)^2}\tilde{\Phi}(x,\lambda)\tilde{\Phi}(y,\lambda)^A + \sum_{n=1}^{N}i\frac{2\ddot{\tilde{a}}(\bar{\lambda}_n)}{\dot{\tilde{a}}(\bar{\lambda}_n)}\tilde{\Phi}(x,\bar{\lambda}_n)\tilde{\Phi}(y,\bar{\lambda}_n)\\ &-\sum_{n=1}^{N}i\frac{2}{\dot{\tilde{a}}(\bar{\lambda}_n)}\{\dot{\tilde{\Phi}}(x,\bar{\lambda}_n)\tilde{\Phi}(y,\bar{\lambda}_n)^A+\tilde{\Phi}(x,\bar{\lambda}_n)\dot{\tilde{\Phi}}(y,\bar{\lambda}_n)^A\}\,.\end{aligned} \tag{21}$$

This is the expansion of the unity.

V. PROOF OF THE COMPLETENESS

To prove Eq.(21), it is necessary to show the completeness of the squared Jost functions. For a given potential u, we have the Marchenko equation and a completeness relation of the Jost functions[15]

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\tilde{\psi}(x,\lambda)\psi(y,\lambda)^T+\psi(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\{r(\lambda)\psi(x,\lambda)\psi(y,\lambda)^T-\tilde{r}(\lambda)\tilde{\psi}(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\sum_{n=1}^{N}\{c_n\psi(x,\lambda_n)\psi(y,\lambda_n)^T+\tilde{c}_n\tilde{\psi}(x,\bar{\lambda}_n)\tilde{\psi}(y,\bar{\lambda}_n)^T\}\sigma_1=\delta(x-y)\,.\end{aligned}\tag{22}$$

For another potential u', we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\tilde{\psi}'(x,\lambda)\psi'(y,\lambda)^T+\psi'(x,\lambda)\tilde{\psi}'(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\{r'(\lambda)\psi'(x,\lambda)\psi'(y,\lambda)^T-\tilde{r}'(\lambda)\tilde{\psi}'(x,\lambda)\tilde{\psi}'(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\sum_{n'=1}^{N'}\{c'_{n'}\psi'(x,\lambda_{n'})\psi'(y,\lambda_{n'})^T+\tilde{c}'_{n'}\tilde{\psi}'(x,\bar{\lambda}_{n'})\tilde{\psi}'(y,\bar{\lambda}_{n'})^T\}\sigma_1=\delta(x-y)\,.\end{aligned}\tag{23}$$

Consider a Kern $M(x,y)$ of integral representation

$$\psi'(x,\lambda)=\psi(x,\lambda)+\int_x^{\infty}\mathrm{d}yM(x,y)\psi(y,\lambda)\,,\tag{24}$$

and a Kern $N(x,y)$

$$\psi(x,\lambda)=\psi'(x,\lambda)+\int_x^{\infty}\mathrm{d}yN(x,y)\psi'(y,\lambda)\,.\tag{25}$$

By means of Eq.(24), replacing $\psi(x,\lambda)$ in Eq.(22) by $\psi'(x,\lambda)$, we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\tilde{\psi}'(x,\lambda)\psi(y,\lambda)^T+\psi'(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\{r(\lambda)\psi(x,\lambda)\psi(y,\lambda)^T-\tilde{r}(\lambda)\tilde{\psi}(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\&+\sum_{n=1}^{N}\{c_n\psi'(x,\lambda_n)\psi(y,\lambda_n)^T+\tilde{c}_n\tilde{\psi}(x,\bar{\lambda}_n)\tilde{\psi}(y,\bar{\lambda}_n)^T\}\sigma_1\\&=\delta(x-y)+\int_x^{\infty}\mathrm{d}sM(x,s)\delta(s-y)\,.\end{aligned}\tag{26}$$

Similarly, by means of Eq.(25), Eq.(23) turns to

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\tilde{\psi}'(x,\lambda)\psi(y,\lambda)^T+\psi'(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\{r'(\lambda)\psi'(x,\lambda)\psi(y,\lambda)^T-\tilde{r}'(\lambda)\tilde{\psi}'(x,\lambda)\tilde{\psi}(y,\lambda)^T\}\sigma_1\mathrm{d}\lambda\\
&+\sum_{n'=1}^{N'}\{c'_{n'}\psi(x,\lambda'_{n'})\psi(y,\lambda'_{n'})^T+\tilde{c}'_{n'}\tilde{\psi}'(x,\bar{\lambda}'_{n'})\tilde{\psi}(y,\bar{\lambda}'_{n'})^T\}\sigma_1\\
&=\delta(x-y)+\int_y^{\infty}\mathrm{d}s\delta(x-s)\sigma_1 N(y-s)^T\sigma_1\ .
\end{aligned}\tag{27}$$

