Volume 8, Number 6 ACTA PHYSICA SINICA June, 1999
1004-423X/1999/08(06)/0416-07 (Overseas Edition) © 1999 Chin. Phys. Soc.

CONTROLLING HOPF BIFURCATIONS: CONTINUOUS-TIME SYSTEMS*

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(Received 5 October 1998)

Bifurcation control has attracted increasing attention in recent years. A simple and unified state-feedback method is developed in this paper for Hopf bifurcation control for continuous-time systems. The control task can be either shifting an existing Hopf bifurcation or creating a new Hopf bifurcation. Some computer simulations are included to illustrate the method and verify the theoretical results.

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I. INTRODUCTION

Bifurcation control involves designing a controller that can alter the bifurcative properties of a given nonlinear system, resulting in some desired dynamical behaviors. Typical examples include delaying the onset of an inherent bifurcation, relocating an existing bifurcation point, modifying the shape or type of a bifurcation chain, introducing a new bifurcation at a preferable parameter value, stabilizing a bifurcated periodic trajectory, changing the multiplicity, amplitude, and/or frequency of some limit cycles emerging from bifurcation, optimizing the system performance near a bifurcation point, or a certain combination of some of these.[1−8] Bifurcation control not only is important in its own right, but also provides an effective strategy for chaos control since period-doubling bifurcation is a typical route to chaos in many nonlinear dynamical systems.

Bifurcation control is useful in many engineering applications. In some physical systems, such as the stressed system, delay of bifurcations offers an opportunity to obtain stable operating conditions for the machine beyond the margin of operability at the normal situation.[9] Sometimes, it is desirable that the stability of bifurcated limit cycles can be modified, with application to some conventional control problems such as thermal convection experiments.[10] Other examples include stabilization via bifurcation control in tethered satellites[11], magnetic bearing systems[12], voltage dynamics of electric power systems[13]; and various mechanical systems such as robotics and electronic systems such as laser machines and nonlinear circuits.[6]

*Project supported by the National Natural Science Foundation of China (Grant No. 19875080).

Bifurcations can be controlled via linear-delayed state-feedback[14,15] or nonlinear state-feedback,[2] using a washout filter,[10] employing harmonic balance approximation,[3,4,7,16] and applying the quadratic invariants in the normal form.[17] In this paper, a unified linear, as well as simple nonlinear, state-feedback technique is developed for Hopf bifurcation control, which by nature is different from the aforementioned existing approaches. Both continuous-time and discrete-time systems are studied, and both problems of shifting and creating a Hopf bifurcation are discussed. Computer simulations are included to illustrate the methodology and to verify the theoretical results.

In the rest of the paper, section Ⅱ briefly summarizes the classical Hopf bifurcation theory for continuous-time systems. section Ⅲ studies the state-feedback control problem for Hopf bifurcations, including computer simulation results. section Ⅳ concludes the investigation with some discussion.

Ⅱ. HOPF BIFURCATIONS: PRELIMINARIES

In this section, the classical criteria for continuous-time Hopf bifurcations are briefly reviewed. This background material is needed in the development of the bifurcation control strategy given in the sequel.

First, consider a two-dimensional continuous-time parametrized autonomous system

$$\begin{cases} \dot{x} = f(x, y; \mu) , \\ \dot{y} = g(x, y; \mu) , \end{cases} \tag{1}$$

where $\mu \in R$ is a real variable parameter, and both $f,\ g \in C^1(R^2)$. Assume that the system has an equilibrium point (x^*, y^*), satisfying $f(x^*, y^*; \mu) = g(x^*, y^*; \mu) = 0$ for all $\mu \in R$. Let $J(\mu)$ be its Jacobian at this equilibrium, and suppose that J has a pair of complex conjugate eigenvalues, $\lambda_{1,2}(\mu)$, with $\lambda_1 = \overline{\lambda}_2$. Assume that λ_1 moves from the left-half plane to the right as μ varies, so does λ_2, and they cross the imaginary axis at the moment $\mu = \mu^*$ in such a way that

