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ANALYSIS ON THE EMBEDDING CHARACTER OF TIME SERIES RECONSTRUCTION*

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A method is proposed to analyze embedding character of the reconstruction by calculating the ratio of two volumes in two reconstructed phase spaces. By this method the embedding dimension and the delay time are found. The Rössler system is discussed as a special model.

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I. INTRODUCTION

The determination of the appropriate phase space by using delay reconstruction is one of the first tasks for all who wish to work with observed data. The reconstruction ought to preserve the dynamical and geometrical properties of the original system, and therefore the reconstructing transformation must be an embedding. Takens[1] provided a sufficient condition for selecting the embedding dimension. Since Takens' theorem only gives a sufficient condition, it motivates the search for identifying a necessary embedding dimension.

Previous works[2−4] have determined the reconstructed dimension and the delay time. However, a rigorous criterion of embedding ought to come from the definition of the embedding. According to this definition, we propose a method to analyze the embedding character of the reconstruction through calculating the ratio of two volumes in two reconstructed phase

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spaces. Our work will show that a necessary embedding dimension is related not only with the reconstructed dimension, but also with the choice of τ, i.e., an improper choice of τ can change the embedding character of the reconstruction.

II. PRINCIPLE OF CALCULATION BY THE PROPOSED METHOD

A reconstruction is a transformation ϕ from an original system M^m to a reconstructed system N^n, where m is the dimension of M and n is that of N. According to Takens' theorem, a good reconstruction must be an embedding. The definition of embedding is[5]:

Let $\phi : M^m \longrightarrow N^n$ be a C^∞-map. The rank of ϕ at $x \in M$ is the rank of the $m \times n$ matrix $(\partial\phi^j/\partial x^i), i = 1, \cdots, m; j = 1, \cdots, n$. ϕ is called an immersion if $\mathrm{d}\phi(x)$ is injective, rank $\phi = m$. ϕ is called an embedding, if ϕ is a 1:1 immersion which is a homeomorphism onto its image $\phi(M)$ endowed with the relative topology.

We consider an original system described by the coordinates (x, y, z). Let delay time be τ, the reconstructed dimension be n. The time series $(y(t_i), i = 1, \cdots)$ is chosen as an example. A transformation of the reconstruction is a map $\phi : (x, y, z) \longmapsto (y, y(\tau), \cdots, y((n-1)\tau))$. The Jacobian matrix of this map is defined by

$$J = \begin{pmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial y(\tau)}{\partial x} & \dfrac{\partial y(\tau)}{\partial y} & \dfrac{\partial y(\tau)}{\partial z} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial y((n-1)\tau)}{\partial x} & \dfrac{\partial y((n-1)\tau)}{\partial y} & \dfrac{\partial y((n-1)\tau)}{\partial z} \end{pmatrix} . \tag{1}$$

If there exists a smallest integer n, such that the matrix J has a nonsingular 3rd-order subdeterminant, then this n is the embedding dimension.[6]

Since the original phase space is not known, the calculation of reconstructed dimension should start from 2. The Jacobian determinant of two-dimensional (2D) reconstruction $(y, y(\tau))$ is

$$\det \mid J_2 \mid = \begin{vmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \\ \dfrac{\partial y(\tau)}{\partial x} & \dfrac{\partial y(\tau)}{\partial y} \end{vmatrix} . \tag{2}$$

In order to calculate numerically,Eq.(2)is rewritten as the following:

$$\det \mid J_2 \mid = \begin{vmatrix} \dfrac{\delta y}{\delta x} & \dfrac{\delta y}{\delta y} \\ \dfrac{\delta y(\tau)}{\delta x} & \dfrac{\delta y(\tau)}{\delta y} \end{vmatrix} ,$$

i.e.,

$$\det \mid J_2 \mid = \frac{1}{\delta x \delta y} \begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(\tau) & \delta y_2(\tau) \end{vmatrix} = \frac{\mathrm{d}s(y, y(\tau))}{\mathrm{d}s_0(x, y)} \quad , \tag{3}$$

where $\mathrm{d}s(y, y(\tau))$ is an area element of the reconstructed space $(y, y(\tau))$, and $\mathrm{d}s_0(x, y)$ is an area element of the original space (x, y).

