Volume 5, Number 7
1004-423X/96/05070481-12

ACTA PHYSICA SINICA
(Overseas Edition)

July, 1996
© 1996 Chin. Phys. Soc.

THE TWO-BOSON SCATTERING PHASE SHIFTS IN THE GENERAL ϕ^6 THEORY WITH THE GAUSSIAN WAVE-FUNCTIONAL APPROACH*

LU WEN-FA(卢文发)
Department of Physics, Fudan University, Shanghai 200433, China

XU BO-WEI(许伯威)
Department of Physics, Shanghai Jiaotong University, Shanghai 200030, China

ZHANG YU-MEI(章豫梅)
Department of Physics, Tongji University, Shanghai 200092, China

(Received 19 July 1995)

The two-particle scattering phase shifts with a $\lambda\phi^4 + \xi\phi^6$ interaction are computed in $(1+1)$ and $(2+1)$ dimensions using the Gaussian wave-functional approach. Through numerical analysis, we give the phase-shift curves with respect to the scattering energy for most cases of this theory. In $(1+1)$ and $(2+1)$ dimensions, for the symmetry vacuum the phase shifts are positive or negative depending on the signs of the renormalized parameter, while for the asymmetric vacuum the phase shifts are negative.

PACC: 1110; 0370

I. INTRODUCTION

Scattering process serves as a major source of information about the properties of various elementary particles, while phase shift is a crucial quantity characterizing this process. In this paper, we shall give phase shifts of the scattering two-particle states in the general ϕ^6 theory with the Gaussian wave-functional (GWF) approach.

The general ϕ^6 theory[1],

$$\mathcal{L}_x = \frac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x - \frac{1}{2}m^2\phi_x^2 - \lambda\phi_x^4 - \xi\phi^6 \ , \tag{1}$$

has three free parameters m, λ and ξ. It is just constructed by adding the ϕ^6 term to the usual $\lambda\phi^4$ field theory.[2] For the latter, due to the requirement from the stability of the vacuum, λ must be positive and accordingly there exist no bound states. However, adding the term $(-\xi\phi^6)$, λ can be either positive or negative without compromising the stability of the general ϕ^6 theory[1,3,4], and consequently the physics of the ϕ^6 theory is quite fruitful. For the multi-component versions of this theory, extensive investigations with

*Project supported by the National Natural Science Foundation of China.

$1/N$ expansion[5-8] or the GWF approach[9] have been made, producing a few interesting results such as the existence of scale-invariant theory and the ultraviolet fixed point. As for the single-component version, Stevenson and Roditi have studied the stability and structure of vacuum as well as bound states in detail by the GWF approach[1], and recently the same problems have been investigated via the coupled cluster method.[10]

On the other hand, the scattering problem is, traditionally, coped with mainly by the covariant perturbation technique. Recently, the developments of the GWF approach has hinted that it is possible to non-perturbatively calculate phase shift in quantum field theory. In the last decade, the GWF approach in Schrödinger picture field theory, which was gradually developed by Schiff[11], Rosen[12], Barnes and Ghandour[13], etc., has become a powerful tool for investigating the vacuum structure[1,2,14-18] and searching for the bound states in quantum field theory.[1,19-21] Furthermore, we feel that the two-particle state energy may be analysed to extract phase shift. As an attempt, we have investigated the scattering state in the sinh-Gordon, sine-Gordon and $\lambda\phi^4$ models with this approach.[22-24] Here, based on Ref. [1], we continue to study the single-component version of the general ϕ^6 theory Eq. (1).

In the following, i.e., in section II, we first introduce some results on the ground state and two-particle state in Ref. [1] (in terms of our notations), and then we give briefly the phase shift expressions (A detailed derivation of the phase-shift expression in $(2+1)$ dimensions will be given in the appendix. The case of $(1+1)$ dimensions has been discussed in Ref. [22]). Following it, the dependence of the phase shift upon the scattering energy will be numerically analyzed in detail in section III. In the end, we give the conclusion.

II. THE PHASE-SHIFT EXPRESSIONS

Corresponding to Eq. (1), the Hamiltonian is

$$H = \int_x \mathcal{H}_x = \int_x \{\frac{1}{2}\Pi_x^2 + \frac{1}{2}(\nabla\phi_x)^2 + \frac{1}{2}m^2\phi_x^2 + \lambda\phi_x^4 + \xi\phi_x^6\} , \tag{2}$$

where $\int_x = \int \mathrm{d}x = \int \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_D$ and ∇ is the gradient operator in D-dimensional space. The parameters m, λ and ξ are the bare mass and the two bare coupling parameters respectively. In Eq. (2), $\Pi_x = -\mathrm{i}\dfrac{\delta}{\delta\phi_x}$ is conjugate to the scalar field $\phi(x) = \phi_x(x = (x_1, x_2, \cdots, x_d)$ is the coordinate in D-dimensional position space).