Substracting Eq.(26) from Eq.(27), using Eq.(24) again, we obtain the generalized Marchenko equation[16,17]

$$\Omega(x,y)+M(x,y)+\int_x^{\infty}\mathrm{d}zM(x,z)\Omega(z,y)=0\ ,\tag{28}$$

where

$$\begin{aligned}
\Omega(x,y)=&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(r'(\lambda)-r(\lambda))\psi(x,\lambda)\psi(y,\lambda)^T\\
&\qquad-(\tilde{r}'(\lambda)-\tilde{r}(\lambda))\tilde{\psi}(x,\lambda)\tilde{(y,\lambda)}^T\}\sigma_1\mathrm{d}\lambda\\
&-\sum_{n'=1}^{N'}\{c'_{n'}\psi(x,\lambda'_{n'})\psi(y,\lambda'_{n'})^T+\tilde{c}'_{n'}\tilde{\psi}(x,\bar{\lambda}'_{n'})\tilde{\psi}(y,\bar{\lambda}'_{n'})^T\}\sigma_1\\
&+\sum_{n=1}^{N}\{c_n\psi(x,\lambda_n)\psi(y,\lambda_n)^T+\tilde{c}_n\tilde{\psi}(x,\bar{\lambda}_n)\tilde{\psi}(y,\bar{\lambda}_n)^T\}\sigma_1\ .
\end{aligned}\tag{29}$$

In the case of small $\delta u(x)=u'(x)-u(x)$, $N=N'$ and $\delta r(\lambda)=r'(\lambda)-r(\lambda)$, etc., are also small, we have

$$\begin{aligned}
\Omega(x,x)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\delta r(\lambda)\Psi(x,\lambda)+\sum_{n=1}^{N}\{\delta c_n\Psi(x,\lambda_n)+c_n\lambda_n\dot{\Psi}(x,\lambda_n)\}\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\delta\tilde{r}(\lambda)\tilde{\Psi}(x,\lambda)+\sum_{n=1}^{N}\{\delta\tilde{c}_n\tilde{\Psi}(x,\lambda_n)+\tilde{c}\lambda_n\dot{\tilde{\Psi}}(x,\lambda_n)\}\ .
\end{aligned}\tag{30}$$

Hence we have

$$\delta u(x)=-2\frac{\mathrm{d}}{\mathrm{d}x}M(x,x)=2\frac{\mathrm{d}}{\mathrm{d}x}\Omega(x,x)\ .\tag{31}$$

From these two equations, we conclude that $\delta u(x)$ can be expanded in terms of $\Psi(x,\lambda)$, $\tilde{\Psi}(x,\lambda_n)$, $\dot{\Psi}(x,\lambda_n)$, etc., thus the squared Jost functions are complete.

VI. SECURITY CONDITIONS

Suppose the unknown function q in Eq.(3) can be expended as that of f in Eq.(20). Then by substituting the expansion into Eq.(3), and evaluating the inner product of $\langle\Psi(\lambda_n)|$ and $\langle\dot{\Psi}(\lambda_n)|$ with the resultant equation, we obtain

$$-\{ig_{nt}+4\lambda_n^2 g_n\}i\frac{1}{2}\dot{a}(\lambda_n)^2=\langle\Psi(\lambda_n)|R\rangle\ ,\tag{32}$$

and

$$-\{if_{nt}+4\lambda_n^2 f_n+8\lambda_n g_n\}i\frac{1}{2}\dot{a}(\lambda_n)^2-\{ig_{nt}+4\lambda_n^2 g_n\}i\frac{1}{2}\dot{a}(\lambda_n)\ddot{a}(\lambda_n)=\langle\dot{\Psi}(\lambda_n)|R\rangle\,. \tag{33}$$

Although the initial values of g_n and f_n vanish, their values may go to infinite if the right hand side of the above equations are not zero. Hence we demand them to vanish,

$$\langle\Psi(\lambda_n)|R\rangle=0,\qquad \langle\dot{\Psi}(\lambda_n)|R\rangle=0\,. \tag{34}$$

Similarly, by evaluating the inner product of $\langle\widetilde{\Psi}(\bar{\lambda}_n)|$ and $\langle\dot{\widetilde{\Psi}}(\bar{\lambda}_n)|$ with the resultant equation, we demand

$$\langle\widetilde{\Psi}(\bar{\lambda}_n)|R\rangle=0,\qquad \langle\dot{\widetilde{\Psi}}(\bar{\lambda}_n)|R\rangle=0\,, \tag{35}$$

to ensure $\tilde{f}_n$ and $\tilde{g}_n$ are finite. These two equations are complex conjugate of each other, one of them can be used as the secularity conditions. By means of them the adiabatic solution can be determined. That is, the time-dependence of the parameters characterizing soliton solution of the order of ϵ can be determined.