If reconstruction $(y, y(\tau))$ is not embedding, ϕ is not full rank, or $\det| J_2 | = 0$. Because the area element of the original space $\mathrm{d}s_0(x, y) = \delta x \delta y \neq 0$,we have $\mathrm{d}s(y, y(\tau))$=0. This means that if the reconstruction$(y, y(\tau))$ is not embedding, the area element in the space $(y, y(\tau))$ equals zero. On the other hand, if $\mathrm{d}s(y, y(\tau)) = 0$, it causes $\det| J_2 |$=0, i.e.,the reconstructon$(y, y(\tau))$ is not embedding. Therefore, one should consider 3,4, $\cdots$dimensional reconstruction until an embedding is arrived at.

Since the area element $\mathrm{d}s_0(x, y)$ is not known, Eq.(3) should be revised. Supposing there is another reconstructed space $(y, y(2\tau))$,its Jacobian determinant is

$$\det \mid J_2' \mid = \frac{1}{\delta x \delta y} \begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(2\tau) & \delta y_2(2\tau) \end{vmatrix} = \frac{\mathrm{d}s(y, y(2\tau))}{\mathrm{d}s_0(x, y)} . \tag{4}$$

Dividing Eq.(3)by Eq.(4)to eliminate $\mathrm{d}s_0(x, y) = \delta x \delta y$,we have

$$w = \frac{\det \mid J_2 \mid}{\det \mid J_2' \mid} = \frac{\mathrm{d}s(y, y(\tau))}{\mathrm{d}s(y, y(2\tau))} = \frac{\begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(\tau) & \delta y_2(\tau) \end{vmatrix}}{\begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(2\tau) & \delta y_2 2\tau \end{vmatrix}} . \tag{5}$$

Similarly, we can obtain a ratio of two Jacobian determinants for two 3D reconstructions $(y, y(\tau), y(2\tau))$and$(y, y(\tau), y(3\tau))$:

$$w = \frac{\det \mid J_3 \mid}{\det \mid J_3' \mid} = \frac{\mathrm{d}v(y, y(\tau), y(2\tau))}{\mathrm{d}v(y, y(\tau), y(3\tau))} = \frac{\begin{vmatrix} \delta y_1 & \delta y_2 & \delta y_3 \\ \delta y_1(\tau) & \delta y_2(\tau) & \delta y_3(\tau) \\ \delta y_1(2\tau) & \delta y_2(2\tau) & \delta y_3(2\tau) \end{vmatrix}}{\begin{vmatrix} \delta y_1 & \delta y_2 & \delta y_3 \\ \delta y_1(\tau) & \delta y_2(\tau) & \delta y_3(\tau) \\ \delta y_1(3\tau) & \delta y_2(3\tau) & \delta y_3(3\tau) \end{vmatrix}} . \tag{6}$$

Elements of Jacobian determinants in Eqs.(5) and (6) are solutions of the linearized ordinary differential equation. Because the ordinary differential equation is not known, we have to replace the ratio of the two Jacobian determinants by the ratio of two areas or volumes in two reconstructed spaces. Here we give a method to calculate the ratio of two areas as follows.