From Ref. [1], the Gaussian vacuum state is

$$|\varphi_0\rangle = N_f \exp\{-\frac{1}{2}\int_{y,z} (\phi_y - \varphi_0) f_{yz} (\phi_z - \varphi_0)\} , \tag{3}$$

where $\int_{y,z} = \int \mathrm{d}y\mathrm{d}z = \int \mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_D\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_D$, N_f is some normalization constant, φ_0 is a classical constant field and

$$f_{yz} = \int \frac{\mathrm{d}p}{(2\pi)^D} f(p) \mathrm{e}^{\mathrm{i}p(y-z)} \tag{4}$$

with the one-particle variational energy

$$f(p) = \sqrt{p^2 + \mu^2(\varphi_0)} \tag{5}$$

and $p = (p_1, p_2, \cdots, p_D)$ a vector in D-dimensional p-space. In Eq. (5),

$$\begin{aligned}\mu^2(\varphi_0) &= m^2 + 6\lambda f_{xx}^{-1} + 12\lambda\varphi_0^2 + 30\xi(\phi_0^4 + 3f_{xx}^{-1}\phi_0^2 + \frac{3}{2}[f_{xx}^{-1}]^2) \\ &= m_R^2 - \frac{3}{\pi}\lambda_r m_R^{D-1} L_1(\frac{\mu^2}{m_R^2}) + \frac{45\xi}{8\pi^2} m_R^{2D-2} L_1^2(\frac{\mu^2}{m_R^2}) \\ &\quad + [12\lambda_r - \frac{45\xi}{\pi} m_R^{D-1} L_1(\frac{\mu^2}{m_R^2})]\varphi_0^2 + 30\xi\varphi_0^4 ,\end{aligned} \tag{6}$$

where f_{xy}^{-1} is the inverse of f_{xy}, $m_R^2 = \mu^2(\varphi_0 = 0)$, $\lambda_r = \lambda + \frac{15}{2}\xi f_{xx}^{-1}|_{\varphi_0=0}$ is the renormalized parameter corresponding to the bare parameter λ and

$$L_1(\frac{\mu^2}{m_R^2}) = \begin{cases} \ln(\frac{\mu^2}{m_R^2}) , & D = 1 , \\ ((\frac{\mu}{m_R}) - 1) , & D = 2 . \end{cases} \tag{7}$$

The quantum vacuum can be obtained by extremizing the Gaussian effective potential[1] with respect to φ_0 and the result is

$$\varphi_0 = 0 \tag{8}$$

or for $\varphi_0 \neq 0$

$$\begin{aligned} & m_R^2 - \frac{3}{\pi}\lambda_r m_R^{D-1} L_1(\frac{\mu^2}{m_R^2}) + \frac{45\xi}{8\pi^2} m_R^{2D-2} L_1^2(\frac{\mu^2}{m_R^2}) \\ & + [4\lambda - \frac{15\xi}{\pi} m_R^{D-1} L_1(\frac{\mu^2}{m_R^2})]\varphi_0^2 + 6\xi\varphi_0^4 = 0 . \end{aligned} \tag{9}$$

When φ_0 satisfies Eq. (8) or (9), $\mu(\varphi_0)$ is the mass of a particle.

Based on the Gaussian vacuum Eq. (3), one can have the functional annihilation operator[4,19,21]

$$A_f(p) = (\frac{1}{2(2\pi)^D f(p)})^{1/2} \int_x e^{-ipx}[f(p)(\phi_x - \varphi_0) + \frac{\delta}{\delta\phi_x}] , \tag{10}$$

and the functional creation operator

$$A_f^\dagger(p) = (\frac{1}{2(2\pi)^D f(p)})^{1/2} \int_x e^{ipx}[f(p)(\phi_x - \varphi_0) - \frac{\delta}{\delta\phi_x}] . \tag{11}$$

Then a trial positronium-like state of two particles with the center of mass in rest can be constructed as

$$|2\rangle = \int dp\Sigma(p)|p, -p\rangle = \int dp\Sigma(p) A_f^\dagger(p) A_f^\dagger(-p)|\varphi_0\rangle , \tag{12}$$

where $\Sigma(p)$ is the Fourier transformation of an s-wave function of the two particles. Using functional integration techniques, one can calculate the total energy of the two-particle system[1]

$$m_2 = \frac{\int_x \langle 2|\mathcal{H}_x|2\rangle}{\langle 2|2\rangle} - \langle\varphi_0|H|\varphi_0\rangle$$
$$= \frac{2\int \mathrm{d}p(\Sigma(p))^2 f(p) + \frac{3\lambda_{\mathrm{eff}}(\varphi_0)}{(2\pi)^D}(\int \mathrm{d}p\frac{\Sigma(p)}{f(p)})^2}{\int \mathrm{d}p(\Sigma(p))^2}, \tag{13}$$

where the finite parameter $\lambda_{\mathrm{eff}}(\varphi_0)$ is defined as

$$\lambda_{\mathrm{eff}}(\varphi_0) = \lambda + 15\xi[\frac{1}{2}f_{xx}^{-1} + \phi_0^2]$$
$$= \lambda_r - \frac{15}{4\pi}\xi m_R^{D-1}L_1(\frac{\mu^2}{m_R^2}) + 15\xi\varphi_0^2 . \tag{14}$$

The two terms in Eq. (13) can be regarded as the kinetic energy of the two constituent particles and their interacting energy respectively. As remarked in Ref. [1], the form of Eq. (13) is exactly as that in the $\lambda\phi^4$ model[2,11,13], except that λ is replaced by $\lambda_{\mathrm{eff}}(\varphi_0)$.