$$\mathrm{Re}\{\lambda_1(\mu^*)\} = 0 \qquad \text{and} \qquad \left.\frac{\partial\, \mathrm{Re}\{\lambda_1(\mu)\}}{\partial \mu}\right|_{\mu=\mu^*} > 0 , \tag{2}$$

where, $\mathrm{Re}\{\cdot\}$ denotes the real part of a complex number. Here, the second condition in Eq.(2) refers to as the transversality condition for the crossing of the eigenlocus at the imaginary axis, which means that the eigenlocus is not tangent to the imaginary axis. Then, the system undergoes a Hopf bifurcation at the bifurcation point (x^*, y^*, μ^*).[11,18] More precisely, in any small left-neighborhood of μ^* (i.e., $\mu < \mu^*$), (x^*, y^*) is a stable focus; and in any small right-neighborhood of μ^* (i.e., $\mu > \mu^*$), this focus changes into an unstable one, surrounded by a limit crcle of amplitude $O(\sqrt{|\mu - \mu^*|})$.

A typical example it the van der Pol circuit

$$\begin{cases} \dot{x} = y , \\ \dot{y} = \mu(1 - x^2)y - x , \end{cases} \tag{3}$$

which has a Hopf bifurcation at (x^*, y^*, μ^*)=(0,0,0), as shown in Figs.1–3. In these figures, initial points are marked$*$. In Fig.1, $\mu = -0.01 < 0.0 = \mu^*$; in Fig.2, $\mu = 0.0 = \mu^*$; and in Fig.3, $\mu = 0.01 > 0.0 = \mu^*$.

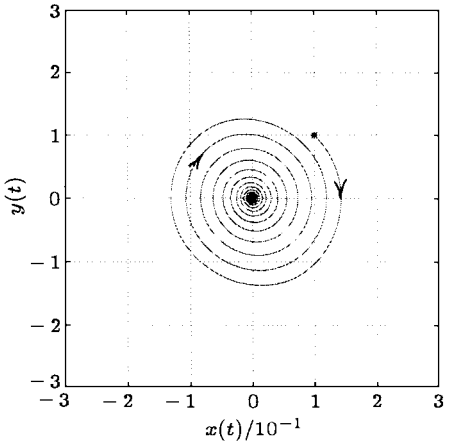

Fig.1. Orbit of the uncontrolled van der Pol circuit ($\mu < \mu^*$).

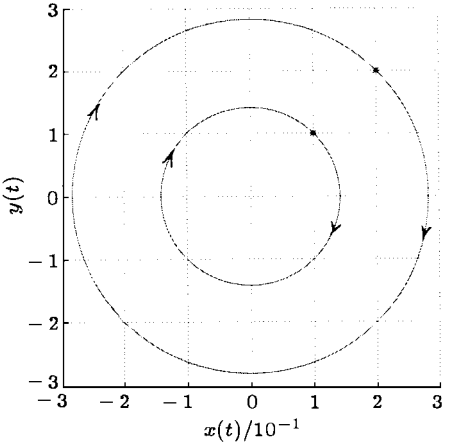

Fig.2. Orbit of the uncontrolled van der Pol circuit ($\mu = \mu^*$).

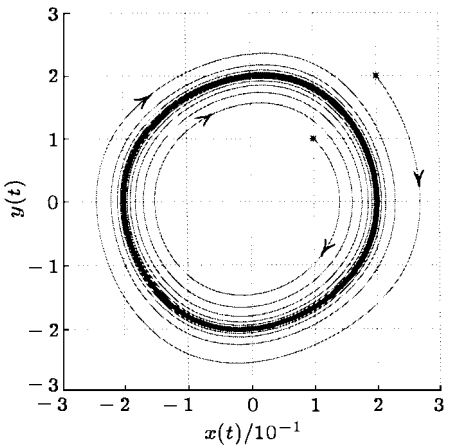

Fig.3. Orbit of the uncontrolled van der Pol circuit ($\mu > \mu^*$).