We take a point $(y_0, y_0(\tau))$ in space $(y, y(\tau))$ as a fiducial point, and then take three points $(y_1, y_1(\tau)), (y_2, y_2(\tau))$ and $(y_3, y_3(\tau))$ as background points close to point $(y_0, y_0(\tau))$. The three points $(y_1, y_1(\tau)), (y_2, y_2(\tau))$ and $(y_3, y_3(\tau))$ constitute a triangle (see Fig.1), whose area is

$$S(y, y(\tau)) = \frac{1}{2}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1(\tau) & y_2(\tau) & y_3(\tau) \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} y_2 - y_1 & y_3 - y_1 \\ y_2(\tau) - y_1(\tau) & y_3(\tau) - y_1(\tau) \end{vmatrix}. \quad (7)$$

The sign of $S(y, y(\tau))$ determines an orientation of the triangle in the space $(y, y(\tau))$. Once the four points in the space $(y, y(\tau))$ are given, we can obtain another four points $(y_0, y_0(2\tau)), (y_1, y_1(2\tau)), (y_2, y_2(2\tau))$ and $(y_3, y_3(2\tau))$ in the space $(y, y(2\tau))$. An area of the triangle in the space $(y, y(2\tau))$ is also given as

$$S(y, y(2\tau))$$

$$= \frac{1}{2}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1(2\tau) & y_2(2\tau) & y_3(2\tau) \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} y_2 - y_1 & y_3 - y_1 \\ y_2(2\tau) - y_1(2\tau) & y_3(2\tau) - y_1(2\tau) \end{vmatrix}. \quad (8)$$

The sign of Eq.(8) determines an orientation of the triangle in the space $(y, y(2\tau))$. If background points are sufficiently close, we have $y_2 - y_1 = \delta y_1, y_3 - y_1 = \delta y_2, y_2(\tau) - y_1(\tau) = \delta y_1(\tau), y_3(\tau) - y_1(\tau) = \delta y_2(\tau), y_2(2\tau) - y_1(2\tau) = \delta y_1(2\tau)$ and $y_3(2\tau) - y_1(2\tau) = \delta y_2(2\tau)$ respectively. Substituting them into Eqs.(7) and (8), and dividing Eq.(7)by Eq.(8), we get

$$w = \frac{S(y, y(\tau))}{S(y, y(2\tau))} = \frac{\begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(\tau) & \delta y_2(\tau) \end{vmatrix}}{\begin{vmatrix} \delta y_1 & \delta y_2 \\ \delta y_1(2\tau) & \delta y_2(2\tau) \end{vmatrix}}. \quad (9)$$

Even if Eq.(9) are similar to Eq.(5), they have an essential distinction: Eq.(5) is a ratio of two differential forms, but Eq.(9) is a ratio of two finite areas. Therefore, we should constitute an approximation by using Eq.(9) to take the place of the differential approach of Eq.(5).

We minimize successively the area of triangle in the space $(y, y(\tau))$ to get a group of triangles (see Fig.1), and get a sequence of ratios from Eq.(9). We use this sequence of ratios to approximate the ratio of Eq.(5).

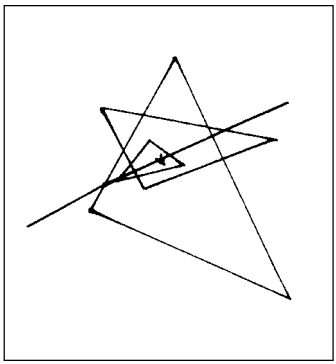

Fig.1. Diagrammatic sketch for calculating area of triangle.

When the sequence of ratios approaches zero, the reconstruction $(y, y(\tau))$ is not embedding, i.e., the reconstructed dimension is too low, or the delay time is not appropriate. To find the position w=0, we calculate a value of w and evolve a step along the attractor orbit, to get another value w'. If $w \times w' < 0$, by the continuity of w, there must exists a point such that $w = 0$. If we do not find w= 0, the reconstruction is an embedding.