Minimizing m_2 with respect to $\Sigma(p)$, one has

$$\Sigma(p) = \frac{A}{f(p)(2f(p) - m_2)}, \tag{15}$$

where A is a normalization constant, and accordingly Eq. (13) reads

$$\int \frac{\mathrm{d}p}{f^2(p)[2f(p) - m_2]} = -\frac{(2\pi)^D}{3\lambda_{\mathrm{eff}}(\varphi_0)} . \tag{16}$$

When $m_2 > 2\mu(\varphi_0)$, the positronium-like state with Eq. (15) is a scattering state and Eq. (16) contains the scattering information of the two particles. As a matter of fact, the integrand in the left hand side of Eq. (16) has a singularity $p_0 = \sqrt{(\frac{m_2}{2})^2 - \mu^2(\varphi_0)}$. So the integral of Eq. (16) can be regarded as the corresponding principal value integral plus the contribution of this pole to the integral. Through the same discretization procedure as used in Refs. [11] and [22], the integrals for $(1+1)$ and $(2+1)$ dimensions can be, respectively, written as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{(p^2 + \mu^2(\varphi_0))(2\sqrt{p^2 + \mu^2(\varphi_0)} - m_2)} = \frac{\pi}{m_2\sqrt{(\frac{m_2}{2})^2 - \mu^2(\varphi_0)}}\cot\delta$$
$$+\mathcal{P}.\mathcal{V}.\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{(p^2 + \mu^2(\varphi_0))(2\sqrt{p^2 + \mu^2(\varphi_0)} - m_2)} , \tag{17}$$

and

$$\int_0^\infty \int_0^{2\pi} \frac{\mathrm{pdpd}\varphi}{(p^2+\mu^2(\varphi_0))(2\sqrt{p^2+\mu^2(\varphi_0)}-m_2)} = \frac{2\pi^2}{m_2}\cot\delta$$
$$+\mathcal{P}.\mathcal{V}.\int_0^\infty \int_0^{2\pi} \frac{\mathrm{pdpd}\varphi}{(p^2+\mu^2(\varphi_0))(2\sqrt{p^2+\mu^2(\varphi_0)}-m_2)}\,. \tag{18}$$

An analysis on the wave function shows that the quantity δ defined above is indeed the phase shift of the two-particle state at a large distance. Thus Eqs. (17) and (18) give the relation between the effects of the singular points and the scattering phase shift. As a result, according to Eq.(16) the expressions of δ for $(1+1)$ dimensions has the form[22]

$$\cot\delta = \frac{1}{2}\sqrt{\widetilde{m}_2^2-1} - \frac{2\widetilde{m}_2\sqrt{\widetilde{m}_2^2-1}}{3\widetilde{\lambda}_{\mathrm{eff}}(\varphi_0)} - \frac{1}{\pi}\ln(\widetilde{m}_2-\sqrt{\widetilde{m}_2^2-1}) \tag{19}$$

with $\widetilde{\lambda}_{\mathrm{eff}}(\varphi_0)=\dfrac{\lambda_{\mathrm{eff}}(\varphi_0)}{\mu^2(\varphi_0)}$, and for $(2+1)$ dimensions (see the appendix)

$$\cot\delta = -\frac{4\widetilde{m}_2}{3\widetilde{\lambda}_{\mathrm{eff}}(\varphi_0)} + \frac{1}{\pi}\ln(\widetilde{m}_2-1) \tag{20}$$

with $\widetilde{\lambda}_{\mathrm{eff}}(\varphi_0)=\dfrac{\lambda_{\mathrm{eff}}(\varphi_0)}{\mu(\varphi_0)}$, where $\widetilde{m}_2=\dfrac{m_2}{2\mu(\varphi_0)}$.

The last two equations give the dependence of the phase shift δ upon the scattering energy m_2 between two particles. Next, we shall discuss it for different dimensions in detail.

