Volume 7, Number 1
1004-423X/98/08010001-11

ACTA PHYSICA SINICA
(Overseas Edition)

January, 1998
©1998 Chin. Phys. Soc.

LIGHT TRACKS IN THE ANISOTROPIC OPTICAL FIBER WITH TWO TYPES OF PARABOLIC REFRACTIVE INDICES*

SHEN WEN-DA(沈文达), MAO LI-YU(毛立宇), and GAO WEI(高 炜)
Department of Physics, Shanghai University, Shanghai 201800, China
ZHU SHI-TONG(朱时通)
Shanghai Institute of Optics and Fine Mechanics, Academia Sinica, Shanghai 201800, China

(Received 4 February 1997; Revised manuscript received 2 June 1997)

The light tracks in an anisotropic optical fiber are studied by the optical metric model. The ordinary and extraordinary light tracks are determined by solving the null geodesic equation. In the paraxial approximation the birefringence of the fiber is analyzed. The focusing and defocusing characteristics of a light beam are described via the geodesic deviation equation

PACC: 4220; 0440

I. INTRODUCTION

Recently, we have successfully studied the optical transmission and light-matter interaction in terms of the optical metric model.[1-8] But the research is limited to an isotropic medium. There is a great need for us to generalize the optical metric model to the anisotropic medium. In this paper we will first give the optical metric which is suitable to an anisotropic medium and then examine the ordinary and extraordinary light tracks in an anisotropic optical fiber with two types of parabolic refractive indices by solving the null geodesic equation. In the paraxial approximation the birefringence of the fiber is analyzed. The focusing and defoc characteristics of a light beam are described via the geodesic dev equation.

II. RELATION OF THE OPTICAL METRIC TO THE REFRACTIVE INDEX

The line element associated with the optical metric in a moving isotropic medium has the following form.[9]

$$\begin{aligned} d\sigma^2 &= \gamma_{\lambda\rho}dx^\lambda - dx^\rho \\ &= g_{\lambda\rho}dx^\lambda dx^\rho + \left(1+\frac{1}{\varepsilon\mu}\right)u_\lambda u_\rho dx^\lambda dx^\rho \\ &= d\tau^2 + \left(1-\frac{1}{\varepsilon\mu}\right)u_\lambda u_\rho dx^\lambda dx^\rho, \end{aligned} \tag{1}$$

where $\gamma_{\lambda\rho}$ is the optical metric, $g_{\lambda\rho}$ is the gravitational metric, ε is the dielectric constant, μ is the magnetic permeability, and u_λ is the four-dimensional velocity of the

* Project supported by the National Hi-Tech ICF Foundation of China and the National Natural Science Foundation of China.

medium relative to the laboratory reference frame. Any Greek index, like ρ in Eq.(1) that appears twice, once as a subscript and once as a superscript, is understood to be summed over the values 1,2,3,0.

In the case of a static non-magnetic isotropic medium with the negligible gravitational field we have

$$g_{\lambda\rho} = \eta_{\lambda\rho} = \begin{cases} +1 & \lambda = \rho = 1,2,3, \\ -1 & \lambda = \rho = 0, \\ 0 & \lambda \neq \rho, \end{cases}$$

$$u^0 = dx^0/\sqrt{-dx^2} = 1, \qquad u^1 = u^2 = u^3 = 0,$$

$$\mu = 1,$$

where $\eta_{\lambda\rho}$ is the Minkowskian metric. Equation (1) thus reduces to

$$d\sigma^2 = -\frac{1}{\varepsilon}(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{2}$$

and Eq.(2) can be written as

$$ds^2 = \varepsilon d\sigma^2 = -(dx^0)^2 + \varepsilon_{ij}dx^i dx^j, \tag{3}$$

where the repeated Latin indices i and j are summed over the values 1, 2, 3, and $\varepsilon_{ij} = \varepsilon\delta_{ij}$.

Now we generalize the optical metric model mentioned above to the case of an anisotropic medium. For a static non-magnetic anisotropic medium with the negligible gravitational field, ds^2 is assumed to take the same form as Eq.(3), and the dielectric constant ε becomes the dielectric tensor $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}.$$

In the system of principal dielectric axes the dielectric tensor $\boldsymbol{\varepsilon}$ has the form

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix},$$

where ε_1, ε_2 and ε_3 are called the principal dielectric constants (or principal permittivities). ε_1, ε_2 and ε_3 are unequal to each other for a biaxial crystal. and $\varepsilon_1 = \varepsilon_2 = \varepsilon_\perp$, $\varepsilon_3 = \varepsilon_{/\!/}$ and $\varepsilon_\perp \neq \varepsilon_{/\!/}$ for a uniaxial crystal.

