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DETERMINATION OF THE PERIOD OF A TORSION PENDULUM BY THE FITTING METHOD IN REAL TIME*

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A phenomenological study on the fitting method in real time has been done, and the method was applied to process and analyze the extensive data of a long period torsion pendulum used to determine Newtonian gravitational constant G. The result shows that the fitting method in real time is effective to determine the period of a torsion pendulum with a relative precision of 10^{-5} orders.

PACC: 0230; 0260; 0480

I. INTRODUCTION

In most gravitational experiments in laboratory, such as the determination of Newtonian gravitational constant G[1−4] and the test of weak equivalence principle[5,6], determining the period of a torsion pendulum with a sufficient precision plays a very important role because the accuracy in measuring the period greatly affects that of experimental results. In the last two decades, the most popular method to determine the period is still the method of fast Fourier transform (FFT)[7], and many hardwares (such as the various spectrum analyzers) and solfwares have been developed and widely used in various fields of research and engineering. Another method which has been used frequently is the "all poles" method.[8]

In the FFT method, a power spectrum associated with the sampled data is calculated by FFT.[9] Then the period would be determined by computing the centroid of a fundamental peak of the spectrum. The algorithms and their accompanied programs need little computing power. Besides, the amplitude and the distribution of interferences during the experiments could be seen from the spectrum. However, there are some limitations to the use of FFT method. It is well known that the accuracy of the method is proportional to the number of sinusoidal cycles involved in the sampled data. Since torsion pendulums are usually designed to have long periods (from several minutes to a few hours) for obtaining higher sensitivities, we have to prolong the time spent on experiments to improve the accuracy of the method, thus more interferences will be introduced simultaneously. Secondly, the "window" effect and the "phase" effect lead to a serious deviation of the period in the method. Though

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Goldblum and Ritter[7] have done some beneficial works on this problem, the two effects cannot be completely eliminated. Moreover, the FFT method does not provide information directly (such as the amplitude of the signal of fundamental frequency) and it is still affected by the interferences in time domain.

Another modern method is the "all poles" method in which a straight line is fitted to a set of extremes of the underlying signals by the method of least squares.[8] The period of the torsion pendulum is proportional to the slope of the straight line. The "all poles" method does not have trouble with the "window" and the "phase" effects and the time spent on calculation is also very little. However, there is a difficulty in choosing the time spent on experiments to meet the requirement of the accuracy and to avoid a great deal of interferences at the same time because the accuracy of the "all poles" method also depends on the number of sinusoidal cycles involved in the sampled data. Furthermore, if there are quirks in the sampled data, the deviation of period will rise sharply.

This article reports a phenomenological study on the fitting method in real time. Then the method is applied to analyze extensive data of a torsion pendulum used to determine the gravitational constant G. The result shows that the fitting method in real time is an effective method to accurately determine the period of a torsion pendulum.

Ⅱ. THE FITTING METHOD IN REAL TIME

The fitting method in real time searches for underlying signals by judging the mean square deviation of an ideal function from the position signals of a pendulum. Using $X(t)$ to describe the ideal function, we can express the root-mean-square (rms) σ as

$$\sigma = \sqrt{\sum_{i=0}^{N-1} (X_{\rm i} - X_{\rm i}^{'})^2/(N-1)}, \tag{1}$$

where $X_{\rm i} = X(t_{\rm i})$ is the value of the ideal function, $X_{\rm i}^{'}$ is the corresponding value in the sampled data, N is the total number of points included in the sampled data. In general, the less the σ, the closer the ideal function $X(t)$ is to the underlying signal. If the fundamental frequency is one of the parameters in the ideal function, we can determine the value of the period of a torsion pendulum when rms σ is the least. Since it is difficult for us to give an analytic expression of the calculation, we have to use a computer to find these values in the following steps.