III. NUMERICAL RESULTS

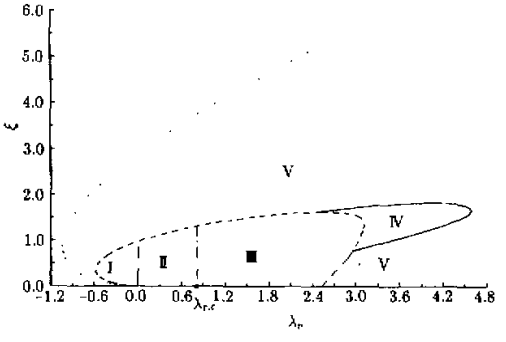

Fig. 1. The λ_r, ξ space for the (1+1)-dimensional $\lambda\phi^6$ theory from Ref. [1]. In this figure, $m_R^2=1$ and we have added to this figure the results for the phase shifts, i.e., the solid curve and the dot-dashed lines. The solid curve represents $\widetilde{\lambda}_{\mathrm{eff}}(\varphi_0)=\lambda_{r,c}$, $\lambda_{r,c}=\dfrac{4\pi}{6+3\pi}=0.81469$.

We begin with (1+1) dimensions. Because of the stability of the vacuum, the values of λ_r, ξ parameters is constrained. From Ref. [1], when $D=1$, the allowed region of the λ_r, ξ

parameter space lies to the right of the dotted curve and above the λ_r axis in Fig. 1. Here, we take (unit mass)^2 as the units of the parameters λ_r and ξ. The inside and outside of the dashed curve correspond to $\varphi_0 = 0$ and $\varphi_0 \neq 0$ vacuum respectively. For the $\varphi_0 = 0$ vacuum, one has $\lambda_{\rm eff}(\varphi_0) = \lambda_r$ and $\mu(\varphi_0) = m_R$. In this case, δ has nothing to do with the parameter ξ, which only contributes to the renormalization of the coupling constant λ. The dependence of δ upon m_2 with $m_R = 1$ is described graphically for some negative and positive values of λ_r in Fig. 2. When $\lambda_r < 0$ (area I in Fig. 1), δ decreases monotonically to zero with the increase of m_2 (the curves I, II, III, in Fig. 2) and the positive phase shift implies that the interaction between two particles is attractive. For $\lambda_r > 0$, the phase shift is negative (the curves IV, V, VI in Fig. 2), which means the interaction between two particles is repulsive, and there is a critical value $\lambda_{r,c} = \dfrac{4\pi}{6+3\pi}$ of the parameter λ_r. When $\lambda_r > \lambda_{r,c}$ (area III in Fig. 1), the phase shift has a minimum value (the curve VI in Fig. 2), while for $\lambda_r < \lambda_{r,c}$ (area II in Fig. 1) the phase shift increases up to zero with the increase of m_2 (the curve IV in Fig. 2). These are similar to those of the sine-Gordon and sinh-Gordon models.[23]

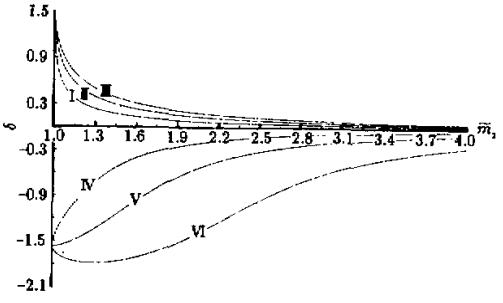

Fig. 2. The phase-shift picture for the (1+1)-dimensional ϕ^6 theory with the symmetric vacuum. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $\lambda_r = -0.20367, -0.40734, -0.61102, 0.40734, 0.81469$ and 1.62937, respectively.

For the $\varphi_0 \neq 0$ vacuum (areas IV and V in Fig. 1), the mass of a particle is $\mu(\varphi_0)$, which satisfies the equation

$$\mu^2(\varphi_0) = m_R^2 - \frac{3}{\pi}\lambda_r \ln(\frac{\mu^2}{m_R^2}) + \frac{45\xi}{8\pi^2}\ln^2(\frac{\mu^2}{m_R^2}) + [12\lambda_r - \frac{45\xi}{\pi}\ln(\frac{\mu^2}{m_R^2})]\varphi_0^2 + 30\xi\varphi_0^4 , \tag{21}$$

and

$$\lambda_{\rm eff}(\varphi_0) = \lambda_r - \frac{15}{4\pi}\xi\ln(\frac{\mu^2}{m_R^2}) + 15\xi\varphi_0^2 , \tag{22}$$

where φ_0 is determined by the equation

$$m_R^2 - \frac{3}{\pi}\lambda_r \ln(\frac{\mu^2}{m_R^2}) + \frac{45\xi}{8\pi^2}\ln^2(\frac{\mu^2}{m_R^2}) + [4\lambda - \frac{15\xi}{\pi}\ln(\frac{\mu^2}{m_R^2})]\varphi_0^2 + 6\xi\varphi_0^4 = 0 . \tag{23}$$