As a specific case we examine the light transmission in a uniaxial crystal optical fiber with cylindrical symmetry. Then we take $x^0 = ct$, $x^1 = r\cos\phi$, $x^2 = r\sin\phi$, $x^3 = z$, and the line element can be rewritten as

$$d\sigma^2 = -\frac{1}{n^2}(cdt)^2 + (dr)^2 + (rd\phi)^2 + \frac{n_3^2}{n^2}(dz)^2, \tag{4}$$

where r, ϕ and z are the radical, azimuthal and axial coordinates, respectively. The principal refractive indices are $n^2 = \varepsilon_2$ and $n_3^2 = \varepsilon_{/\!/}$, and c is the velocity of light. The optical metric $\gamma_{\lambda\rho}$ has the following nonvanishing components

$$\begin{aligned}
&\gamma_{00} = -\frac{c^2}{n^2}, \quad \gamma^{00} = -\frac{n^2}{c^2};\\
&\gamma_{11} = 1, \qquad \gamma^{11} = 1;\\
&\gamma_{22} = r^2, \qquad \gamma^{22} = \frac{1}{r^2};\\
&\gamma_{33} = \frac{n_3^2}{n^2}, \qquad \gamma^{33} = \frac{n^2}{n_3^2}.
\end{aligned} \tag{5}$$

The refractive indices $n(t, r, \phi, z)$ and $n_3(t, r, \phi, z)$ are related to the metric coefficients γ_{33} and γ^{00} by

$$n^2 = -c^2\gamma^{00} \quad \text{and} \quad n_3^2 = -c^2\gamma_{33}\gamma^{00}.$$

III. LIGHT TRACKS AND NULL GEODESICS

The light tracks can be determined from Fermat's principle. In accordance with the general relativistic version of Fermat's principle, the light tracks are just the null geodesics.[10] Then the light tracks can be found by solving the null geodesic equation

$$\frac{d^2x^\mu}{dq^2} + \Gamma^\mu_{\lambda\gamma}\frac{dx^\lambda}{dq}\frac{dx^\gamma}{dq} = 0, \tag{6}$$

where q is the parameter describing the light track, and the affine connection $\Gamma^\mu_{\lambda\gamma}$ is related to the optical metric $\gamma_{\lambda\rho}$ by

$$\Gamma^\mu_{\lambda\gamma} = \frac{1}{2}\gamma^{\mu\rho}\left[\frac{\partial\gamma_{\rho\lambda}}{\partial x^\gamma} + \frac{\partial\gamma_{\rho\gamma}}{\partial x^\lambda} - \frac{\partial\gamma_{\lambda\gamma}}{\partial x^\rho}\right]. \tag{7}$$

By substituting Eq. (5) into Eq. (7), the nonvanishing components of $\Gamma^\mu_{\lambda\gamma}$ are obtained

$$\begin{aligned}
&\Gamma^0_{00} = -\frac{n_{,t}}{n}, \qquad \Gamma^0_{01} = \Gamma^0_{10} = -\frac{n_{,r}}{n},\\
&\Gamma^0_{02} = \Gamma^0_{20} = -\frac{n_{,\phi}}{n}, \qquad \Gamma^0_{03} = \Gamma^0_{30} = -\frac{n_{,z}}{n},\\
&\Gamma^0_{33} = \frac{1}{c^2}n_3 n_{3,t} - \frac{n_3^2}{c^2 n}n_{,t}\\
&\Gamma^1_{00} = -\frac{c^2}{n^3}n_{,r}, \qquad \Gamma^1_{22} = -r,\\
&\Gamma^1_{33} = \frac{n_3^2}{n^3}n_{,r} - \frac{n_3}{n^2}n_{3,r},\\
&\Gamma^2_{00} = -\frac{c^2}{n^3r^2}n_{,\phi}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\\
&\Gamma^2_{33} = -\frac{n^3}{r^2n^2}n_{3,\phi} + \frac{n_3^2}{n^3r^2}n_{,\phi},\\
&\Gamma^3_{00} = -\frac{c^2}{n_3^2 n}n_{,z}, \qquad \Gamma^3_{03} = \Gamma^3_{30} = \frac{n_{3,t}}{n_3} - \frac{n_{,t}}{n},\\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{n_{3,r}}{n_3} - \frac{n_{,r}}{n}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{n_{3,\phi}}{n_3} - \frac{n_{,\phi}}{n},
\end{aligned} \tag{8}$$