Finally, by evaluating the inner product of $\langle\Psi(\lambda)|$ and $\langle\widetilde{\Psi}(\lambda)|$, with the resultant equation, we obtain

$$-\{if_t(\lambda)+4\lambda^2 f(\lambda)\}a(\lambda)^2=\langle\Psi(\lambda)|R\rangle\,, \tag{36}$$

and

$$\{i\tilde{f}_t(\lambda)-4\lambda^2\tilde{f}(\lambda)\}\tilde{a}(\lambda)^2=\langle\widetilde{\Psi}(\lambda)|R\rangle\,. \tag{37}$$

Equation (36) is the complex conjugate of Eq.(37). After the adiabatic solution is determined, the right hand side of Eq.(36) is known, thus f and $\tilde{f}$ can be determined.

VII. APPLICATION IN THE 1-SOLITON CASE

In this case, $a(\lambda)$ has one simple pole at $\lambda_1=\mu_1+i\nu_1$ (the subscript 1 will be dropped in the following for simplity). For the damping case

$$\epsilon r[u]=-i\Gamma u,\qquad 0<\Gamma<<1\,. \tag{38}$$

By employing the developed formulae, one can easily find out the adiabatic solution

$$u_1^a=2\nu\mathrm{sech}(2\nu z)\mathrm{e}^{-i\phi}\,, \tag{39}$$

where

$$\nu=\nu_0\mathrm{e}^{-2\Gamma t},\quad \mu=\mu_0 \tag{40}$$

and

$$z=x+4\mu_0 t-x_0\bar{x},\qquad \phi=2\mu z-4\mu_0^2 t-\nu_0^2\left(1-\frac{\mathrm{e}^{-4\Gamma t}}{\Gamma}\right)+\alpha_0\,. \tag{41}$$

We know that for optical soliton transmission[18], $\mu_1=0$, then by Eq.(36), one can find

$$f(t)=\frac{i\Gamma\pi}{4(\lambda-i\nu_1)^2}(\mathrm{e}^{-i4(\lambda^2+\nu_1^2)t}-1)\mathrm{sech}\left(\frac{\lambda\pi}{2\nu_1}\right)\,. \tag{42}$$

In the following, the developed theory will be used to treat the evolution of a Raman loss compensated soliton in an optical fiber. In this case, the damping coefficient in Eq.(38) is given by[18]

$$\Gamma = \frac{\alpha_s}{2}\left\{1 - \frac{\alpha_p L}{2\mathrm{sh}(\alpha_p L/2)}\mathrm{ch}(\alpha_p(t - L/2))\right\}, \tag{43}$$

where α_s and α_p are the damping coefficient of the optical fiber and the damping coefficient at the place of repeaters, respectivly, and L represents the distance between the repeaters. According to the formulae developed, one can find that the relative change of the soliton area is given by

$$\frac{\delta A}{A} = \mathrm{Re}\left\{\frac{\int_{-\infty}^{\infty} v\mathrm{d}\theta_1}{2\nu_1\pi\mathrm{e}^{-i2\alpha}} - 1\right\} = \frac{\Gamma}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}\lambda F(z,\lambda), \tag{44}$$

where

$$\begin{aligned} F(z,\lambda) = \frac{\sin(4\nu_1^2 t + 4\lambda^2 t)}{(\lambda^2+\nu_1^2)^2\mathrm{ch}\left(\dfrac{\pi\lambda}{2\nu_1}\right)}&\left\{2\nu_1^2\cos(2\lambda z)\left[\mathrm{sech}(2\nu_1 z) - \frac{1}{2}\right]\right. \\ &\left. + 2\lambda^2\cos(2\lambda z) - 2\nu_1\lambda\sin(2\lambda z) - \mathrm{th}(2\nu_1 z)\mathrm{th}(2\nu_1 z)\right\}, \end{aligned} \tag{45}$$

in which Z_c is the characteristic length. The relative soliton area change is thus determined by using the α_s, α_p, and Z_c provided in numerical simulation.[13,18]

VIII. CONCLUSION AND DISCUSSION

In previous sections, an exact formalism of the direct perturbation theory of the NLS equation has been developed. Based on the result of IST, the completeness of the squared Jost functions is shown by the generalized Marchenko equation, thus the basis of the direct perturbation theory for the perturbed NLS equation is constructed.

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