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EVOLUTION OF ENTROPY IN THE NONLINEAR JAYNES-CUMMINGS MODEL

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In this paper we study the correlation between the external and the internal state of the trapped atoms by means of the nonlinear Jaynes-Cummings Hamiltonian, in which we pay special attention to the influence of the nonlinearity. We find that the correlation is reflected by the entropy of the vibration mode, which is strongly affected by the nonlinearity. The nonlinearity, embodied by the nonlinear Rabi frequency, exhitits various effects, of which the most important feature is the appearance of the second local minimum value of the entropy in each collapse-revival cycle of the atomic inversion.

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I. INTRODUCTION

The Jaynes-Cummings (JC) model is a fundamental, fully quantized model of the interaction between a two-level system and a boson mode, being of both theoretical and experimental importance. In the optical domain, this model gives rise to a realistic description of the dipole interaction of a single two-level atom with a single mode quantized electromagnetic wave. Moreover, recent work about the motion of a trapped atom or ion interacting simultaneously with the electromagnetic wave has shown that in the Lamb-Dicke limit, where the particle's characteristic length of quantum mechanical state of motion is small compared with the wavelength of light field[1], the particle motion is described by the Jaynes-Cummings dynamics[2]; beyond the Lamb-Dicke limit, the particle motion is governed by a nonlinear coupled Jaynes-Cummings Hamiltonian[3], in which the nonlinearity obviously affects the dynamics of the particle. In these JC-like Hamiltonians, the quantized center-of-mass motion of the particle, named as the vibration mode throughout this paper, plays the role of the boson mode. Because of the entanglement between the internal and the external states, various schemes to control one freedom by another are studied.[2,4–7]

It has shown that examination of the atomic inversion alone is not sufficient to give a clear insight into the interaction, which leads to the study of the entanglement between the atom and the field.[8] Since the entanglement between the internal and external degrees of freedom is of the central position in the above studies, it is necessary to determine the entanglement between these two degrees of freedom, and it is the establishment of the entanglement that leads to the strong correlation between properties of these two component

systems. The purpose of this paper is to study the correlation between the internal and the external degrees of freedom of the atom, to study the effect of the nonlinearity that works when the motion of the atom is far from the Lamb-Dicke limit, on the entanglement of the two degrees of freedom of the atom.

Ⅱ. EVOLUTION OF THE ENTROPY

Consider an atom of mass m, transition angular frequency ω_0 and dipole moment D, where $D = \mu(\sigma_x, \sigma_y, \sigma_z)$ (σ_i $[i = x, y, z]$ are the Pauli spin operators and here denote the orientations of the dipole moment). The atom experiencing harmonic motion at frequency ν in z direction is irradiated by a circularly polarized electromagnetic wave of wave vector k, angular frequency ω_L. The electromagnetic wave propagates along the positive z direction and its electric field $\boldsymbol{E}$ is assumed to be of the form $\boldsymbol{E}$=(E_x, E_y), where $E_x = A\cos(\omega_L t - \kappa z)$, $E_y = A\sin(\omega_L t - \kappa z)$, and A is the amplitude of $\boldsymbol{E}$. Using the circularly polarized electromagnetic wave allows one to avoid the extensively used rotating-wave-approximation.[9] Choosing the electromagnetic wave to the first-higher-sideband, which means $\omega_0 = \omega_L + \nu$, in a frame rotating with the atom's transition frequency ω_0, the effective Hamiltonian becomes

$$H_{int} = \frac{1}{2}\eta\hbar\lambda[f^{(n)}(a^\dagger, a)a\sigma_+ + H.c.] \tag{1}$$

with

$$f^{(n)}(a^\dagger, a) = \sum_l \frac{(-1)^l\eta^{2l}}{l!(l+1)!}a^{\dagger l}a^l \ , \tag{2}$$

where the dipole approximation is made and we have used the JC model approximation to neglect the time-dependent terms. Moreover, we have set $\lambda = \Omega_L \exp(-\eta^2/2)$, where η is the Lamb-Dicke parameter defined as $\eta = \omega_L/c(\hbar/2m\nu)^{1/2}$, and $\Omega_L = \mu \cdot A/\hbar$ is related to the coupling between the atom and the electromagnetic wave. The boson operators a, $a^\dagger$ are, respectively, the annihilation and creation operators of the vibration mode. This Hamiltonian is named as the nonlinear JC Hamiltonian because of the appearance of the normal-ordered operator function $f^{(n)}(a^\dagger, a)$. From this Hamiltonian we can see that the internal and the external degrees of freedom of the atom inevitably couples to each other, so a study of their correlation is necessary.