In contrast to the $\varphi_0 = 0$ case, the phase shift in the present case is related not only to λ_r but also to ξ. Moreover, Ref. [1] has proven that for the $\varphi_0 \neq 0$ vacuum $\lambda_{\text{eff}}(\varphi_0)$ is solely greater than zero. So, the phase shift is always negative and accordingly the interaction between two particles with the asymmetric vacuum is repulsive. According to Eqs. (19), (21), (22) and (23), the δ-m_2 curves is plotted for some values of (λ_r, ξ) with $m_R^2 = 1$ in Fig. 3. The change of δ with m_2 in this case is still analogous to those for the case $\lambda_r > 0$ with the $\varphi_0 = 0$ vacuum. However, $\widetilde{\lambda}_{\text{eff}}(\varphi_0) = \dfrac{4\pi}{6+3\pi}$ corresponds not to a vertical line but to a curve in λ_r-ξ space, which is illustrated in Fig. 1 as a solid curve. In the region between the dashed and the solid curves (area IV in Fig. 1), $\widetilde{\lambda}_{\text{eff}}(\varphi_0)$ is greater than $\lambda_{r,c}$ and the corresponding phase shift (the curve IV in Fig. 3) has a minimum. Outside this region, i.e., in the area V in Fig. 1, $\widetilde{\lambda}_{\text{eff}}(\varphi_0)$ is smaller than $\lambda_{r,c}$ and the phase shift is increased monotonously.

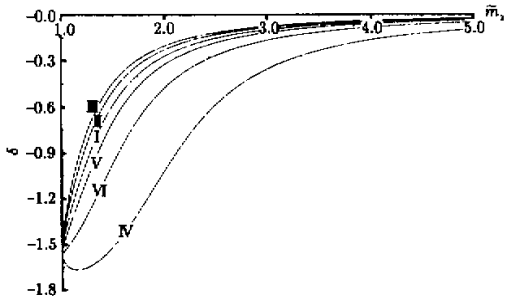

Fig. 3. The phase-shift picture for (1+1)-dimensional, asymmetric vacuum. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $(\lambda_r, \xi) = (-0.7, 0.5), (0.0, 2.0), (3.1045, 4.5), (3.1045, 1.4349), (3.1045, 0.5)$ and $(5.0, 1.4349)$, respectively.

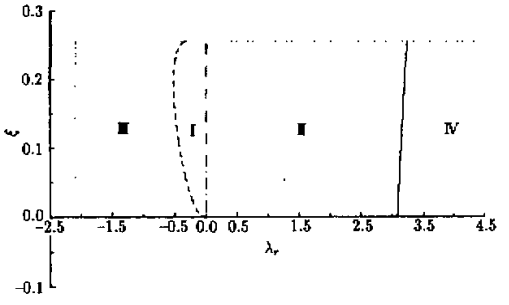

Fig. 4. The λ_r, ξ space for the (2+1)-dimensional $\lambda\phi^6$ theory from Ref. [1]. In this figure, $m_R^2 = 1$ and we have added to this figure the results for the phase shifts, i.e., the solid curve and the dot-dashed line. The dotted horizontal line corresponds to $\xi = 0.254916$. The dotted vertical line corresponds to $\lambda_r = -2.09439$.

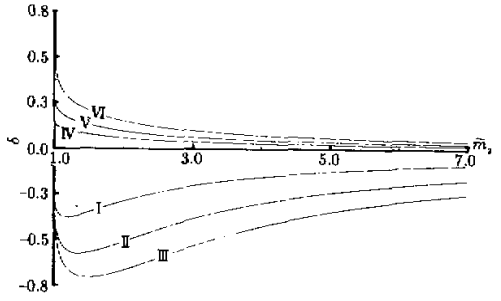

Fig. 5. The phase-shift picture for the (2+1)-dimensional ϕ^6 theory with the symmetric vacuum. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $\lambda_r = 0.8, 1.5, 2.1, -0.15, -0.25$ and -0.4, respectively.

Now we turn to discuss (2+1) dimensions. In this case, the allowed region of the λ_r-ξ space is reproduced from Ref. [1] in Fig. 4, where λ_r is regarded as divided by unit mass. The region is the semi-infinite strip $\lambda_r > -2.0944$ and $0 < \xi < 0.2549$ (enclosed by the dotted line and λ_r axis). In the aeras I and II, the $\varphi_0 = 0$ vacuum is stable, while in the aeras III and IV, the stable vacuum locates at $\varphi_0 \neq 0$. For the $\varphi_0 = 0$ vacuum, when $m_R^2 \neq 0$ or -1, one has also $\lambda_{\text{eff}}(\varphi_0) = \lambda_r, \mu(\varphi_0) = m_R$. This case is the same as in the corresponding case in (1+1) dimensions, except that when $\lambda_r > 0$ the phase shift has a minimum. These are illustrated in Fig. 5 according to Eq. (20). For the $\varphi_0 \neq 0$ vacuum, i.e., the spontaneous symmetry broken phase, $\lambda_{\text{eff}}(\varphi_0)$ is again positive. Equations (6), (9) and (14) can be explicitly written, respectively, as

