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ON A SUFFICIENT CONDITION FOR THE METRIC COEFFICIENTS TO BE CONFORMALLY TIME-INDEPENDENT*

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The statement in Rindler's book "Essential Relativity" [(Springer-Verlag, New York, 1977), p. 119] claiming that "the metric coefficients are time-independent if light signals tracing out the same path always take the same coordinate time" has two problems, the first one has been pointed out by us in a recent paper "relation between choosing 'good' coordinates and the behavior of light signals" [*Acta Physica Sinica (Overseas Edition)*, **3**(1994), 881], while the second is the main focus of the present article.

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I. INTRODUCTION

There is a statement proved in Ref.[1] (p.119): "a necessary and sufficient condition for the metric coefficients to be time-independent is that light signals tracing out the same path always take the same coordinate time". There are two problems involved in the sufficient part. The first one is that the requirement "time-independent" should be replaced by "conformally time-independent", as pointed out in Ref. [2]. The second problem involves the behavior of light signals, and is the focus of the present paper. Since the electric charge of a photon is zero, photons do not interact directly with electromagnetic fields, and other matter can influence a photon only by destroying it, thus each photon of interest is subject only to gravity and hence "light signals" can only be null geodesics.[3] The behavior of null geodesics is different from that of nongeodedic null curves. Let $\{x^\mu\}$ be any coordinate system with x^0 time-like and x^1, x^2, x^3 space-like in an arbitrary spacetime. Given any null geodesic $\eta(\lambda)$, there always exist null curves that "trace out the same spatial path as that of $\eta(\lambda)$" (by "a spatial path of a null curve" we mean its projection onto a space-like hypersurface Σ defined by x^0=const), but it is possible that there exist no null geodesics admitting this property (for examples see section II). Therefore the only viable meaning of the sufficient condition in Ref. [1] is: "Given any null geodesic $\eta(\lambda)$, whenever there exist null geodesics $\widehat{\eta}(\lambda)$ tracing out the same path as that of $\eta(\lambda)$, $\widehat{\eta}(\lambda)$ and $\eta(\lambda)$ always take the

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same coordinate time". However, we will give a counterexample showing that this condition is not sufficient to guarantee the conformal time-independence of the metric coefficients.

II. THE COUNTEREXAMPLE

Consider the metric

$$ds^2 = (\alpha t + \beta x)^{-2}(-dt^2 + dx^2) + e^{2A(y,z)}(dy^2 + dz^2) , \tag{1}$$

where α and β are constants with $\alpha^2 \neq \beta^2$, and $A(y, z)$ is an arbitrary function of y and z. Note that $e^{2A(y,z)}(dy^2 + dz^2)$ is the most general line element for a two-dimensional (2D) Riemannian metric since all 2D metrics are locally conformally flat. The metric coefficients g_{tt} and g_{xx}, being functions of the time coordinate t, are not time-independent, we will, however, show that for any null geodesic $\eta(\lambda)$, whenever there exist null geodesics $\widehat{\eta}(\lambda)$ tracing out the same path as that of $\eta(\lambda)$, then $\widehat{\eta}(\lambda)$ and $\eta(\lambda)$ always take the same coordinate time, therefore giving a counterexample of the sufficient part of the quoted statement in section I.

Let $\eta(\lambda)$ be a null geodesic, $\eta'(\lambda)$ and $\eta''(\lambda)$ be, respectively, its projections on the t-x plane ($y = z = 0$) and y-z plane ($t = x = 0$), then it follows from Eq. (1) that both $\eta'(\lambda)$ and $\eta''(\lambda)$ are geodesics. we will write $\eta = (\eta', \eta'')$ in short. Let $\widehat{\eta} = (\widehat{\eta}', \widehat{\eta}'')$ be any null geodesic tracing out the same spatial path as that of $\eta(\lambda)$. In the special case where $\eta''(\lambda)$ reduces to a single point, the issue reduces essentially to that involving only null geodesics in a 2D Minkowski spacetime and it is obvious that $\widehat{\eta}(\lambda)$ and $\eta(\lambda)$ take the same coordinate time. To obtain a counterexample of the quoted statement, it is then sufficient to show that $\widehat{\eta}'(\lambda) = \eta'(\lambda)$ whenever $\eta''(\lambda)$ is not a point, i.e., in the case where $\eta''(\lambda)$ is not a point, there exist no null geodesics other than $\eta(\lambda)$ itself which trace out the same spatial path as that of $\eta(\lambda)$. The following is our proof.