Ⅲ. CONTROLLING HOPF BIFURCATIONS

Now, consider a Hopf bifurcation control problem as follows: Design a simple controller,

$$u(t; \mu) = u(x, y; \mu) \,, \tag{4}$$

which does not change the original equilibrium point at (x^*, y^*) but can move the Hopf bifurcation point (x^*, y^*, μ^*) to a new position, $(x^0, y^0, \mu^0 \neq (x^*, y^*, \mu^*)$. Clearly, the controller must satisfy

$$u(x^*, y^*; \mu) = 0 \tag{5}$$

for all μ, in order not to change the original equilibrium (x^*, y^*).

For the sake of determination, suppose that the controller is added to the second equation of the given system, namely

$$\begin{cases} \dot{x} = f(x, y; \mu) , \\ \dot{y} = g(x, y; \mu) + u(x, y; \mu) . \end{cases} \tag{6}$$

This controlled system has the Jacobian at (x^0, y^0)

$$J(\mu) = \left[\begin{array}{cc} f_x & f_y \\ g_x + u_x & g_y + u_y \end{array} \right]_{x=x^0, y=y^0} , \tag{7}$$

where $f_x = \partial f / \partial x_k$ and $g_y = \partial g / \partial y$, etc., with eigenvalues

$$\lambda^c_{1,2}(\mu) = \frac{1}{2}(f_x + g_y + u_y) \pm \frac{1}{2}\sqrt{(f_x + g_y + u_y)^2 - 4[f_x(g_y + u_y) - f_y(g_x + u_x)]} , \tag{8}$$

where $f_x := f_x|_{x=x^0, y=y^0}$ and $g_y := g_y|_{x=x^0, y=y^0}$, etc., for notational simplicity.

To have a Hopf bifurcation at $(x^0, y^0; \mu^0)$ as required, the classical Hopf bifurcation theory leads to the following conditions:

(i) (x^0, y^0) is an equilibrium point of the controlled system (6), namely

$$\begin{cases} f(x^0, y^0; \mu) = 0 , \\ g(x^0, y^0; \mu) + u(x^0, y^0; \mu) = 0 , \end{cases} \tag{9}$$

for all $\mu \in R$.

(ii) The eigenvalues $\lambda^c_{1,2}(\mu)$ of the controlled system (6) are purely imaginary at the point $(x^0, y^0; \mu^0)$ and are complex conjugate

$$(f_x + g_y + u_y)\Big|_{\mu=\mu^0} = 0 , \tag{10}$$

$$f_x(g_y + u_y) - f_y(g_x + u_x)\Big|_{\mu=\mu^0} > 0 , \tag{11}$$

$$(f_x + g_y + u_y)^2 - 4[f_x(g_y + u_y) - f_y(g_x + u_x)]\Big|_{\mu \neq \mu^0} < 0 . \tag{12}$$

(iii) The crossing of the eigenlocus at the imaginary axis is transversal, namely

$$\frac{\partial \operatorname{Re}\{\lambda^c_1(\mu)\}}{\partial \mu}\Big|_{\mu=\mu^0} = \frac{\partial (f_x + g_y + u_y)}{\partial \mu}\Big|_{\mu=\mu^0} > 0 . \tag{13}$$

For the van der Pol circuit (3), if the goal of control is to move the bifurcation value of μ from $\mu^* = 0$ to a $\mu^0 > 0$ while maintaining the equilibrium point of the system to be unchanged: $(x^0, y^0) = (x^*, y^*) = (0, 0)$, then condition (10) yields a linear controller of the form

$$u = -\mu^0(1 - (x^0)^2)y , \tag{14}$$

which satisfies all conditions required by Eqs. (5) and (9)–(13), stated in (i)–(iii) above. More specifically, since (x^0, y^0)=(0,0) and $u(x^*, y^*; \mu) = -\mu^0(1 - (x^0)^2)y^* = 0$, conditions

(5) and (9) are satisfied; since $u_y = -\mu^0(1-(x^0)^2)$, condition (5) and (9) are satisfied; since $f_x(g_y+u_y) = 0 > -\mu^0 = f_y(g_x+u_x)$, condition (11) is satisfied; similarly, condition (12) is satisfied; and finally, since $(f_x+g_y+u_y)_\mu = 2\mu^0 x^0 y^0 + 1 = 1 > 0$, condition (12) is satisfied.