We study the correlation by means of the von Neumann entropy, which is defined through the density operator of a given system. Entropies of the electronic state and the vibrational state of the atom are defined through their respective reduced density operators as[10]

$$S_j = -\mathrm{Tr}_j\rho_j(t)\ln\rho_j(t) \ , \tag{3}$$

where the subscript j is taken to imply one of the electronic component, the vibrational, and the total (component) system, and ρ_j means the density operator of the jth (component) system. Generally, the entropies of a two-component quantum system are linked by a remarkable theorem given by Araki and Leib[11], which reads

$$|S_e - S_v| \geq S \geq S_e + S_v \ . \tag{4}$$

The subscripts e and v denote any two general quantum systems and here, the electronic and vibrational component systems respectively. The total entropy of the system S evolves a unitary evolution governed by the interaction Hamiltonian (1), which leads to S keeping its initial value all the time. For simplicity, we consider the case of the atom being initially prepared in a pure state and its electronic and vibrational degrees of freedom are decoupled, that is, the initial value of the total entropy is zero. One immediate consequence of this assumption is, by means of Eq.(4), that $S_v \equiv S_e$: the entropy of the vibrational state is always equivalent to that of the electronic state. If we adopt the correlation index I_c, which is based on the von Neumann entropy[12],

$$I_c = S_e + S_v - S \tag{5}$$

to mark the degree of the correlation between these two component systems, we are easy to show $I_c = 2S_v$. Then we can safely conclude that the entropy of each component system also corresponds to the correlation.

Using the standard techniques[11], the time evolution operator governed by Hamiltonian (1) is given by

$$U(t) = \begin{pmatrix} \cos\lambda T(faa^\dagger f)^{1/2} & -ifa(a^\dagger ffa)^{-1/2}\sin\lambda T(a^\dagger ffa)^{1/2} \\ -ia^\dagger f(faa^\dagger f)^{-1/2}\sin\lambda T(faa^\dagger f)^{1/2} & \cos\lambda T(a^\dagger ffa)^{1/2} \end{pmatrix}, \tag{6}$$

where $U(t)$ is written in the atomic basis and for simplicity, we have written $f^{(n)}(a^\dagger, a) = f^{\dagger(n)}(a^\dagger, a) = f$, $T = 1/2\eta t$. By assuming that initially the atom is fully inverted and its vibrational state is a pure state $|\psi_f>$, the initial state of the atom can be described by

$$|\psi(0)> = |e> \otimes |\psi_f> , \tag{7}$$

where $|e>$ ($|g>$) is the electronic excited (ground) state. Following Phoenix and Knight, the time-evolved state of the atom becomes[8]

$$|\psi(t)> = |C> \otimes |e> + |S> \otimes |g> , \tag{8}$$

with $|C>$ and $|S>$ being given by

$$\begin{aligned} |C> &= \cos\lambda t(faa^\dagger f)^{1/2}|\psi_f> , \\ |S> &= -ia^\dagger f\frac{\sin\lambda t(faa^\dagger f)^{1/2}}{(faa^\dagger f)^{1/2}}|\psi_f> \ . \end{aligned} \tag{9}$$

The density operator of the total system is $\rho(t) = |\psi(t)><\psi(t)|$, from which we are easy to verify $\rho^2(t) \equiv \rho(t)$ and $S \equiv 0$. Tracing over the internal degree of freedom of the atom, we can get the reduced density operator of the external state as follows:

$$\rho_v(t) = |C><C| + |S><S| . \tag{10}$$

The eigenstate of this operator must be of the form

$$|\phi_v> = \phi_c|C> + \phi_s|S> \ . \tag{11}$$

With acting the density operator on this state, for $|\phi_v>$ to be an eigenstate of ρ_v, ϕ_c and ϕ_s must satisfy

$$< C|C> + \frac{\phi_s}{\phi_c} < C|S> = < S|S> + \frac{\phi_c}{\phi_s} < S|C> \ . \tag{12}$$