$$\Gamma^3_{33} = \frac{n_{3,z}}{n_3} - \frac{n_{,z}}{n},$$

where $n_{,t} = \frac{\partial n}{\partial t}$, $n_{,r} = \frac{\partial n}{\partial r}$, $n_{,\phi} = \frac{\partial n}{\partial \phi}$, $n_z = \frac{\partial n}{\partial z}$.

By substituting Eq. (8) into Eq. (6), the geodesic equation becomes

$$\ddot{t} - \frac{1}{n}n_{,t}\dot{t}^2 - 2\frac{1}{n}n_{,r}\dot{t}\dot{r} - 2\frac{1}{n_{,\phi}}\dot{t}\dot{\phi} - 2\frac{1}{n}n_{,z}\dot{t}\dot{z} + \left(\frac{1}{c^2}n_3 n_{3,t} - \frac{n_3^2}{c^2 n}n_{,t}\right)\dot{z}^2 = 0, \quad (9)$$

$$\ddot{r} - \frac{c^2}{n^3}n_{,r}\dot{t}^2 - r\dot{\phi}^2 + \left(\frac{n_3^2}{n^3}n_{,r} - \frac{n_3}{n^2}n_{3,r}\right)\dot{z}^2 = 0, \quad (10)$$

$$\ddot{\phi} - \frac{c^2}{n^3 r^2}n_{,\phi}\dot{t}^2 + \frac{2}{r}\dot{r}\dot{\phi} + \left(\frac{n_3^2}{n^3}n_{3,\phi} - \frac{n_3}{r^2 n^2}n_{3,\phi}\right)\dot{z}^2 = 0, \quad (11)$$

$$\ddot{z} - \frac{c^2}{n_3^2 n}n_{,z}\dot{t}^2 + 2\left(\frac{n_{3,t}}{n_3} - \frac{n_{,t}}{n}\right)\dot{t}\dot{z} + 2\left(\frac{n_{3,r}}{n_3} - \frac{n_{,r}}{n}\right)\dot{r}\dot{z} + 2\left(\frac{n_{3,\phi}}{n_3} - \frac{n_{,\phi}}{n}\right)\dot{\phi}\dot{z}$$
$$+ \left(\frac{n_{3,z}}{n_3} - \frac{n_{,z}}{n}\right)\dot{z}^2 = 0. \quad (12)$$

where the dot "." denotes the differentiation with respect to q. We know that the light tracks are just the null geodesics $d\sigma^2 = 0$. We have

$$-\frac{c^2}{n^2}\dot{t}^2 + \dot{r}^2 + r^2\dot{\phi}^2 + \left(\frac{n_3}{n}\right)^2\dot{z}^2 = 0. \quad (13)$$

Then the light tracks can be found by solving Eqs. (9) – (12) which determine the light tracks in a medium with the general refractive indices $n(t, r, \phi, z)$ and $n_3(t, r, \phi, z)$. Now we try to exactly solve Eqs. (9) – (12) for the optical fiber with the radial-dependent refractive indices $n(r)$ and $n_3(r)$. Equations (9) – (12) are reduced to

$$\ddot{t} - 2\frac{n_{,r}}{n}\dot{t}\dot{r} = 0, \quad (14)$$

$$\ddot{r} - \frac{c^2}{n^3}n_{,r}\dot{t}^2 - r\dot{\phi}^2 + \left(\frac{n_3^2}{n^3}n_{,r} - \frac{n_3}{n^2}n_{3,r}\right)\dot{z}^2 = 0, \quad (15)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0, \quad (16)$$

$$\ddot{z} + 2\left(\frac{n_{3,r}}{n_3} - \frac{n_{,r}}{n}\right)\dot{r}\dot{z} = 0. \quad (17)$$

Solving Eqs. (14), (16)and(17), we obtain

$$\dot{t} = \frac{n^2}{c^2}\omega, \quad (18)$$

$$\dot{\phi} = \frac{J}{r^2}, \quad (19)$$

$$\dot{z} = k_0\left(\frac{n}{n_3}\right)^2, \quad (20)$$

where k_0, J and ω are the integration constants.

