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A DESCRIPTION OF THE THERMO-ELASTIC RESPONSE OF LAYERED MEDIA BY A MATRIX FORMALISM

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A matrix formalism is introduced here for the analytical calculation of the thermo-elastic response of one-dimensional optically inhomogeneous layered structures exposed to the periodic laser heating. The multilayer structure consists of thermally and mechanically homogeneous solid layers which can be interleaved by liquid or gaseous interlayers. The method enables the modelling of photoacoustic and photothermal measurements in a variety of engineering applications.

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I. INTRODUCTION

Application of the photoacoustic and photothermal techniques[1] to the analysis of layered materials requires determination of the thermo-elastic response of such media induced by the absorption of modulated light. For an n-layer system this problem amounts to solving the $(4n-4)\times(4n-4)$ set of equations describing boundary conditions. Using the Gauss method Glorieux *et al.*[2] solved it numerically for some special cases and analyzed the possibility of using inversion procedure for obtaining characteristics of layers. Recently, Bozoki *et al.*[3] proposed a transfer matrix approach for modeling of the thermo-elastic response of layered media. Here we present a generalization of this method for multilayer structures, which can include layers with spatially varying optical absorption and thin (in comparison with the thermal diffusion length) gas layers.

II. THEORY

Consider a one-dimensional(1D)structure consisting of $n+1$ thermally and mechanically homogeneous layers: $-\infty<x<x_1$, $x_0<x<x_1$, $\cdots$, $x_{n-1}<x<+\infty$ labeled by numbers 0, 1, $\cdots$, n(see Fig.1). Solid layers can be interleaved by liquid or gaseous interlayers. The incident light of intensity I_0, sinusoidally chopped with an angular frequency ω, is absorbed in the optically inhomogeneous layer j (other layers are assumed to be transparent for optical radiation) and acts as a time-dependent source of heat density $Q_j(x)(1+e^{i\omega t})$, $x_{j-1}<x<x_j$. If reflection and scattering in the layer j with spatially varying optical absorption coefficient $\beta_j(x)$ can be neglected, the following equation for $Q_j(x)$ is valid:

$$Q_j(x) = \frac{1}{2} I_0 \frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(\int_{x_j}^{x} \beta_j(\xi)\mathrm{d}\xi\right)\right).$$

Here it is assumed, for simplicity, that all absorbed optical energy is converted to thermal energy. The sinusoidally variable part of the optically excited thermo-elastic response is characterized by a vector $\boldsymbol{W}(x) = (w_1, w_2, w_3, w_4,)^{\mathrm{T}} = (\tau(x), h(x), u(x), \sigma(x))^{\mathrm{T}}$, where $\tau(x), h(x), u(x)$ and $\sigma(x)$ denote the complex amplitudes of the temperature variation, the heat flux, the displacement and the stress, respectively. The superscript T denotes the transposing.

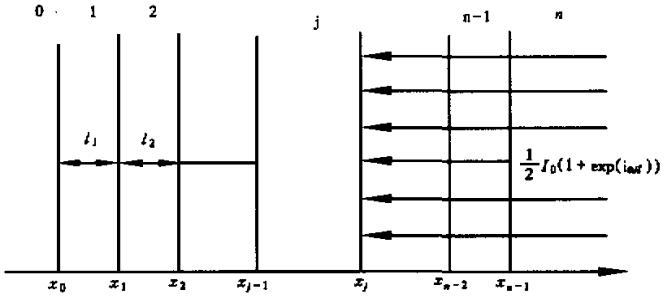

Fig. 1. A multilayer structure exposed to an optical heating.

Using a matrix formalism, the following expression for thermo-elastic response $\boldsymbol{W}_j(x)$ in the transparent layers excited by the heat source $Q_j(x)$ can be obtained:

$$\boldsymbol{W}_j(x) = \boldsymbol{A}(x, x_{n-1})\,\boldsymbol{L}^{(n)}\,\boldsymbol{G}_n(\boldsymbol{D}(0,n)\,\boldsymbol{G}_n - \boldsymbol{G}_0)^{-1}\,\boldsymbol{D}(0,j)\,\boldsymbol{R}(x_{j-1}), \quad x > x_j,$$

$$\boldsymbol{W}_j(x) = \boldsymbol{A}(x, x_0)\,\boldsymbol{L}^{(0)}\,\boldsymbol{G}_0(\boldsymbol{G}_n - \boldsymbol{D}(n,0)\,\boldsymbol{G}_0)^{-1}\,\boldsymbol{D}(n,j)\,\boldsymbol{R}(x_j), \quad x > x_{j-1}.$$