The nonvanishing Christoffel symbols relevant to $\eta'(\lambda)$ are

$$\Gamma^t_{tt} = \Gamma^t_{xx} = \Gamma^x_{tx} = \Gamma^x_{xt} = -\alpha(\alpha t + \beta x)^{-1} ,$$
$$\Gamma^t_{tx} = \Gamma^t_{xt} = \Gamma^x_{tt} = \Gamma^x_{xx} = -\beta(\alpha t + \beta x)^{-1} ,$$

therefore the equations of the time-like geodesic $\eta'(\lambda)$ read

$$\frac{d^2t}{d\lambda^2} - \alpha(\alpha t + \beta x)^{-1}\left(\frac{dt}{d\lambda}\right)^2 - 2\beta(\alpha t + \beta x)^{-1}\frac{dt}{d\lambda}\frac{dx}{d\lambda} - \alpha(\alpha t + \beta x)^{-1}\left(\frac{dx}{d\lambda}\right)^2 = 0 , \tag{2}$$

$$\frac{d^2x}{d\lambda^2} - \beta(\alpha t + \beta x)^{-1}\left(\frac{dt}{d\lambda}\right)^2 - 2\alpha(\alpha t + \beta x)^{-1}\frac{dt}{d\lambda}\frac{dx}{d\lambda} - \beta(\alpha t + \beta x)^{-1}\left(\frac{dx}{d\lambda}\right)^2 = 0 . \tag{3}$$

Introducing new coordinates

$$t' = \alpha t + \beta x , \quad x' = \beta t + \alpha x , \tag{4}$$

one obtains from Eqs. (2) and (3) that

$$\frac{\mathrm{d}^2t'}{\mathrm{d}\lambda^2}-\frac{1}{t'}\left[\left(\frac{\mathrm{d}t'}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}x'}{\mathrm{d}\lambda}\right)^2\right]=0\,, \tag{5}$$

and

$$\frac{\mathrm{d}^2x'}{\mathrm{d}\lambda^2}-\frac{2}{t'}\frac{\mathrm{d}t'}{\mathrm{d}\lambda}\frac{\mathrm{d}x'}{\mathrm{d}\lambda}=0\,. \tag{6}$$

The general solution to Eq. (6) is

$$\frac{\mathrm{d}x'}{\mathrm{d}\lambda}=ct'^2\,, \tag{7}$$

where c is a constant. On the other hand, since $\eta''(\lambda)$ is a geodesic, the length L of its tangent vector is a constant, i.e.,

$$L^2\equiv \mathrm{e}^{2A}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right]=\text{const}\,, \tag{8}$$

and $L\neq 0$ due to the assumption that $\eta''(\lambda)$ does not reduce to a point. It then follows from the null condition

$$(\alpha t+\beta x)^{-2}\left[-\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2\right]+\mathrm{e}^{2A}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right]=0$$

that

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2=t'^2L^2\,. \tag{9}$$

Equation (4) gives rise to

$$\left(\frac{\mathrm{d}t'}{\mathrm{d}\lambda}\right)^2-\left(\frac{\mathrm{d}x'}{\mathrm{d}\lambda}\right)^2=(\alpha^2-\beta^2)\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2\right]\,,$$