After some algebraic calculation, we find ϕ_c and ϕ_s satisfying

$$\begin{aligned} \phi_c &= \exp(\pm\theta/2)\exp(i\phi/2) \ , \\ \phi_s &= \pm\exp(\mp\theta/2)\exp(-i\phi/2) \ , \end{aligned} \tag{13}$$

with

$$\begin{aligned} \theta &= \sinh^{-1}(\Omega/2) \ , \\ \Omega &= \frac{1}{|<C|C> |}(<C|C> - <S|S>) \ , \\ <C|S> &= |<C|S>|\exp(i\phi) \ . \end{aligned} \tag{14}$$

Thus the eigenstates and the corresponding eigenvalues of ρ_v can be worked out to be (see Ref.[8] for more details)

$$\begin{aligned} |\phi_v^{\pm}> &= \frac{1}{(2\pi^{\pm}\cosh\theta)^{1/2}}(\mathrm{e}^{(1/2)(i\phi\pm\theta)}|C> \pm \mathrm{e}^{-(1/2)(-i\phi\mp\theta)}|S>) \ , \\ \pi^{\pm} &= <C|C> \pm \mathrm{e}^{\mp\theta}|<C|S>| = <S|S> \pm \mathrm{e}^{\pm\theta}|<C|S>| \ . \end{aligned} \tag{15}$$

These quantities are formally the same as those in Ref.[11], but with different $|C>$ and $|S>$. The entropy of the reduced vibration system is given by

$$S_v = -\pi^+\ln\pi^+ - \pi^-\ln\pi^- \ . \tag{16}$$

When the vibrational state is initially in a coherent state described by $|\alpha>$, after a straightforward calculation, we find these quantities become

$$\begin{aligned} \pi^+ &= \mathrm{e}^{-\bar{n}}\sum_n \frac{\bar{n}^n}{n!}\cos^2\Omega_n t + \exp(-\theta)\mathrm{e}^{-\bar{n}}|\bar{\alpha}\sum_n \frac{\bar{n}^n}{n!}\frac{\cos\Omega_{n+1}t\sin\Omega_n t}{\sqrt{n+1}}| \ , \\ \pi^- &= \mathrm{e}^{-\bar{n}}\sum_n \frac{\bar{n}^n}{n!}\cos^2\Omega_n t - \exp(\theta)\mathrm{e}^{-\bar{n}}|\bar{\alpha}\sum_n \frac{\bar{n}^n}{n!}\frac{\cos\Omega_{n+1}t\sin\Omega_n t}{\sqrt{n+1}}| \ , \end{aligned} \tag{17}$$

with

$$\Omega = \frac{1}{2}\sum_n \frac{\bar{n}^n}{n!}\cos^2\Omega_n t/|\bar{\alpha}\sum_n \frac{\bar{n}^n}{n!}\frac{\cos\Omega_{n+1}t\sin\Omega_n t}{\sqrt{n+1}}| \ , \tag{18}$$

in which

$$\Omega_n = \lambda\sum_{l=0}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\frac{(-\eta^2)^l}{(l+1)!}\sqrt{1+n} \tag{19}$$

is the nonlinear Rabi frequency, the summation comes from the effect of the nonlinear function $f^{(n)}(a^\dagger, a)$

$$f^{(n)}(a^\dagger, a)|n> = \sum_{l=0}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\frac{(-\eta^2)^l}{(l+1)!}|n> \ , \tag{20}$$

where $|n>$ is the Fock state of the vibration mode and $\begin{pmatrix} n \\ l \end{pmatrix}$ means the binomial coefficient.