Consider matrices defining $\boldsymbol{W}_j(x)$. $\boldsymbol{A}(\xi, \eta)$ is the transfer matrix relating the thermo-elastic variables in the planes $x = \xi$ and $x = \eta$ of the 1D structure free of the heat sources: $\boldsymbol{W}(\xi) = \boldsymbol{A}(\xi, \eta)\,\boldsymbol{W}(\eta)$. For a single layer k taking into account the influence of both thermal and acoustical waves on all the components of $\boldsymbol{W}(x)$, the following expressions for the elements a_{ij} of the matrix $A^{(k)} = A(x_k, x_{k-1})$ are obtained:

$$a_{11} = a_{22} = \cosh ql - \delta_1 \cos pl, \qquad a_{12} = -\frac{1}{\kappa q}(\sinh ql - \delta_2 \sin pl),$$

$$a_{21} = -\kappa q(\sinh ql + \delta_1\delta_3 \sin pl), \qquad a_{13} = \frac{(\gamma-1)p}{\alpha}(\sin pl - \delta_3 \sinh ql),$$

$$a_{14} = \frac{(\gamma-1)p^2}{\alpha\rho\omega^2}(\cosh ql - \cos pl), \qquad a_{23} = \frac{(\gamma-1)\kappa p^2}{\alpha}(\cosh ql - \cos pl),$$

$$a_{24} = -\frac{(\gamma-1)\kappa p^2 q}{\alpha\rho\omega^2}(\sinh ql + \delta_3 \sin pl), \qquad a_{31} = \frac{a_l}{q}(\sinh ql + \delta_3 \sin pl),$$

$$a_{32} = \frac{\alpha}{\kappa q^2}(\cos pl - \cosh ql), \qquad a_{41} = \frac{a_l \rho\omega^2}{q^2}(\cos pl - \cosh ql),$$

$$a_{42} = -\frac{\alpha_l \rho\omega^2}{\kappa p q^2}(\sin pl - \delta_3 \sinh ql), \qquad a_{33} = a_{44} = \cos pl - \delta_1 \cosh ql,$$

$$a_{34} = \frac{p}{\rho\omega^2}(\sin pl + \delta_2 \sinh ql), \qquad a_{43} = -\frac{\rho\omega^2}{p}(\sin pl - \delta_1\delta_3 \sinh ql),$$

where $\delta_1 = \alpha_l(\gamma-1)p^2/(\alpha q^2)$, $\delta_2 = \alpha_l(\gamma-1)p/(\alpha q)$, $\delta_3 = p/q$ are the small parameters due to the fact that for sound frequencies the thermal wave number $q = (1+i)\sqrt{\omega/2x}$ (x is the thermal diffusivity) is large in comparison with the acoustic wave $p = \omega/c$ (c is the sound speed), α is the coefficient of thermal expansion, $\alpha_l = \alpha(3\lambda + 2\mu)/(3\lambda + 6\mu)$ with λ and μ being the two Lamé constants, γ is the specific heat ratio, ρ is the density, κ is the thermal conductivity, l is the thickness of the layer k. For arbitrary planes $x = x_i$ and $x = x_{i+s}$, the transfer matrix $\boldsymbol{A}(x_{i+s}, x_i)$ can be expressed as the product of those of the individual layer:

$$\boldsymbol{A}(x_{i+s}, x_i) = \boldsymbol{A}^{(i+s)} \cdots \boldsymbol{A}^{(i+2)} \cdot \boldsymbol{A}^{(i+1)}.$$