Inserting Eqs. (18) – (20) into Eq. (13) we can get

$$\dot{r} = \pm \left[\frac{n^2}{c^2}\omega^2 - \frac{J^2}{r^2} - (k_0)^2 \frac{n^2}{n_3^2} \right]^{\frac{1}{2}}. \tag{21}$$

From Eqs. (20) and (21) the orbit equation for the light beam propagating in an inhomogeneous medium with refractive indices $n(r)$ and $n_3(r)$ is derived

$$\frac{dz}{dr} = \pm \left[\frac{n_3^4 \omega^2}{n^2 c^2 k_0^2} - \frac{J}{r^2}\left(\frac{n_3^4}{k_0^2 n^4} \right) - \left(\frac{n_3^2}{n^2} \right) \right]^{-\frac{1}{2}}. \tag{22}$$

IV. ANALYTICAL SOLUTION FOR THE ANISOTROPIC OPTICAL FIBER WITH REFRACTIVE INDICES $n(r)$ AND $n_3(r)$

In order to obtain the exact dependence of r on z and compare it with the results given in the paraxial approximation, $n(r)$ and $n_3(r)$ are assumed to be of parabolic types, i.e.,

$$n = n_0\left(1 - \frac{r^2}{2l^2}\right), \quad n_3 = n_{30}\left(1 - \frac{r^2}{2l_3^2}\right), \tag{23}$$

where n_0 and n_{30} represent the refractive indices at $r = 0$. When l and l_3 are real, expression(23) corresponds to the first medium discussed in Ref. [8]. The refractive indices decrease with r. When l and l_3 are imaginary numbers($l = is$, $l_3 = is_3$ and s, s_3 are real), Eq. (23) can be written as

$$n = n_0\left(1 + \frac{r^2}{2s^2}\right), \qquad n_3 = n_{30}\left(1 + \frac{r^2}{2s_3^2}\right). \tag{24}$$

Expression (24) corresponds to the second medium given in Ref. [8], in which the refractive indices increase with r. In Eqs. (23) and (24) l and s are the factors governing the variation of $n(r)$ and $n_3(r)$ with r.

Substituting Eq. (23) into Eq. (22) and then integrating the obtained equation, we have

$$z = \pm (z_1 + z_2), \tag{25}$$

$$z_1 = \left(\frac{n_0 l_3^2}{n_{30} l^2} \right) \sqrt{\frac{2}{A(A+1)}}\, l_3 [F(\theta, k) - F(\theta_1, k)], \tag{26}$$

$$z_2 = \frac{n_0}{n_{30}}\left(1 - \frac{l_3^2}{l^2}\right) \sqrt{\frac{2}{A(A+1)}}\, l_3 [\Pi(\theta, V, k) - \Pi(\theta_1, V, k)], \tag{27}$$

$$A = \frac{\omega n_{30}}{c k_0}, \tag{28}$$

$$k^2 = \frac{A-1}{A+1}, \tag{29}$$

$$\theta = \arcsin\left(\sqrt{\frac{A}{2(A-1)}}\, \frac{r}{l_3} \right), \tag{30}$$

$$V = \frac{\left(-\frac{r^2}{2l_3^2}\right)}{\left(\sqrt{\frac{A}{2(A-1)}}\,\frac{r}{l_3}\right)^2} = \frac{1-A}{A}, \tag{31}$$

$$F(\theta, k) = \int_0^{\theta} \frac{\mathrm{d}\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}}, \tag{32}$$

$$\Pi(\theta, V, k) = \int_0^{\theta} \frac{\mathrm{d}\theta}{(1+V\sin^2\theta)(1-k^2\sin^2\theta)^{\frac{1}{2}}}. \tag{33}$$

Substituting Eq. (24) into Eq. (22) and integrating the obtained equation, we have

$$z = Z_1 + Z_2, \tag{34}$$

$$Z_1 = \frac{n_0}{n_{30}}\frac{s_3^2}{s^2}\sqrt{\frac{2}{A(A+1)}}\,is_3[F(\beta, k) - F(\beta_2, k)], \tag{35}$$

$$Z_2 = \frac{n_0}{n_{30}}\left(1-\frac{s_3^2}{s^2}\right)\sqrt{\frac{A}{A(A+1)}}\,is_3[\Pi(\beta, V, k) - \Pi(\beta_2, V, k)], \tag{36}$$

$$\beta = \arcsin\left(\sqrt{\frac{A}{2(A-1)}}\,\frac{r}{is_3}\right). \tag{37}$$