It is clear that $\mu^0 > 0$ can be arbitrary, which implies that the simple linear feedback controller can significantly extend the operational range of the key system parameter, μ, of the van der Pol circuit, at one's will. And such extension can be very important in some practical applications: it may actually enhance the stability and/or performance robustness of the dynamical systems. The resulting "shifted" Hopf bifurcation phenomenon for $\mu^0 = 2$ is shown in Figs. 4–6: in Fig.4 $\mu = 1.99 < 2.0 = \mu^0$, in Fig.5 $\mu = 2.0 = \mu^0$, and in Fig.6 $\mu = 2.01 > 2.0 = \mu^0$.

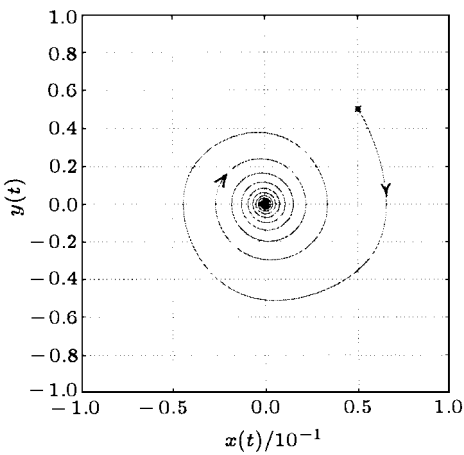

Fig.4. Orbit of the controlled van der Pol circuit ($\mu < \mu^0 = 2$).

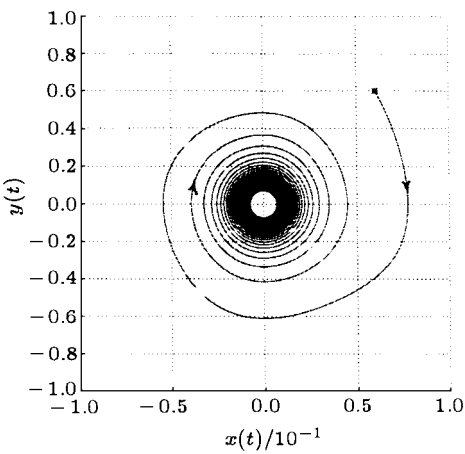

Fig.5. Orbit of the controlled van der Pol circuit ($\mu = \mu^0 = 2$).

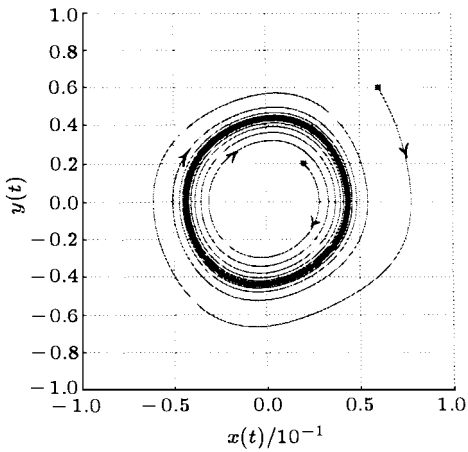

Fig.6. Orbit of the controlled van der Pol circuit ($\mu > \mu^0 = 2$).

Note that different bifurcation control tasks can be performed via different controllers for the van der Pol circuit. For instance, if the goal for control is not to change the bifurcation value of μ (namely, $\mu^0 = \mu^* = 0$ is retained), but to shift the equilibrium point of the system from (x^*, y^*)=(0,0) to, say, (x^0, y^0)=$(x^0, 0)$ with $0 < x^0 < 1$, then another linear feedback controller can be designed. Actually, in this case, even the very simple constant controller $u = x^0$ can do the job. This can be easily verified by going through the same procedure discussed above. In this example with $x^0 = 0.9$, the three corresponding bifurcation phase portraits are shown in Figs. 7–9, which demonstrate that the Hopf bifurcation point has indeed been changed to (0.9,0,0).