Comparing these quantities with those of the general JC model, we find that they are formally the same, but with a more complicated Rabi frequencies. Recalling that in the general JC model the Rabi frequency is given by $\Omega_n = \lambda\sqrt{1+n}$ (where λ is the coupling constant of atom with the electromagnetic wave), the frequency in Eq.(19) is also parameterized by the Lamb-Dicke parameter η, which leads to a varying frequency with different η for the same n. Figure 1 depicts the nonlinear Rabi frequency as a function of n for various η. From Fig.1 we find that in the Lamb-Dicke limit ($\eta \ll 1$), the nonlinear frequency is a monotonic function of n, which results from the fact that only the leading term in the summation works when η is small; for large η, huge nonlinearity appears. Thus, it is reasonable to anticipate that within the Lamb-Dicke limit we may get a result similar to that of the general JC model, while the nonlinearity works only when η is not too small.

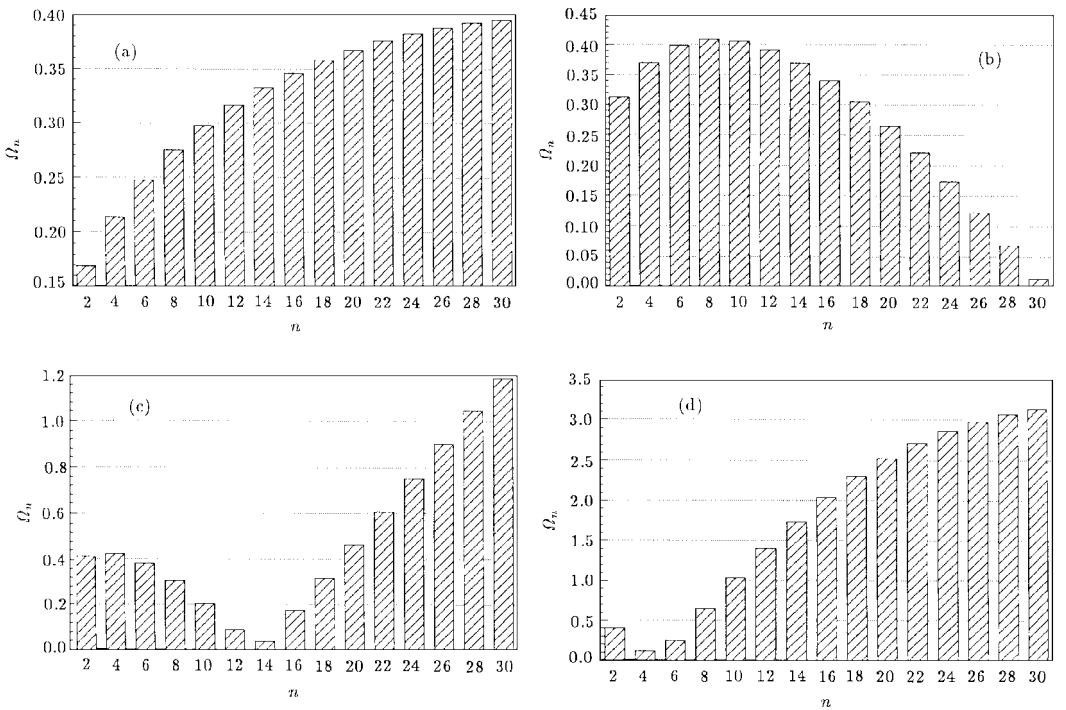

Fig.1. Nonlinear Rabi frequency Ω_n as a function of n for various η: (a) η=0.01; (b) η=0.1; (c) η=0.3; (d) η=0.5.

Due to the close relation between the atomic inversion and the entropy, we also compare the evolution of entropy with atomic inversion, which is given by

$$< \sigma_z >= \mathrm{e}^{-\bar{n}} \sum_n \frac{\bar{n}^n}{n!} \cos 2\,\Omega_n t\,, \tag{21}$$

where Ω_n is the nonlinear Rabi frequency defined by Eq.(19). Physically, the atomic inversion means the occupation probability difference of the atoms lying in the excited and ground electronic states. The collapse and revival phenomenon of atomic inversion is one of

the most attracting prediction of the JC model, which gives an unambiguous configuration of the quantum nature of a coherent field. Contrary to the expectation that the field and the atom would return most closely to their initial states at the peak of the revival, Phoenix and Knight have shown that the atom and the field return most closely to a pure state in the collapse region.[8]