V. EXTRAORDINARY AND ORDINARY LIGHT RAYS IN THE PARAXIAL APPROXIMATION

To compare expressions (25) and (34) with the results obtained in the paraxial approximation, we write the elliptical integrals in the series form[11]

$$F(\theta, k) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!}(k)^{2n}2^{-2n}\left[\binom{2n}{n}\theta + \sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\sin(2m\theta)}{m}\right], \tag{38}$$

$$\Pi(\theta, V, k) = \sum_{n=0}^{\infty}(-V)^n\sum_{m=0}^{n}\binom{-\frac{1}{2}}{m}\left(\frac{k^2}{V}\right)^m 2^{-2n}\left[\binom{2n}{n}\theta + \sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\sin(2m\theta)}{m}\right], \tag{39}$$

$$|k| < 1, \quad |V| < 1.$$

In the paraxial approximation, $A \approx 1$ and $k^2 \approx 0$. Only the term with $m = 0$ remains in these series, then we have

$$F(\theta, k) = \theta,$$

$$\Pi(\theta, V, k) = \binom{-\frac{1}{2}}{0}\theta = -\frac{1}{2}\theta.$$

We first consider the light tracks in the first medium. Then Eq. (25) becomes

$$\frac{z}{l_3} = \pm M_1(\theta - \theta_1) = \pm M_1\left(\arcsin\left(\sqrt{\frac{1}{2(A-1)}}\,\frac{r}{l_3}\right) - \theta_1\right), \tag{40}$$

with

$$M_1 = -\frac{1}{2}\frac{n_0}{n_{30}}\left(1 - \frac{3l_3^2}{l^2}\right); \tag{41}$$

and in turn we have

$$r = \pm\sqrt{2(A-1)}\,l_3\sin\left(\frac{z}{l_3M_1} \pm \theta_1\right), \tag{42}$$

$$r' = \frac{dr}{dz} = \pm\frac{\sqrt{2(A-1)}}{M_1}\cos\left(\frac{z}{l_3M_1} \pm \theta_1\right), \tag{43}$$

$$r_1 = r\big|_{z=0} = \sqrt{2(A-1)}\,l_3\sin\theta_1, \tag{44}$$

$$r_1' = r'\big|_{z=0} = \pm\frac{\sqrt{2(A-1)}}{M_1}\cos\theta_1, \tag{45}$$

where the prime "′" denotes the differentiation with respect to z; r_1 and r_1' are the position and slope of the light ray at $z=0$, respectively. Equation (42) can be written as

$$r_e = r_1\cos\frac{z}{M_1l_3} + r_1'M_1l_3\sin\frac{z}{M_1l_3}. \tag{46}$$

For the extraordinary light from Eqs. (44) and (45) we obtain

$$A_e = 1 + \frac{1}{2}\left(\frac{r_1}{l_3}\right)^2 + \frac{1}{2}(M_1r_1')^2, \tag{47}$$

$$\mathrm{tg}\,\theta_{1e} = \pm\frac{r_1}{l_3M_1r_1'}. \tag{48}$$

It is easier to derive the ordinary light track in the uniaxial-crystal optical fiber with the refractive indices n and n_3. In this case $n = n_3$ and then $M_1 = 1$, so Eqs. (46), (47) and (48) reduce

$$r_o = r_1\cos\frac{z}{l} + r_1'l\sin\frac{z}{l}, \tag{49}$$

$$A_o = 1 + \frac{1}{2}\left(\frac{r_1}{l}\right)^2 + \frac{1}{2}(r_1')^2, \tag{50}$$

$$\mathrm{tg}\theta_{1o} = \pm\frac{r_1}{lr_1'}. \tag{51}$$

Now we examine the extraordinary light track in the second medium. As seen, Eqs. (35) – (37) have the same forms as Eqs. (26) – (30) except that is, is_3, β and β_2 replace l, l_3, θ and θ_1, respectively. By the replacement, Eqs. (46), (47) and (48) become

$$r_e = r_2\cosh\frac{z}{M_2s_3} + r_2'M_2s_3\sinh\frac{z}{M_2s_3}, \tag{52}$$

$$A_e = 1 - \frac{1}{2}\left(\frac{r_2}{s_3}\right) + \frac{1}{2}(M_2r_2')^2, \tag{53}$$