which, on account of Eq. (9), yields

$$\left(\frac{\mathrm{d}t'}{\mathrm{d}\lambda}\right)^2-\left(\frac{\mathrm{d}x'}{\mathrm{d}\lambda}\right)^2=\sigma^2t'^2\,, \tag{10}$$

where

$$\sigma=(\alpha^2-\beta^2)^{1/2}I\,. \tag{11}$$

It follows from Eqs. (5) and (10) that

$$\frac{\mathrm{d}^2t'}{\mathrm{d}\lambda^2}-\frac{1}{t'}\left[2\left(\frac{\mathrm{d}t'}{\mathrm{d}\lambda}\right)^2-\sigma^2t'^2\right]=0\,, \tag{12}$$

which can be integrated to give

$$t'(\lambda)=-2\sigma\left(k\mathrm{e}^{\sigma\lambda}+l\mathrm{e}^{-\sigma\lambda}\right)^{-1}\,, \tag{13}$$

where k and l are integral constants. Substitution of Eq. (13) into (7) then yields

$$\frac{\mathrm{d}x'}{\mathrm{d}\lambda} = 4c\sigma^2 \mathrm{e}^{2\sigma\lambda} \left(k\mathrm{e}^{2\sigma\lambda} + 1\right)^{-2} , \tag{14}$$

and hence

$$x'(\lambda) = -2c\sigma k^{-1} \left(k\mathrm{e}^{2\sigma\lambda} + 1\right)^{-1} + m , \quad m = \text{const} . \tag{15}$$

Setting

$$a = mk , \quad b = ml - 2c\sigma k^{-1} , \tag{16}$$

one gets

$$x'(\lambda) = \left(a\mathrm{e}^{\sigma\lambda} + b\mathrm{e}^{-\sigma\lambda}\right) \left(k\mathrm{e}^{\sigma\lambda} + l\mathrm{e}^{-\sigma\lambda}\right)^{-1} . \tag{17}$$

It is now clear that $(t'(\lambda), x'(\lambda))$ expressed by Eqs. (13) and (17) comprises the general solution to Eq. (6), and it can be checked that Eq. (5) is also satisfied by this $(t'(\lambda), x'(\lambda))$ if and only if

$$(al - bk)^2 = -4kl\sigma^2 . \tag{18}$$

Therefore a null geodesic $\eta(\lambda)$ is characterized by five constants k, l, a, b and σ restricted by Eq. (18), where σ corresponds to $\eta''(\lambda)$ while (k, l, a, b) to $\eta'(\lambda)$. Indeed, denoting the initial data of $\eta'(\lambda)$ by $t'_0 \equiv t'(0), x'_0 \equiv x'(0), T'_0 \equiv (\mathrm{d}t'/\mathrm{d}\lambda)|_{\lambda=0}$, and $x'_0 \equiv (\mathrm{d}x'/\mathrm{d}\lambda)|_{\lambda=0}$, one obtains from Eqs. (13) and (17) that

$$\begin{aligned} &t'_0 = -2\sigma(k+l)^{-1} , \quad x'_0 = (a+b)(k+l)^{-1} , \\ &T'_0 = 2\sigma^2(k-l)(k+l)^{-2} , \quad x'_0 = 2\sigma(al-bk)(k+l)^{-2} , \end{aligned} \tag{19}$$

and hence

$$\begin{aligned} &k = t'^{-2}_0(T'_0 - \sigma t'_0) , \quad l = -t'^{-2}_0(T'_0 + \sigma t'_0) , \\ &a = \nu\sigma - t'^{-1}_0 T'_0(\mu + \nu) , \quad b = \nu\sigma + t'^{-1}_0 T'_0(\mu + \nu) , \end{aligned} \tag{20}$$