$$\mu^2(\varphi_0) = m_R^2 - \frac{3}{\pi}\lambda_r m_R\left(\frac{\mu}{m_R} - 1\right) + \frac{45\xi}{8\pi^2}m_R^2\left(\frac{\mu}{m_R} - 1\right)^2 + \left[12\lambda_r - \frac{45\xi}{\pi}m_R\left(\frac{\mu}{m_R} - 1\right)\right]\varphi_0^2 + 30\xi\varphi_0^4 , \tag{24}$$

$$m_R^2 - \frac{3}{\pi}\lambda_r m_R\left(\frac{\mu}{m_R} - 1\right) + \frac{45\xi}{8\pi^2}m_R^2\left(\frac{\mu}{m_R} - 1\right)^2 + \left[4\lambda_r - \frac{15\xi}{\pi}m_R\left(\frac{\mu}{m_R} - 1\right)\right]\varphi_0^2 + 6\xi\varphi_0^4 = 0 \tag{25}$$

and

$$\lambda_{\text{eff}}(\varphi_0) = \lambda_r - \frac{15}{4\pi}\xi m_R\left(\frac{\mu}{m_R} - 1\right) + 15\xi\varphi_0^2 . \tag{26}$$

From Eq. (20), together with Eqs. (24), (25) and (26), the phase shifts for some values of the parameters λ_r and ξ is shown in Fig. 6. It is clear that δ is negative and has always a minimum. According to Ref. [1], as far as the (2+1)-dimensional ϕ^6 theory is concerned, we have $m_R^2 = 0$, and the $m_R^2 = -1$ cases are also meaningful. When $m_R^2 = 0$, there are three

cases; $\lambda_r < 0, \lambda_r > 0$ and $\lambda_r = 0$. Here, we consider the former two cases. The case $\lambda_r < 0$ is identical to the case $\lambda_r = -\pi(1 + \frac{45\xi}{8\pi^2})$ with $m_R^2 = 1$ (see Ref. [1]). In this case, $\lambda_r < -\frac{15\xi}{4\pi}$ always holds and therefore the vacuum is always spontaneous symmetry broken. This case already included in the above analysis of the $\varphi_0 \neq 0$ vacuum. For the sake of completeness,

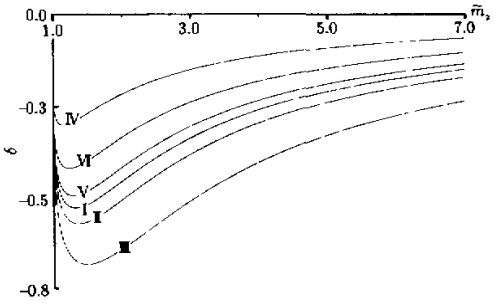

Fig. 6. The phase-shift picture for the (2+1)-dimensional ϕ^6 theory with the asymmetric vacuum. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $(\lambda_r, \xi) = (-2.0, 0.095), (-1.3, 0.12), (-0.6, 0.21), (-0.3, 0.02), (3.25, 0.21)$ and $(4.0, 0.12)$, respectively.

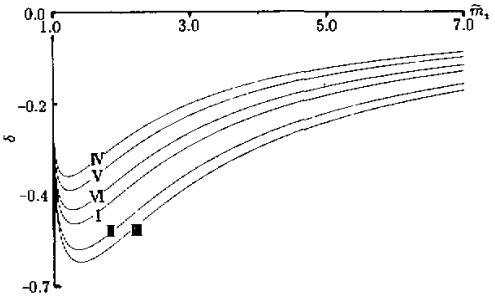

Fig. 7. The phase-shift picture for the (2+1)-dimensional ϕ^6 theory with $m_R^2 = 0$ and $\lambda_r \neq 0$. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $(\lambda_r, \xi) = (-2.1, 0.085), (-1.3, 0.12), (-0.3, 0.14), (1.5, 0.115), (3.25, 0.15)$ and (4.0, 0.2), respectively.

we have still plotted three δ-m_2 curves for this case in Fig. 7 (the curves I, II and III). As for the case $\lambda_r > 0$, the vacuum is also spontaneous symmetry broken and the mass of a particle is $\mu(\varphi_0)$, which is the solution of the equation

$$\mu^2(\varphi_0) = -\frac{3}{\pi}\lambda_r\mu + \frac{45\xi}{8\pi^2}\mu^2 + [12\lambda_r - \frac{45\xi}{\pi}\mu]\varphi_0^2 + 30\xi\varphi_0^4 , \tag{27}$$

where φ_0 is determined by

$$-\frac{3}{\pi}\lambda_r\mu + \frac{45\xi}{8\pi^2}\mu^2 + [4\lambda_r - \frac{15\xi}{\pi}\mu]\varphi_0^2 + 6\xi\varphi_0^4 = 0 . \tag{28}$$