$$\mathrm{tg}\beta_{2e} = \pm\frac{r_2}{is_3M_2r_2'}, \tag{54}$$

with

$$M_2 = -\frac{1}{2}\frac{n_0}{n_{30}}\left(1 - \frac{3s_3^2}{s^2}\right), \tag{55}$$

$$r_2 = r'_{z=0} = \sqrt{2(A-1)}is_3\sin\beta_2, \tag{56}$$

$$r'_2 = r'|_{z=0} = \pm\frac{\sqrt{2(A-1)}}{M_2}\cos\beta_2. \tag{57}$$

For the ordinary light track, $M_2 = 1$ and we have

$$r_o = r_2\cosh\frac{z}{s} + r'_2 s\sinh\frac{z}{s}, \tag{58}$$

$$A_o = 1 - \frac{1}{2}\left(\frac{r_2}{s}\right)^2 + \frac{1}{2}(r'_2)^2, \tag{59}$$

$$tg\beta_{2o} = \pm\frac{r_2}{isr'_2}. \tag{60}$$

Here r_2 and r'_2 are the position and slope of the light ray at $z=0$ for the second medium. As is seen from Eqs. (47) and (50) or Eqs. (53) and (59), the condition $A\simeq 1$ demands that $(r_1/l_3)^2 \ll 1$ and $(M_1 r'_1)^2 \ll 1$ or $(r_2/s_3)^2 \ll 1$ and $(M_2 r'_2)^2 \ll 1$. This shows that both the extraordinary and the ordinary light rays are just paraxial and the inhomogeneity is very weak.

Next we discuss some special cases:

(1) $r_1 = r'_1 = 0$ for the first medium, or $r_2 = r'_2 = 0$ for the second medium.

Equations (46) and (49) or Eqs. (52) and (58) give

$$r = r_e = r_o = 0.$$

In this case the ordinary and extraordinary light rays will propagate with a momentum of $k_0 = \omega n_0/c$ along the z axis which is the optical axis, and no refraction will occur.

(2) $r'_1 = 0$, $r_1 \neq 0$ for the first medium, or $r'_2 = 0$, $r_2 \neq 0$ for the second medium.

We obtain from Eqs. (46), (49), (52) and (58)

$$r_e = r_1\cos\frac{z}{M_1 l_3}, \tag{46'}$$

$$r_o = r_1\cos\frac{z}{l} \tag{49'}$$

for the first medium; and

$$r_e = r_2\cosh\frac{z}{M_2 s_3}, \tag{52'}$$

$$r_o = r_2\cosh\frac{z}{s} \tag{58'}$$

for the second medium.

This means that when the incident light ray is parallel to the z axis, the birefringence will occur due to the anisotropy and the inhomogeneity of the medium. If $l_3 \to \infty$, $l \to \infty$ or $s_3 \to \infty$, $s \to \infty$, the medium is homogeneous and no birefringence will occur. In fact, from Eqs. (46'), (49'), (52') and (58') we obtain

$$r_e = r_o = r_1$$

for the first medium, and

$$r_e = r_o = r_2$$

for the second medium. The ordinary and extraordinary light rays are the same straight lines parallel to the z axis.

(3) $r_1 = 0$, $r_1' \neq 0$ for the first medium, or $r_2 = 0$, $r_2' \neq 0$ for the second medium.

From Eqs. (46), (49), (52) and (58) we obtain

$$r_e = r_1' M_1 l_3 \sin \frac{z}{M_1 l_3}, \tag{46''}$$

$$r_o = r_1' l \sin \frac{z}{l} \tag{49''}$$

for the first medium, and

$$r_e = r_2' M_2 s_3 \sinh \frac{z}{M_2 s_3}, \tag{52''}$$

$$r_o = r_2' s \sinh \frac{z}{s} \tag{58''}$$

for the second medium.

From Eqs. (46''), (49''), (52'') and (58'') we find that when the incident light ray is not parallel to the z axis, the birefringence will occur due to the anisotropy and inhomogeneity of the medium. If $l_3 \to \infty$, $l \to \infty$ or $s_3 \to \infty$, $s \to \infty$, the medium is homogeneous, and no birefringence will occur. In fact, from Eqs. (46''), (49''), (52'') and (58'') we obtain

$$r_e = r_o = r_1' z$$

for the first medium, and

$$r_e = r_o = r_2' z$$

for the second medium. The ordinary and extraordinary light rays are the same oblique lines.