Similarly, we deal with 3D reconstruction $(y, y(\tau), y(2\tau))$. We take four background points, which constitute a small tetrahedron, close to a fiducial point $(y_0, y_0(\tau), y_0(2\tau))$. The volume of this small tetrahedron is

$$V(y, y(\tau), y(2\tau)) = \frac{1}{6}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1(\tau) & y_2(\tau) & y_3(\tau) & y_4(\tau) \\ y_1(2\tau) & y_2(2\tau) & y_3(2\tau) & y_4(2\tau) \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ y_2(\tau) - y_1(\tau) & y_3(\tau) - y_1(\tau) & y_4(\tau) - y_1(\tau) \\ y_2(2\tau) - y_1(2\tau) & y_3(2\tau) - y_1(2\tau) & y_4(2\tau) - y_1(2\tau) \end{vmatrix}. \qquad (10)$$

The sign of Eq.(10) determines an orientation of the tetrahedron in the space $(y, y(\tau), y(2\tau))$. In another reconstructed space $(y, y(\tau), y(3\tau))$, we get a corresponding small tetrahedron, whose volume is

$$V(y, y(\tau), y(3\tau)) = \frac{1}{6}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1(\tau) & y_2(\tau) & y_3(\tau) & y_4(\tau) \\ y_1(3\tau) & y_2(3\tau) & y_3(3\tau) & y_4(3\tau) \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ y_2(\tau) - y_1(\tau) & y_3(\tau) - y_1(\tau) & y_4(\tau) - y_1(\tau) \\ y_2(3\tau) - y_1(3\tau) & y_3(3\tau) - y_1(3\tau) & y_4(3\tau) - y_1(3\tau) \end{vmatrix} . \qquad (11)$$

The sign of Eq.(11) determines an orientation of the tetrahedron in the space $(y, y(\tau), y(3\tau))$. Dividing Eq.(10) by Eq.(11), we obtain

$$w_3 = \frac{V(y, y(\tau), y(2\tau))}{V(y, y(\tau), y(3\tau))} = \frac{\begin{vmatrix} \delta y_1 & \delta y_2 & \delta y_3 \\ \delta y_1(\tau) & \delta y_2(\tau) & \delta y_3(\tau) \\ \delta y_1(2\tau) & \delta y_2(2\tau) & \delta y_3(2\tau) \end{vmatrix}}{\begin{vmatrix} \delta y_1 & \delta y_2 & \delta y_3 \\ \delta y_1(\tau) & \delta y_2(\tau) & \delta y_3(\tau) \\ \delta y_1(3\tau) & \delta y_2(3\tau) & \delta y_3(3\tau) \end{vmatrix}} . \qquad (12)$$

Equation (12) is similar to Eq.(6). We constitute an approximation by using Eq.(12) to replace the differential approach of Eq.(6). In order to obtain an embedding reconstruction, we increase the reconstructed dimension and adjust the delay time until $w \neq 0$ for all points. Thus we can arrive at an embedding reconstruction.

III. NUMERICAL EXAMPLE AND ANALYSIS

We consider Rössler equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z, \qquad (13)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + ay, \qquad (14)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = b + (x - c)z, \tag{15}$$

where the parameters are $a = 0.15, b = 0.20,$ and $c = 10$. A unit of one orbit time is $193.3\pi/100$. We integrate Eqs.(13)−(15) with initial conditions $t = 0, x(0) = 10, y(0) = z(0) = 0$, and time step $\Delta t = \pi/100 \doteq 0.0314$ to obtain an attractor after canceling the first 2×10^3 steps. The attractor is shown in Fig.2.

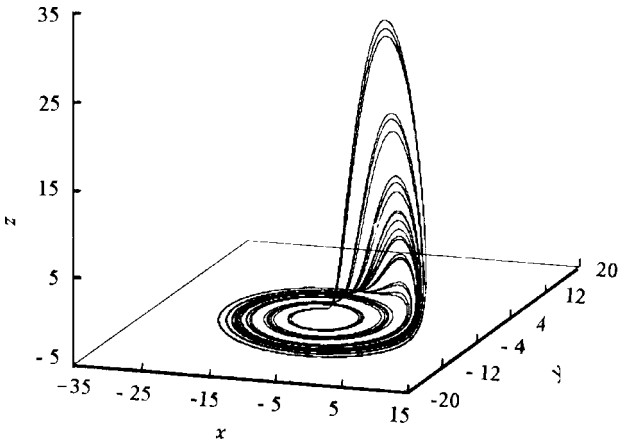

Fig.2. The Rössler attractor.