Our numerical study shows that the pure state is reached mostly in the collapse region of the atomic inversion, and in each of the collapse-revival cycle, the entropy would arrive at its local minimum value. The degree to which the atom can be regarded in a pure state at the point also increases when the vibrational energy, embodied by $\bar{n}$, becomes larger. When the Lamb-Dicke parameter is small, i.e., the effect of the nonlinearity is weak, the entropy of the vibration mode is the same as that of the field in general JC model, and sometimes the internal and the external states of the atom would return most closely to a pure state in the collapse region. The nonlinearity works in the case of large η: the revival region of the atomic inversion is elongated, with its collapse and revival phenomenon getting weak, the corresponding minimum value of the entropy becomes less predetermined; however, the most prominent effect of the nonlinearity is that in each collapse-revival cycle, the entropy would arrive at its local minimum value twice: one is the collapse region, the other in the revival region, corresponding to the peak of the atomic inversion. Figures 2–4 plot the atomic inversion and the entropy of the vibration mode for different $\bar{n}$ with η=0.1 and η=0.2 respectively. From Fig.4, we see clearly the appearance of the second local minimum value of the entropy. We also notice that for some special combination of $\bar{n}$ and η, for instance,

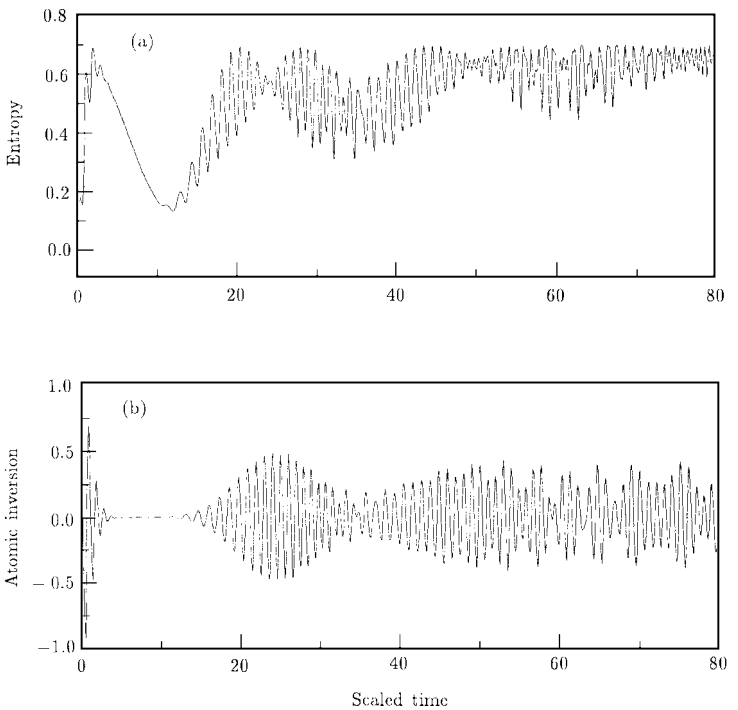

Fig.2. (a) The entropy S_v of the vibration model and (b) the atomic inversion $< \sigma_z >$ versus scaled time $\frac{1}{2}\eta\Omega_L t$ for $\bar{n}$=10 and η=0.1.