Now, $\lambda_{\rm eff}(\varphi_0)$ becomes

$$\lambda_{\rm eff}(\varphi_0) = \lambda_r - \frac{15}{4\pi}\xi\mu + 15\xi\varphi_0^2 \,. \tag{29}$$

As an illustration, three phase-shift curves are plotted in Fig. 7. When $m_R^2 = -1$, taking $\lambda_r = -\frac{15\xi}{4\pi}m_R$, the theory is still made sense and there exists only the spontaneous symmetry broken vacuum from Ref. [1]. In this case, we have

$$\mu^2(\varphi_0) = -1 + \frac{45\xi}{8\pi^2} + \frac{45\xi}{8\pi^2}\mu^2 - \frac{45\xi}{\pi}\mu\varphi_0^2 + 30\xi\varphi_0^4 \,, \tag{30}$$

$$-1 + \frac{45\xi}{8\pi^2} + \frac{45\xi}{8\pi^2}\mu^2 - \frac{15\xi}{\pi}\mu\varphi_0^2 + 6\xi\varphi_0^4 = 0 \,, \tag{31}$$

and

$$\lambda_{\rm eff}(\varphi_0) = -\frac{15}{4\pi}\xi\mu + 15\xi\varphi_0^2 \,. \tag{32}$$

The δ-m_2 curves are shown in Fig. 8. In all the cases involved in Figs. 6, 7 and 8, the phase shifts have minima.

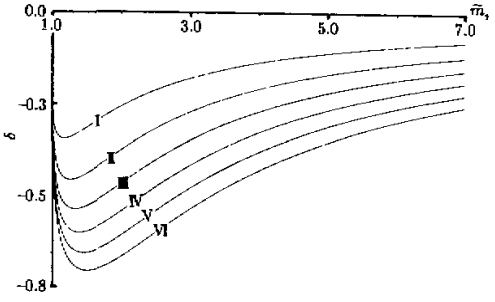

Fig. 8. The phase-shift picture for the (2+1)-dimensional ϕ^6 theory with $m_R^2 = -1$. The curves I, II, III, IV, V and VI represent δ-m_2 curves with $\xi = 0.04, 0.08, 0.12, 0.16, 0.20$ and 0.24, respectively.

IV. CONCLUSIONS

In this paper, we have computed the phase shifts for most cases in the general ϕ^6 model with the GWF approach. From the phase shifts, one can easily obtain a closed expression for the differential scattering cross section in the center-of-mass system. The method in this paper is non-perturbative. Its extensions to the theories which contain the scalar Fermi and gauge fields will be viable and meaningful. In addition, the (2+1)-dimensional case of the general ϕ^6 theory with $m_R^2 = 0$ and $\lambda_r = 0$ will be interesting and important because it amounts to a scale-invariant massless theory. How to tackle the infrared singularity appearing in the massless theory is crucial to this case, and we have no more discussions about it in this paper.

APPENDIX A

In this appendix, we shall derive Eq. (20). This derivation mainly consists in coping with the integral in Eq.(16) ($m_2 > 2\mu^2(\varphi_0)$)

$$I = \int_0^\infty \frac{2\pi p\mathrm{d}p}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]} , \tag{A1}$$

and analysing the large-x behaviours of the wave function

$$\Sigma(x) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}^2 p \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\Sigma(p) . \tag{A2}$$

Let us begin with the integral I. We appeal to Schiff's procedure of discretization.[10,21] Imaging that the system is limited in a sphere with an arbitrarily large radius R, one can have the integral I over p replaced by a summation over the discrete momenta $p_n = \dfrac{n\pi}{R}$ (n is an integer):

$$I = \frac{2\pi^2}{R}\sum_n \frac{p_n}{[p_n^2+\mu^2(\varphi_0)][2\sqrt{p_n^2+\mu^2(\varphi_0)}-m_2]} . \tag{A3}$$

Singling out the term with p_N closest to the pole p_0 from the summation in Eq. (A3), one can see that the rest of the summation is to be restored to a principal value integral, i.e.,

$$\begin{aligned} I =& \frac{\pi^2}{\sqrt{\bar{p}^2+\mu^2(\varphi_0)}}\frac{2\bar{p}}{R}\frac{1}{\sqrt{\bar{p}^2+\mu^2(\varphi_0)}[2\sqrt{\bar{p}^2+\mu^2(\varphi_0)}-m_2]} \\ &+ \mathcal{P}\int_0^\infty \frac{2\pi p\mathrm{d}p}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]} . \end{aligned} \tag{A4}$$

Defining

$$\cot\delta = \frac{2\bar{p}}{R}\frac{1}{\sqrt{\bar{p}^2+\mu^2(\varphi_0)}[2\sqrt{\bar{p}^2+\mu^2(\varphi_0)}-m_2]} \tag{A5}$$

we obtain Eq. (18). Analysis of the asymptotic behaviours of the wave function Eq. (A2) at infinity will show that δ in Eq. (A5) is just the phase shift. to this point, we rewrite $\Sigma(x)$ as

$$\Sigma(x) \propto \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}pr\cos\theta}p\mathrm{d}p\mathrm{d}\theta}{\sqrt{p^2+\mu^2(\varphi_0)}[2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]} . \tag{A6}$$