First, we discuss the 2D reconstructions $(y, y(\tau))$and$(y, y(2\tau))$. We take two background points which are the nearest neighbours of the fiducial point. Two background points and one fiducial point in the space $(y, y(\tau))$ constitute a triangle whose area $S_1(y, y(\tau))$ is calculated from Eq. (7). Meanwhile, the area $S_2(y, y(2\tau))$ in the space $(y, y(2\tau))$ is calculated from Eq.(8). By Eq.(9), we obtain a ratio w_1. Then we take another two points which are the next nearest neighbours of the fiducial point $(y_0, y_0(\tau))$. These two background points and one fiducial point constitute a new triangle. From Eqs.(7) and (8) we get two new areas $S_2(y, y(\tau))$ and $S_2(y, y(2\tau))$. Here $\mid S_2(y, y(\tau)) \mid > \mid S_1(y, y(\tau)) \mid$. Again we increase successively the area of the triangle close to the fiducial point to obtain a group of growing triangles. In this paper we take six triangles at a fiducial point in the space $(y, y(\tau))$, and demand that $\mid S_6(y, y(\tau)) \mid > \mid S_5(y, y(\tau)) \mid > \mid S_4(y, y(\tau)) \mid \gg \mid S_3(y, y(\tau)) \mid > \mid S_2(y, y(\tau)) \mid > \mid S_1(y, y(\tau)) \mid$. At the ith fiducial point in the orbit,we obtain an approximant

$$w_6(i) \longmapsto w_5(i) \longmapsto w_4(i) \longmapsto w_3(i) \longmapsto w_2(i) \longmapsto w_1(i). \tag{16}$$

Similarly,at the $(i+1)$th fiducial point in the orbit,we get another approximant

$$w_6(i+1) \longmapsto w_5(i+1) \longmapsto w_4(i+1) \longmapsto w_3(i+1) \longmapsto w_2(i+1) \longmapsto w_1(i+1). \tag{17}$$

If the sign of the approximant (16) differs from that of the approximant (17), there exists a point between the ith and the $(i+1)$th fiducial points such that $w = 0$. When $\mid w \mid$ decreases,

the singularity appears in the space $(y, y(\tau))$; when $\mid w \mid$ increases, the singularity appears in the space $(y, y(2\tau))$. If the signs of the approximants (16) and (17) show no difference, there will be no singularity in the reconstructed space, and the reconstruction is embedding.

Since the approximant is not exact differential, it may cause error which brings additional singular points. To reduce the additional singular points, two algorithms are applied to select neighbouring points. The first one is to select background points in a circle with radius $\varepsilon(> 0)$ without considering their relative positions and angles between them. Here we choose ε=0.15. The calculated singular points would include intrinsic singular points and additional singular points. The second algorithm is tried to select a triangle whose sides have a certain angle, e.g., a rectangular triangle. We use this triangle to re-calcuate the singular points. Because additional singular points are produced from computing error, they are different for different algorithms, but intrinsic singular points are invariant when algorithms are varied. Finally, we find out the intrinsic singular points. The more the background points are taken, the more exact the approximant is. In this paper, we choose 6×10^3 fiducial points and 24×10^4 background points, and use τ =0.12 (orbits) to calculate the position at which w =0. We find out many positions where w=0. To verify our method, we have calculated singular points by using Eq.(6), whose elements were obtained from Eqs.(13)−(15), and their linearized ordinary differential equations. Also we use the present method to find the positions where w =0, which are the same as the above results. The agreement of the singular positions shows that our method is practicable.