$\boldsymbol{L}^{(k)}$ is the matrix defined by the thermo-elastic parameters of the layer k:

$$\boldsymbol{L}^{(k)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i\kappa p & -i\kappa p & \kappa q & -\kappa q \\ -\dfrac{i\alpha}{(\gamma-1)p} & \dfrac{i\alpha}{(\gamma-1)p} & -\dfrac{\alpha_l}{q} & \dfrac{\alpha_l}{q} \\ -\dfrac{\alpha\rho\omega^2}{(\gamma-1)p^2} & -\dfrac{\alpha\rho\omega^2}{(\gamma-1)p^2} & -\dfrac{\alpha_l\rho\omega^2}{q^2} & -\dfrac{\alpha_l\rho\omega^2}{q^2} \end{pmatrix}.$$

Characteristic matrix $\boldsymbol{D}(k, s)$ is defined by the following equation:

$$\boldsymbol{D}(k, s) = (\boldsymbol{L}^{(k)})^{-1} \boldsymbol{A}(x_{k-1}, x_s)^{(s)}, \quad k > s \text{ and } \boldsymbol{D}(s, k) = \boldsymbol{D}^{-1}(k, s).$$

Note that

$$\boldsymbol{D}(k, s) = \boldsymbol{D}(k, m)\, \boldsymbol{P}^{(m)}(x_m - x_{m-1})\, \boldsymbol{D}(m, s) \qquad k > m > s.$$

The vector $\boldsymbol{R}(x) = \int_{x_{j-1}}^{x_j} \boldsymbol{P}^{(j)}(x-\xi)(\boldsymbol{L}^{(j+1)} \boldsymbol{E} Q_i(\xi)\, d\xi$, with the vector $\boldsymbol{E} = (0, 1, 0, 0)^T$. Other matrices are:

$$\boldsymbol{G}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{G}_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{P}^{(j)}(x) = \begin{pmatrix} e^{-p_j x} & 0 & 0 & 0 \\ 0 & e^{p_j x} & 0 & 0 \\ 0 & 0 & e^{-q_j x} & 0 \\ 0 & 0 & 0 & e^{q_j x} \end{pmatrix},$$

where the subscript j identifies the layer j.

It is worth noting that due to the linearity of the process of the thermo-elastic excitation, the following expression $\boldsymbol{W}(x) = \sum_{j=k}^{m} \boldsymbol{W}_j(x)$ for $x \le x_{k-1}$, $x \ge x_m$ can be obtained in the more general case when the optical radiation is absorbed in the layer $k, k+1, \cdots, m$.

The obtained spatial variation of the components of the vector $\boldsymbol{W}(x)$ enables the modeling of different photoacoustic and photothermal measurements. "Acoustical" components w_3 and w_4 can be used for the evaluation of the signal in the photoacoustic measurements

with microphone and piezoelectric detection, as well as in the photothermal displacement detection scheme[4]. "Thermal" components w_1 and w_2 define the signal in the different photothermal detection schemes: mirage effect (or photothermal optical beam deflection) in its perpendicular and collinear configurations, infrared radiometry, and photothermal reflectance. For example, if the probe beam is sent through the optically homogeneous layer i characterized by the index or refraction n_i, then the mirage deflection is given by[5]

$$\phi = -\frac{1}{n_i}\frac{\partial n_i}{\partial w_1}\int_{\text{opticalpath}} \nabla w_1 \times \mathrm{d}l.$$

III. DISCUSSION

The advantage of using the matrix formalism is that the transfer and characteristic matrices of a layered structure can be expressed in terms of those of the individual layers. These enable to derive an analytical result in the form suitable for further analysis.

Bozoki *et al.*[3] calculated the transfer matrix assuming that temperature and heat flux distributions can be obtained by neglecting the influence of acoustic waves. According to the generalized composite-piston model proposed by McDonald *et al.*[4], it is not valid for a thin (in comparison with the thermal diffusion length) gas layers.

In this work the transfer matrix was calculated in the more general form using the 1D solution of the coupled thermal-diffusion and elastic wave equations. So the derived expression or $\boldsymbol{W}(x)$ is valid for a multilayer structure containing solid, liquid and gas layers of an arbitrary thickness.

IV. CONCLUSION

The proposed matrix formalism enables the modeling of photoacoustic and photothermal nondestructive evaluation of layered media in a variety of engineering applications: detections of adhesively bonded joints, protective paint coating on automobile bodies and aerospace structures, trapped corrosion layers in aging-aircraft structures, delaminations (and other defects) in modern layered materials (*e. g.*, aluminum/epoxy), and so on.

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