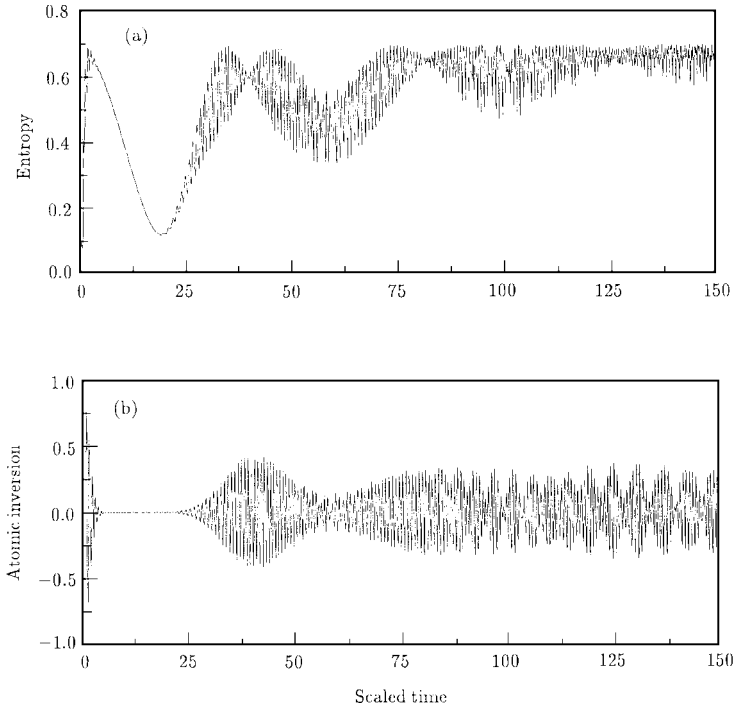

Fig.3. (a) The entropy S_v of the vibration model and (b) the atomic inversion $< \sigma_z >$ versus scaled time $\frac{1}{2}\eta\Omega_L t$ for $\bar{n}$=20 and η=0.1.

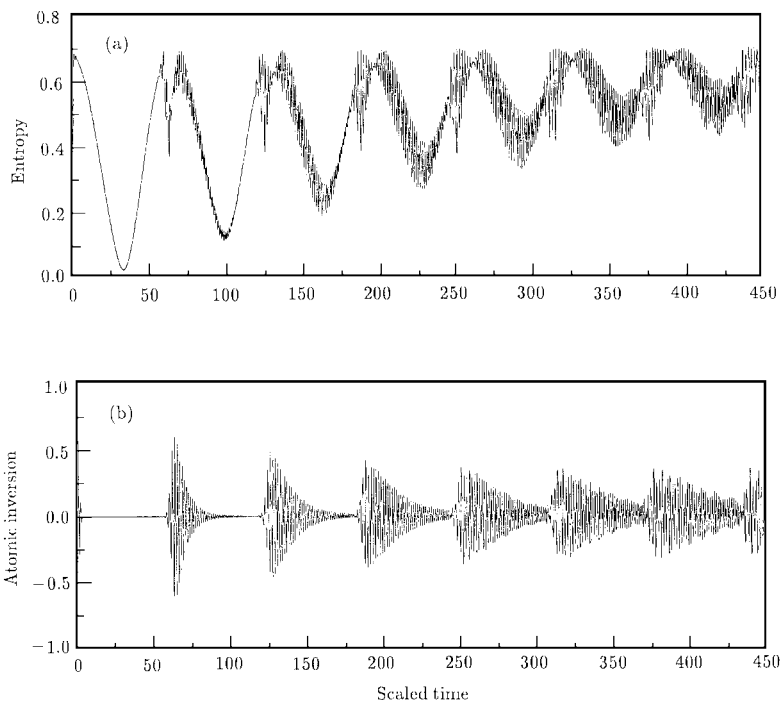

Fig.4. (a) The entropy S_v of the vibration model and (b) the atomic inversion $< \sigma_z >$ versus scaled time $\frac{1}{2}\eta\Omega_L t$ for $\bar{n}$=50 and η=0.2.

$\bar{n}$=20 and η=0.2, the collapse and revival of the atomic inversion is obscure, the entropy loses its local minimum value, thus the correlation between the two degrees of freedom of the atom always keeps in a higher level.

Ⅲ. CONCLUSION

In summary, we have found that the correlation between the external and the internal states is reflected by the entropy of one degree of freedom. Although we have studied the case in which the initial entropy of the total system is zero, the conclusion does not lose its generality. By means of the nonlinear Hamiltonian, we studied the effect of the nonlinearity which comes from the fact that the atomic motion is far beyond the Lamb-Dicke limit. Numerical study shows that the nonlinearity, embodied by the nonlinear Rabi frequency, has various effects, in which the appearance of the second local minimum value in each collapse-revival cycle is the most important phenomenon. The stronger the nonlinearity, the more complicated the behavior of entropy.

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