By calculating 2D reconstruction $(y, y(\tau))$ with the delay time τ=0.12 (orbits), the conclusion is drawn that 2D reconstruction of Rössler system is not embedding. The corresponding picture is shown in Fig.3. To find the relation between topological character of

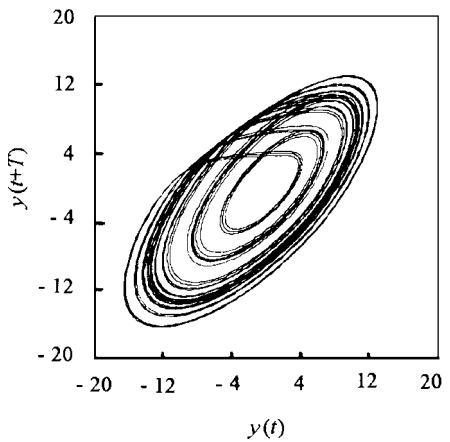

Fig.3. The reconstructed attractor in $(y, y(\tau)), \tau = 0.12$ (orbits).

2D reconstruction and delay time τ, calculation is practiced by using the same data and same method except for $\tau \in$(0,0.4). The result is shown in Fig.4. It is clear that whatever the value of τ is taken, there are always enough singular points to demonstrate that 2D

reconstruction is not embedding. This indicates that the reconstructed dimension is too small. In Ref.[6] we showed that if the delay time increases, the reconstruction turns worse. So the following conclusion is drawn: The 2D reconstruction $(y, y(\tau))$ of Rössler system is not embedding, which agrees with the results in Refs.[3,6].

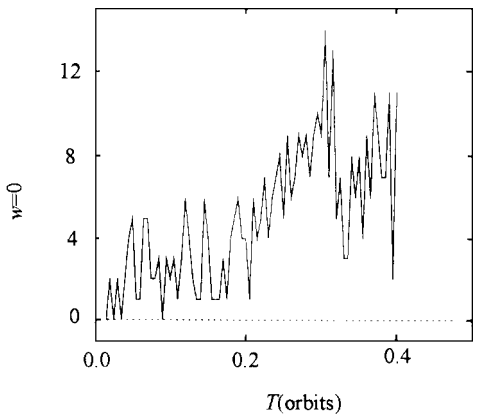

Fig.4. Diagram of points of w=0 versus delay time in reconstructed space $(y, y(\tau)), \tau = 0 - 0.4$ (orbits).

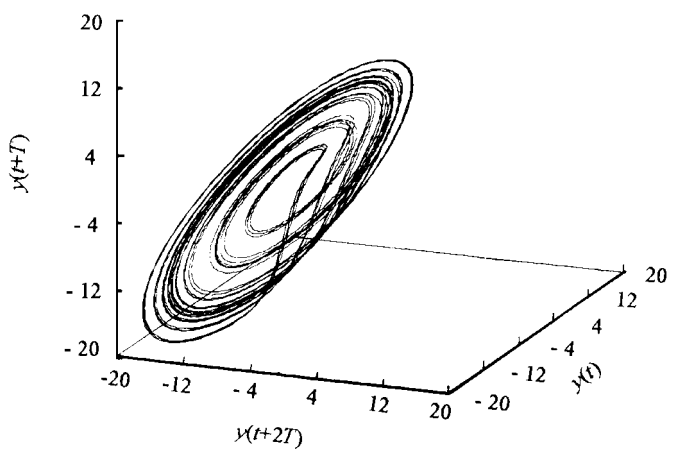

Fig.5. The reconstructed attractor in $(y, y(\tau), y(2\tau))$, τ=0.12 (orbits).