Employing the formula

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}pr\cos\theta}\mathrm{d}\theta = 2\pi J_0(pr) , \tag{A7}$$

with $J_0(pr)$ the zero-order Bessel function, and dealing with the pole as above lead to

$$\Sigma(x) \propto \frac{J_0(pr)}{4}\cot\delta + \frac{1}{2\pi}\mathcal{P}\int_0^\infty \frac{J_0(pr)p\mathrm{d}p}{\sqrt{p^2+\mu^2(\varphi_0)}[2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]} . \tag{A8}$$

When $r\to\infty$, $J_0(pr)\to\sqrt{\dfrac{2}{\pi pr}}\cos(pr-\dfrac{\pi}{4})$, the contributions from the pole dominate the

principal-value integral, and therefore we have

$$\begin{aligned}\Sigma(x) &\propto \frac{1}{4}\cot\delta\sqrt{\frac{2}{\pi p_N r}}\cos(p_N r-\frac{\pi}{4})+\frac{1}{4\pi}\sqrt{\frac{2}{\pi p_0 r}}\mathcal{P}\int_0^\infty \frac{\cos(pr-\frac{\pi}{4})\mathrm{d}p}{p-p_0}\\ &=\frac{1}{4}\cot\delta\sqrt{\frac{2}{\pi p_0 r}}\cos(p_0 r-\frac{\pi}{4})-\frac{1}{4}\sqrt{\frac{2}{\pi p_0 r}}\sin(p_0 r-\frac{\pi}{4})\,.\end{aligned} \tag{A9}$$

Because the s-wave function of a free particle in polar coordinates system is just

$$\sqrt{\frac{2}{\pi pr}}\cos(pr-\frac{\pi}{4})\,, \tag{A10}$$

it is evident that δ is the s-wave phase shift with the scattering energy m_2.

REFERENCES

[1] P. M. Stevenson and I. Roditi, *Phys. Rev.*, **D33**(1986), 2305.
[2] P. M. Stevenson, *Phys. Rev.*, **D32**(1985), 1389.
[3] J. Dimock and J. P. Eckmann, *Commun. Math. Phys.*, **51**(1976), 41.
[4] T. Barnes and G. J. Daniell, *Phys. Lett.*, **B142**(1984), 188.
[5] P. K. Townsend, *Phys. Rev.*, **D12**(1976), 2269; *ibid.*, **D14**(1976), 1715.
[6] T. Appelquist and U. Heinz, *Phys.* Rev., **D25**(1982). 2620.
[7] R. D. Pisarski, *Phys. Rev. Lett.*, **48**(1982), 574.
[8] R. Gudmunlsdottir, G. Rydnell and P. Salomonson, *Phys. Rev. Lett.*, **53**(1984), 2529.
[9] W. A. Bardeen, M. Moshe and M. Bander, *Phys. Rev. Lett.*, **52**(1984), 1188.
[10] M. Funke and H. G. Kümmel, *Phys. Rev.*, **D50**(1994), 991.
[11] L. I. Schiff, *Phys. Rev.*, **130**(1963), 458.
[12] G. Rosen, *Phys.* Rev., **173**(1968), 1632.
[13] T. Barnes and G. I. Ghandour, *Phys. Rev.*, **D22**(1980), 924.
[14] P. M. Stevenson, G. A. Hajj and J. F. Reed, *Phys.* Rev., **D34**(1986), 3117.
[15] P. M. Stevenson, B. Allés and R. Tarrach, *Phys. Rev.*, **D35**(1987), 2407.
[16] R. Ingermanson, *Nucl. Phys.*, **B266**(1986), 620.
[17] S. K. Kim, J. Yang, K. S. Soh and J. H. Yee, *Phys. Rev.*, **D40**(1989), 2647.
[18] S. K. Kim, K. S. Soh and J. H. Yee, *Phys. Rev.*, **D41**(1990), 1345.
[19] J. W. Darewych, M. Horbatsch and R. Koniuk, *Phys. Rev. Lett.*, **54**(1985), 2188; *Phys. Rev.*, **D33** (1986), 2316.
[20] S. K. Kim, W. Namgung, K. S. Soh and J. H. Yee, *Ann. Phys.*, **214**(1992), 142.
[21] W. F. Lu and B. W. Xu, *Chin. Phys. Lett.*, **10**(1993), 8.
[22] W. F. Lu, B. W. Xu and Y. M. Zhang, *Z. Phys.*, **C58**(1993), 499.
[23] W. F. Lu, B. W. Xu and Y. M. Zhang, *Phys. Lett.*, **B309**(1993), 109.
[24] W. F. Lu, B. W. Xu and Y. M. Zhang, *Phys. Rev.*, **D49**(1994), 5625.