(4) $r_1 \neq 0$, $r_1' \neq 0$, but $n_0 = n_{30}$, $l = l_3$, $M_1 = 1$ for the first medium, or $r_2 \neq 0$, $r_2' \neq 0$, but $n_0 = n_{30}$, $s = s_3$, $M_2 = 1$ for the second medium.

Then Eqs. (46) and (49), (52) and (58) give

$$r_e = r_o = r_1 \cos \frac{z}{l} + r_1' l \sin \frac{z}{l}$$

for the first medium, and

$$r_e = r_o = r_2 \cosh \frac{z}{s} + r_2' s \sinh \frac{z}{s}.$$

for the second medium.

These results represent the isotropic case, and no birefringence will occur. They are in agreement with the results in Ref. [8].

VI. GEODESIC DEVIATION EQUATION AND THE FOCUSING AND DEFOCUSING CHARACTERISTICS OF A LIGHT BEAM

Based on the optical metric theory, we can determine the focusing and defocusing characteristics of a light beam in an inhomogenous medium by solving the geodesic deviation equation[12]

$$\frac{D^2N^\alpha}{dq^2} + R_{\beta\delta}\frac{dx^\beta}{dq}N^\gamma\frac{dx^\delta}{dq} = 0, \tag{61}$$

where N^α is the geodesic separation vector.

For the above optical fibers with two types of parabolic refractive indices, the Riemann curvature tensor has the non-vanishing components

$$\begin{aligned}
R^\alpha_{\beta\gamma\delta} &= \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta}\Gamma^\mu_{\beta\gamma},\\
R^0_{101} &= \frac{n_{,rr}}{n} - 2\frac{n^2_{,r}}{n^2},\\
R^0_{202} &= \frac{r}{n}n_{,r},\\
R^1_{010} &= -\frac{c^2}{n^3}n_{,rr} + 2\frac{c^2}{n^4}n^2_{,r},\\
R^1_{313} &= \left(\frac{n_3^2}{n^3}n_{,rr} - \frac{n_3}{n^2}n_{3,rr}\right) - 2\frac{n_3^2}{n^4}n^2_{,r} - 2\frac{n_3}{n^3}n_{3,r}\,n_{,r} + 4\frac{n_3}{n^3}n_{,r}\,n_{3,r},\\
R^2_{020} &= -\frac{c^2}{n^3 r}n_{,r},\\
R^2_{323} &= \frac{n_3^2}{rn^3}n_{,r} - \frac{n_3}{rn^2}n_{3,r},\\
R^3_{030} &= \frac{c^2}{n_3 n^3}n_{,r}\,n_{3,r} - \frac{c^2}{n^4}n^2_{,r},\\
R^3_{131} &= \frac{1}{n}n_{,rr} - \frac{1}{n_3}n_{3,rr} + \frac{2}{n_3^2}n^2_{3,r} - \frac{2}{nn_3}n_{,r}n_{3,r},\\
R^3_{232} &= \frac{n_{,r}}{nr} - \frac{n_{3,r}}{n_3 r}.
\end{aligned} \tag{62}$$

Then the geodesic deviation equation for the extraordinary light has the form

$$\begin{aligned}
&\frac{D^2N^0}{dq^2} + \left(\frac{n_{,rr}}{n} - 2\frac{n^2_{,r}}{n^2}\right)\left(\frac{dx^1}{dq}\right)^2N^0 + \frac{r}{n}n_{,r}\left(\frac{dx^2}{dq}\right)^2N^0 = 0,\\
&\frac{D^2N^1}{dq^2} + \left(-\frac{c^2}{n^3}n_{,rr} + 2\frac{c^2}{n^4}n^2_{,r}\right)\left(\frac{dx^0}{dq}\right)^2N^1\\
&\quad + \left(\frac{n_3^2}{n^3}n_{,rr} - \frac{n_3}{n^2}n_{3,rr} - 2\frac{n_3^2}{n^4}n^2_{,r} + 2\frac{n_3}{n^3}n_{3,r}\,n_{,r}\right)\left(\frac{dx^3}{dq}\right)^2N^1 = 0,\\
&\frac{D^2N^2}{dq^2} + \left(\frac{c^2}{n^3 r}n_{,r}\right)\left(\frac{dx^0}{dq}\right)^2N^2 + \left(\frac{n_3^2}{rn^3}n_{,r} - \frac{n_3}{rn^2}n_{3,r}\right)\left(\frac{dx^3}{dq}\right)^2N^2 = 0,\\
&\frac{D^2N^3}{dq^2} + \left(\frac{-c^2}{n_3 n^3}n_{,r}n_{3,r} - \frac{c^2}{n^4}n^2_{,r}\right)\left(\frac{dx^0}{dq}\right)N^3\\
&\quad + \left(\frac{n_{,rr}}{n} - \frac{n_{3,rr}}{n_3} + \frac{2}{n_3^2}n^2_{3,r} - \frac{2}{nn_3}n_{,r}n_{3,r}\right)\left(\frac{dx^1}{dq}\right)^2N^3\\
&\quad + \left(\frac{n_{,r}}{nr} - \frac{n_{3,r}}{n_3 r}\right)\left(\frac{dx^2}{dq}\right)^2N^3 = 0.
\end{aligned} \tag{63}$$