In the following we consider the 3D reconstruction. In reconstruction of $(y, y(\tau), y(2\tau))$ and $(y, y(\tau), y(3\tau))$, we use 6×10^3 fiducial points, 24×10^4 background points and delay time τ=0.12 (orbits). In the neighbourhood of each fiducial points, we select 20 small tetrahedrons to calculate the approximant with Eqs.(10)–(12). The number of singular points are not equal to zero. This shows that there is no position of w =0, and 3D reconstruction $(y, y(\tau), y(2\tau))$ with τ=0.12 (orbits) is embedding, which agrees with the analysis in Ref.[6]. Figure 5 is the reconstruction picture.

To compare with reconstruction of the other delay time, the same calculation is practiced with τ=0.65 (orbits). The result shows that the number of singular points is not equal to zero, and the positions of singular points agree with that computed with Eq.(7) through the differential equation. This demonstrates that the 3D reconstruction with unsuitable delay time τ is not embedding. Figure 6 is the reconstruction $(y, y(\tau), y(2\tau))$ with τ =0.65 (orbits). This result is also the same as the analysis in Ref. [6].

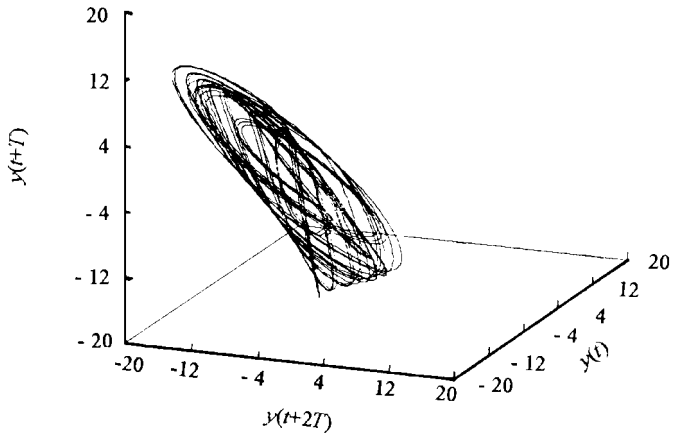

Fig.6. The reconstructed attractor in $(y, y(\tau), y(2\tau)), \tau = 0.65$ (orbits).

To find the interval of suitable τ, we select 4×10^2 fiducial points and delay time τ=0–0.8 (orbits) to calculate the number of singular points versus τ. The result is shown in Fig.7.

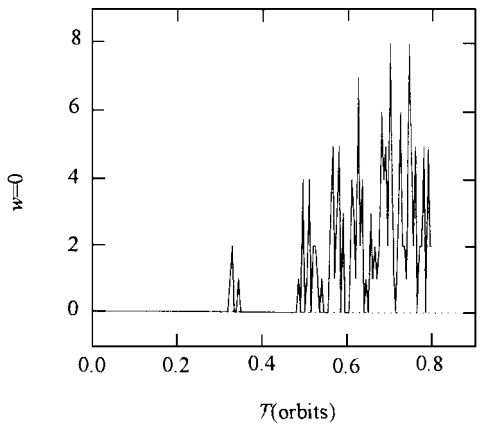

Fig.7. Diagram of points of w=0 versus delay time in reconstructed space $(y, y(\tau), y(2\tau)), \tau = 0-0.8$ (orbits).

Figure 7 demonstrates that for delay time in $0 < \tau \leq 0.29$ (orbits), the number of singular points is equal to zero, so the reconstruction is embedding; whereas for $\tau > 0.29$

(orbits), the number of singular points is not equal to zero, hence the reconstruction is not embedding. The result here agrees with that in Ref.[6]. And it indicates that one must be careful in choosing τ, avoiding to change the topological property of reconstruction.

IV. CONCLUSION

In this paper, we generalized Jacobian determinant method suggested in Refs. [3,6]. To analyze embedding character of reconstruction, we calculate the rank of the reconstruction map by using the ratio of the volumes in two reconstructed spaces. Taking Rössler system as an example, we calculate the reconstruction character with time series of y variable, and determine the embedding parameter. The method gives a criterion of embedding for reconstruction of chaotic system without differential equations. The results agree with that calculated through ordinary differential equations.

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