Above all, the ideal function should be chosen according to the practical sitution. If we only use $X(t) = a\cos(\omega t + \phi)$ as the ideal function in which a, ω and ϕ are the parameters to be determined by the fitting method, then we are just taking the signal of fundamental frequency into consideration, and probably ignore some interferences which will greatly affect the accuracy of the method. For example, we use the ideal function $X(t) = a\cos(\omega t + \phi)$ to fit the experiment data (as shown in Fig.1) of our precise torsion pendulum in time domain and get the residual data as shown in Fig.2 Obviously, there are interferences of high-order harmonic frequencies included in this residual curve. As will be seen, this kind of interferences has an important effect on the accuracy in determining the periods. Therefore, interferences of high-order harmonic frequencies should be present in the ideal function with

suitable parameters. Taking $X(t) = a\cos(\omega t + \phi) + a_2\cos(2\omega t + \phi_2) + a_3\cos(3\omega t + \phi_3)$ as the ideal function, we fit the same data again and get the residual data as shown in Fig.3. It is clear that the deviation of the residual data in Fig.3 is smaller than that in Fig.2. The accuracy in determining the periods has also been improved. In fact, we usually take $X(t) = a\cos(\omega t + \phi)$ as the ideal function at first and then analyze the residual data after subtracting the ideal function from the raw data. Based on the analysis, we change the ideal function until the major factors which affect the accuracy of the periods have been taken into consideration. On the other hand, the number of parameters in the ideal function should not be too many, otherwise it will take us a very long time to obtain the result with a personal computer. For example, if there are seven parameters in the ideal function and 20000 points in the sampled data, the time spent on calculation may be nearly four days with a 586 computer. We should make the number of parameters as small as possible, so long as the accuracy of the periods meets our requirement.

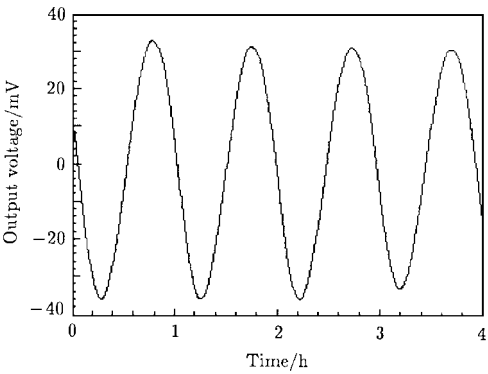

Fig.1. Output voltage curve of a free oscillating torsion pendulum without processing.

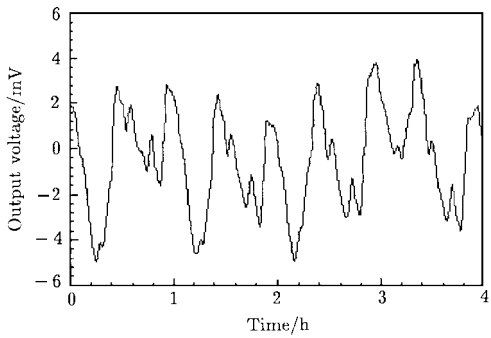

Fig.2. The residual curve after subtracting the ideal function $X(t) = a\cos(\omega t + \phi)$ from raw data of the torsion pendulum.

Thus, a suitable region for the variation of parameters in the ideal function should be given. First, the region must involve the accurate values of the parameters, otherwise there will be departure from the correct result by the fitting method in real time. For instance, the amplitude of the signal of fundamental frequency can be read from the analogous curve of the sampled data on a chart recorder and the period can also be estimated by averaging long time interval which includes several periods of the torsion pendulum, and then the region should contain these estimated values of the parameters concerned. Second, it is also difficult to choose the size of the region. If the size is too large, there might be several minimums for rms σ in which we can hardly distinguish them. Figure 4 shows an ordinary relationship of σ with the fundamental frequency. If and only if we take the size of fundamental frequency from f_a to f_b, we can easily find the minimum value of σ. On the other hand, if the size is too small, the accurate values of the parameters may be lost especially that of the period. In our case, we choose the different sizes for the parameters in the experiments and take the estimated values of the parameters as the centers of the region. Particularly, it should be made sure that the sizes of the different parameters do not have common part, otherwise

the fitting method will give a wrong result.

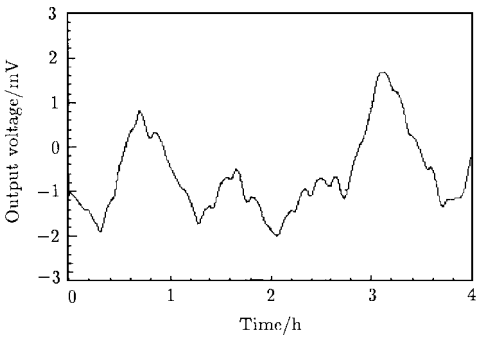

Fig.3. The residual curve after subtracting the ideal function $X(t) = a\cos(\omega t + \phi) + a_2\cos(2\omega t + \phi_2) + a_3\cos(3\omega t + \phi_3)$ from raw data of the torsion pendulum.