For the ordinary light the corresponding equation is

$$\begin{aligned}
&\frac{D^2N^0}{dq^2}+\left(\frac{n_{,rr}}{n}-2\frac{n_{,r}^2}{n^2}\right)\left(\frac{dx^1}{dq}\right)^2N^0+\frac{r}{n}n_{,r}\left(\frac{dx^2}{dq}\right)^2N^0=0,\\
&\frac{D^2N^1}{dq^2}+\left(-\frac{c^2}{n^3}n_{,rr}+2\frac{c^2}{n^4}n_{,r}^2\right)\left(\frac{dx^0}{dq}\right)^2N^1=0,\\
&\frac{D^2N^2}{dq^2}+\left(-\frac{c^2}{n^3r}n_{,r}\right)\left(\frac{dx^0}{dq}\right)^2N^2=0,\\
&\frac{D^2N^3}{dq^2}=0.
\end{aligned}\tag{64}$$

If $\left(\frac{dx^0}{dq}\right)$, $\left(\frac{dx^1}{dq}\right)$ and $\left(\frac{dx^2}{dq}\right)$ are substituted into Eqs. (63) and (64), then the separation vector (N^0, N^1, N^2, N^3) can be obtained and the focusing and defocusing characteristics of a light beam in a crystal with the refractive indices $n(r)$ and $n_3(r)$ can be determined. From Eqs. (63) and (64) it is found that the time component N^0 of the separation vector obeys the same equation but the space components (N^1, N^2, N^3) are different for the extraordinary and ordinary light rays. The extraordinary and ordinary light rays have the same focusing and defocusing characteristics in the isotropic medium ($n = n_3$) or in the homogeneous medium (the derivatives of n and n_3 with respect to r are zero).

REFERENCES

[1] Zhu Shi-tong, Shen Wen-da, *Acta Physica Sinica*, **42**(1993), 1483. (in Chinese)

[2] Shitong Zhu, Qizhi Guo, Wenda Shen *et al.*, *International J. Theoret. Phys.*, **34**(1995), 169.

[3] Zhu Shi-tong, Shen Wen-da, Guo Qi-zhi, *Acta Physica Sinica* **42**(1993), 1471. (in Chinese)

[4] Wenda Shen, Shitong Zhu, *International J. Theoret. Phys.*, **34**(1995), 2085.

[5] Wenda Shen, Shitong Zhu, Qizhi Guo, *International J. Theoret. Phys.*, **34**(1995), 2095.

[6] Guo Qi-zhi, Shen Wen-da, Zhu Shi-tong, *Acta Physica Sinica*, **44**(1995), 210. (in Chinese)

[7] Guo Qi-zhi, Shen Wen-da, Zhu Shi-tong, *Acta Physica Sinica* **44**(1995), 396. (in Chinese)

[8] Wenda Shen, Shitong Zhu and Ximing Deng, *Chin. J. Lasers*, **B5**(1996), 516.

[9] W. Gordon, *Ann. der Phys.*, **72**(22)(1923), 421.

[10] C. W. Misner, K. S. Thorne, J. A. Wheeler. Gravitation (W. H. Freeman and Company San Francisco, 1973), p. 1108.

[11] A. Erdelyi. Higher transcendental functions Ⅱ (McGraw-Hill Book Co. Inc., 1995).

[12] C. W. Misner, K. S. Thorne, J. A. Wheeler. Gravitation (W. H. Freeman and Company San Francisco, 1973), pp. 219, 293.