Note also that the above control strategy can be applied to other dynamical systems, for which different controllers can be designed according to conditions (9)–(14), where,

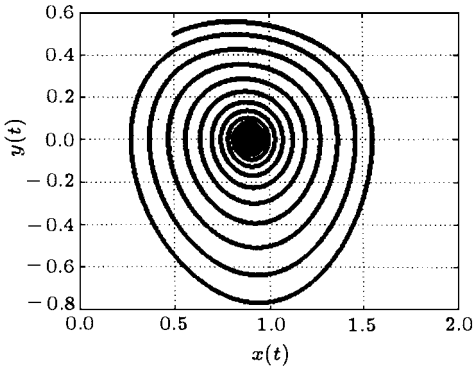

Fig.7. Orbit of the controlled van der Pol circuit ($\mu < \mu^0 = 0$).

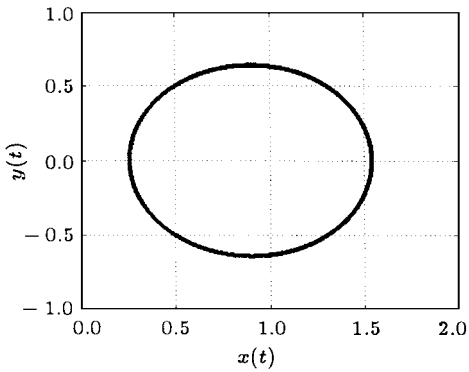

Fig.8. Orbit of the controlled van der Pol circuit ($\mu = \mu^0 = 0$).

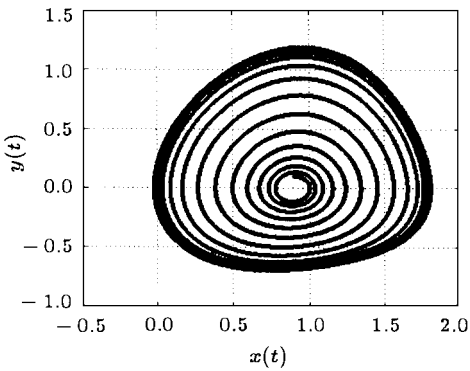

Fig.9. Orbit of the controlled van der Pol circuit ($\mu > \mu^0 = 0$).

particularly, condition (14) suggests a basic form (structure) for the controller. The following two systems have been also tested by simulations: the first one is

$$\begin{cases} \dot{x} = \mu x - y + xy^2 \,, \\ \dot{y} = x + \mu y + y^3 \,, \end{cases} \tag{15}$$

for which the linear controller $u = -2(\mu^0 + 2(y^0)^2)y$ moves the original Hopf bifurcation point (0, 0, 0) to (0.1, 0.1, 0.9), and the nonlinear controller $u = -2\mu^0 - 4y^3/3$ moves the original bifurcation point to (0.1, 0.1, 0.99), respectively; and the second system is

$$\begin{cases} \dot{x} = \mu x - y + xy^2 \,, \\ \dot{y} = x + \mu y - x^2 \,, \end{cases} \tag{16}$$

for which the linear controller $u = -2(\mu^0 + 2(y^0)^2)y$ moves the original Hopf bifurcation point (0, 0, 0) to $(-1/2, \sqrt{6}/4, -(4\sqrt{6}+3)/8)$, and the nonlinear controller $u = x^2 - 2\mu^0 - y^3/3$ moves the original bifurcation point to (0.1, 0.1, 0.99), respectively.

IV. CONCLUSIONS

In this paper, a simple and unified state-feedback control method has been developed for Hopf bifurcations for continuous-time systems, and for both problems of shifting and creating a Hopf bifurcation point in the controlled system. The discussion in this paper has been restricted to two-dimensional systems, but the basic idea and the proposed approach can be extended to higher-dimensional dynamical systems. However, it must be noted that the higher-dimensional setting is much more complicated than it may appear to be. In a higher-dimensional system, there are many intrinsic situations where one may face with multiple eigenvalues and degenerate Hopf bifurcations[7,19] (e. g., the transversal condition is not satisfied in some direction, and/or the so-called curvature coefficients vanish). Moreover, global bifurcations and nonautonomous systems remain to be a great challenge for both bifurcation and chaos controls,[20] which calls for further research efforts from the communities of nonlinear dynamics and control systems.

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