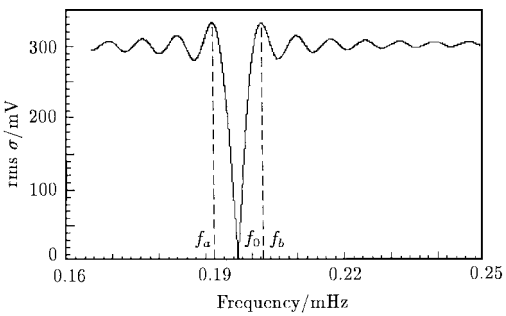

Fig.4. An ordinary relationship of rms σ with the fundamental frequency. The parameter f_0 is the accurate vaule of the fundamental frequency.

Having determined the sizes of the parameters, we divide them into small even parts and compute σ at each crossing point. Around the crossing point whose σ is the least, new sizes of the parameters are taken which are smaller than that in the last step and then repeat the above process until rms σ comes near a stable extreme value. At this time, the values of the parameters are what we search for. Theoretically, the larger the number of division, the more accurate the result, and the longer time it takes to come near the stable extreme value. Dealing with the same data, we found that the computation time of five divisions will be twice as long as that of three divisions under the same accuracy requirement. Therefore we take three divisions in our data process, and it has been proved to be an effective division.

Ⅲ. EFFECTS OF THE INTERFERENCES

The fitting method in real time suffers various interferences such as a monotonic drift, the interferences of high-order harmonic frequencies, the phase effect, etc.

A. Deviation due to monotonic drift

Due to the secular creep of the fiber, there is always a monotonic base-line drift in the sampled data. Regarding the drift as a linear change, we can use a straight line $kt + b$ to express it, where k is the rate of drift and b is the initial equilibrium position. A group of simulation experiments have been carried out to test the effect of such a drift on a sinusoid curve. The function $a_0\cos(\omega_0 t+\phi_0)+kt+b$ is used to generate a data-series by the computer and then the ideal function $a\cos(\omega t + \phi)$ is used to fit this data-series in time domain. Similar to that in our experiments, the period of the torsion pendulum is either 3560.0 s or 4770.0 s and the corresponding ω_0 is either 0.001765 rad/s or 0.001317 rad/s; the value of a_0 is $200.0 \le a_0 \le 500.0$, that of ϕ_0 is $0 \le \phi_0 \le 2\pi$; the value of k is $1.0\times10^{-7} \le k \le 1.0\times10^{-3}$ while that of b is $0.0 \le b \le 50.0$. Table 1 shows the results of the fitting method in real time. From Table 1, we find that the monotonic drift has an obvious effect on the fitting method. Therefore it is necessary to take the drift into consideration. In practice, the drift

is eliminated from the sampled data by the least-square method at first in order to reduce the number of the parameters in the ideal function. The relative deviation of the results of the fitting method using the modified data is then reduced to less than 1 ppm.

Table 1. The effect of monotonic drift.

	Fundamental period of computer-generated data-series/s	Period obtained by the fitting method in real time without considering monotonic drift/s	Maximum relative deviation
Croup 1	3560.000	3558.677	3.7×10^{-4}
Group 2	4770.000	4767.864	4.5×10^{-4}

★ Computer-generated data with monotonic drift. Each group has ten separate sets of data-series, whose fundamental period is either 3560.0 or 4770.0 s; their phase ϕ_0, amplitude a_0, rate of drift k and initial position of equilibrium b are randomly selected in the value mentioned in the text. Sampling interval of each data-series is 1 s and there are twenty sinusoidal cycles in each data-series.

B. Deviation due to interferences of high frequencies

Interferences of high frequencies are the most common ones in our experiments. The fluctuation of the power supply, the shot noise in the circuits of the amplifier and recording system, and the vibrations of the ground are all sources of these kinds of interferences. Although we can cut down most of them by experimental techniques, such as the mechanical isolator, the analogous filter and the digital filter, etc., the effects of the interferences of the high frequencies on the fitting method in real time still need be studied.