where $\mu \equiv x'_0/T'_0$, $\nu \equiv -x'_0/t'_0$. Therefore one can obtain the characterizing group (k, l, a, b) of a given null geodesic $\eta(\lambda)$ from its initial data via Eq. (20). Suppose $\widehat{\eta}(\lambda)$ is any null geodesic tracing out the same spatial path as that of $\eta(\lambda)$, then $\widehat{\eta}''(\lambda) = \eta''(\lambda)$ and $\widehat{x}(\lambda) = x(\lambda)$ implies

$$[\widehat{\alpha}(\widehat{a}\mathrm{e}^{\sigma\lambda} + \widehat{b}\mathrm{e}^{-\sigma\lambda}) + 2\beta\sigma](\widehat{k}\mathrm{e}^{\sigma\lambda} + \widehat{l}\mathrm{e}^{-\sigma\lambda})^{-1} = [\alpha(a\mathrm{e}^{\sigma\lambda} + b\mathrm{e}^{-\sigma\lambda}) + 2\beta\sigma](k\mathrm{e}^{\sigma\lambda} + l\mathrm{e}^{-\sigma\lambda})^{-1}$$

or

$$(\widehat{k}\mathrm{e}^{2\sigma\lambda} + \widehat{l})[\alpha(a\mathrm{e}^{2\sigma\lambda} + b) + 2\beta\sigma\mathrm{e}^{\sigma\lambda}] = (k\mathrm{e}^{2\sigma\lambda} + l)[\alpha(\widehat{a}\mathrm{e}^{2\sigma\lambda} + \widehat{b}) + 2\beta\sigma\mathrm{e}^{\sigma\lambda}] .$$

The two sides of this equation are both polynomials of $\mathrm{e}^{\sigma\lambda}$, and comparision of coefficients gives

$$\widehat{k} = k , \quad \widehat{l} = l , \quad \widehat{a} = a , \quad \widehat{b} = b .$$

This implies that $\widehat{\eta}'(\lambda) = \eta'(\lambda)$ and hence $\widehat{\eta}(\lambda) = \eta(\lambda)$. We therefore conclude that there exists no null geodesic other than $\eta(\lambda)$ itself that traces out the same spatial path as that of $\eta(\lambda)$ unless $\eta''(\lambda)$ reduces to a point. Consequently, we have found a spacetime in which null geodesics tracing out the same spatial path always take the same coordinate time, yet the metric coefficients are not even conformally time-independent. In order to obtain a revised version of the sufficient part of the quoted statement, one should replace not only "time-independent" by "conformally time-independent", but also "null signals" by "null curves". This, however, has much less physical significance.

III. FURTHER DISCUSSION

The metric (1) can be rewritten as

$$ds^2 = (\alpha t + \beta x)^{-2}[-dt^2 + dx^2 + (\alpha t + \beta x)^2 e^{2A(y,z)}(dy^2 + dz^2)] ,$$

and hence is conformal to the following metric:

$$d\bar{s}^2 = -dt^2 + dx^2 + (\alpha t + \beta x)^2 e^{2A(y,z)}(dy^2 + dz^2) . \tag{21}$$

It can be shown that metric (21) is flat if and only if

$$\frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = (\alpha^2 - \beta^2)e^{2A} . \tag{22}$$

There exist plenty of functions $A(y, z)$ that satisfy Eq. (22). Since null geodesics are invariant under conformal transformations, the conclusion about the behavior of null geodesics obtained in section II also applies to metric (21), i.e., null geodesics of metric (21) tracing out the same spatial path always take the same coordinate time, yet the metric coefficients are not even conformally time-independent. The important point is that metric (21) can be made flat by taking $A(y, z)$ to be a solution to Eq. (22), i.e., the line element (21) can be made essentially the same as the explicitly flat line element $-dt^2 + dx^2 + dy^2 + dz^2$, the apparent difference being simply due to the different choices of coordinates. We then conclude that the "same path same time" condition is not sufficient to guarantee the time-independence of the metric coefficients even in Minkowski spaectime! This further reveals the drawback of expressing statements in terms of coordinate language.

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