The function by which we generated the data-series for simulation experiments is $X(t) = a_0 \cos(\omega_0 t + \phi_0) + a_1 \cos(\omega_1 t + \phi_1)$, where ω_0, a_0, and ϕ_0 are the parameters of the signal of fundamental frequency; ω_1, a_1, and ϕ_1 are those of the high frequencies. The choice of ω_0, a_0, and ϕ_0 is the same as that in the last simulation experiment while the values of ω_1, a_1 and ϕ_1 are changed to $0.02\pi \leq \omega_1 \leq 2\pi$, $1.0 \leq a_1 \leq 50.0$, and $0 \leq \phi_1 \leq \pi$, respectively. The ideal function is still $X(t) = a\cos(\omega t + \phi)$. The relative deviation of the period is less than one part in 10^7. This means that the interferences of high frequencies have little effects on the fitting method. Hence the parameters of interferences of high frequencies need not be taken into account in the ideal function. Of course, we can remove most of the interferences of high frequencies away from the sampled data by the digital filter before the data is used in the fitting method.

C. Deviation due to the interferences of low frequencies

Different from the interferences of high frequencies, it is difficult to eliminate those of low frequencies because the frequency of underlying signal is also very low. Therefore we must study the dependence of the accuracy of the fitting method in real time on the interferences of low frequencies.

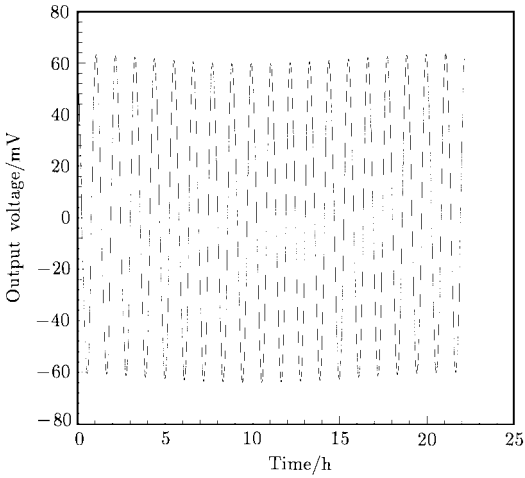

Fig.5. An experiment data of the torsion pendulum.

The function that we use to generate the data by the computer is the same as that in the last simulation experiment and most of the parameters are kept unchanged. The value of ω_1 is changed to $7.270 \times 10^{-5} \leq \omega_1 \leq 5.816 \times 10^{-4}$, which is corresponding to very low frequencies. Taking $X(t) = a\cos(\omega t + \phi)$ as the ideal function in the fitting method, we find that the relative deviation of the period is in the order of 10^{-5}. Clearly, the parameters to fit the interferences of low frequencies must be taken into account in the ideal function. The residual curve after subtracting the ideal function $X(t) = a\cos(\omega t + \phi) + a_1\cos(\omega_1 t + \phi_1)$ from the data shown in Fig.5 (as shown in Fig.6) is smaller than that after subtracting the ideal function $X(t) = a\cos(\omega t + \phi)$ (as shown in Fig.7). The corresponding rms σ is reduced from 2.032 mV to 1.311 mV. Therefore, if there is any obvious interference of low frequency in the sampled data, taking $X(t) = a\cos(\omega t + \phi) + a_1\cos(\omega_1 t + \phi_1)$ as the ideal function is proper.

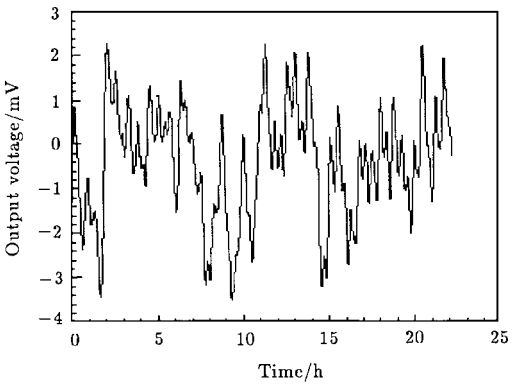

Fig.6. The residual curve after subtracting the ideal function $X(t) = a\cos(\omega t + \phi) + a_1\cos(\omega_1 t + \phi_1)$ from the experimental data of the torsion pendulum.

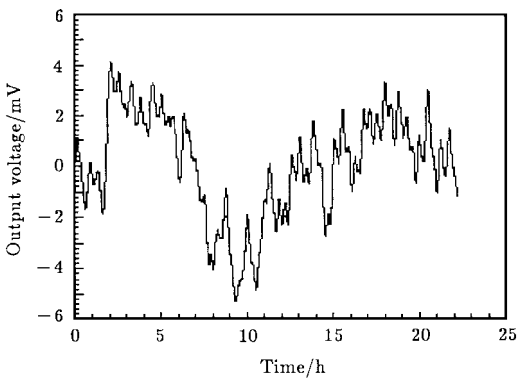

Fig.7. The residual curve after subtracting the ideal function $X(t) = a\cos(\omega t + \phi)$ from the experimental data of the torsion pendulum.

D. Deviation due to the high-order harmonic frequencies

In the experiments of determining the gravitational constant G with the swing time method, the change of period of the torsion pendulum at different configurations of attracting masses is detected.[1−4] Usually, the gravitational fields produced by the attracting masses and the surrounding masses are inhomogenous and asymmetrical. The asymmetry and inhomogeneity of gravitational fields will lead to a systematic error in the experiment.

Because the torsion pendulum used in our experiment is non-symmetrical, there will be a nonlinear effect in the movement of the torsion pendulum, and the interferences of high-order harmonic frequencies cannot be neglected in the sampled data.

The simulation experiment is similar to that of testing the effects of interferences of high frequencies. The only difference is that ω_1, a_1, and ϕ_1 are the parameters for high-order harmonic frequencies. Supposing the interference of triple-frequency is the main one, we set $\omega_1 = 3\omega_0$ in the ideal function. The result is shown in Table 2.

Table 2. The effect of interferences of high-order harmonic frequencies.

	Fundamental period of computer-generated data-series/s	Period obtained by the fitting method in real time without considering high-order harmonic frequencies/s	Relative deviation
Croup 1	3560.0000	3559.9186	2.0×10^{-5}
Group 2	4770.0000	4769.9981	5.0×10^{-6}

★Computer-generated data with interferences of high-order harmonic frequencies. These data are similar to that in the last simulation experiment. The values of ω_1, a_1 and ϕ_1 are $1.342 \times 10^{-4} \leq \omega_1 \leq 5.817 \times 10^{-4}$, $1.0 \leq a_1 \leq 50.0$, and $0 \leq \phi_1 \leq \pi$, respectively.

Therefore, as shown in Table 2, the interferences of high-order harmonic frequencies greatly affect the accuracy of the periods. This means that we should add the parameters of high-order harmonic frequencies in the ideal function to improve the accuracy of the measurement of periods.

E. Deviation due to the starting phase

In FFT method, the starting phase of the sampled data has an important effect on the accuracy of the periods. This is the so-called "phase effect" and can hardly be removed away. This problem does not exist in the fitting method in real time. A group of simulation experiments with the fitting method in real time have proved that the deviation with different starting phases is less than one part in 10^9. The reason in that the staring phase ϕ_0 has been a parameter to be determined in the ideal function.

IV. DISCUSSION

In our recent experiment of measuring gravitational constant G with swing time method, we have used the fitting method in real time to determine the periods of the torsion pendulum in different gravitational fields. Considering the nonlinear effect of the pendulum as the main interferences, we have taken $X(t) = a\cos(\omega t + \phi) + a_2\cos(2\omega t + \phi_2) + a_3\cos(3\omega t + \phi_3)$ as the ideal function to fit the raw data. The residual curve has been partly shown in Fig.3, and the accuracy of determining the period of the pendulum is about 75 ppm.[4]

Besides, the accuracy of the fitting method in real time depends on the number of points involved in the sampled data. In order to improve the accuracy of the method, we can increase the frequency of sample and decrease the time spent on experiments. Although the fitting method may spend a lot of time on data processing, it does not introduce any external

noise to affect the accuracy of the experimental result, and the time spent on calculation will be greatly reduced with the improvement of the personal computer's power. Therefore, the fitting method in real time will be more convenient and practical in determining the period of a torsion pendulum in gravitational experiments.

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