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CONJUGATE POINTS ALONG NULL GEODESICS IN A CLASS OF SPACETIMES*

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The existence and location of conjugate points along null geodesics in Taub's vacuum spacetime is investigated in detail. It is shown that every null geodesic η not confined in a t-z plane contains two pairs of segments $(\mathcal{M}, \widetilde{\mathcal{M}})$ and $(\mathcal{N}, \widetilde{\mathcal{N}})$ such that each point p in $\mathcal{M}$ (resp. $\mathcal{N}$) has a unique conjugate point $\widetilde{p}$ along η that is located in $\widetilde{\mathcal{M}}$ (resp. $\widetilde{\mathcal{N}}$) and vice versa, and what is more interesting, if p and $\widetilde{p}$ are conjugate points along η with $\widetilde{p} \in J^+(p)$, then $\widetilde{p} \in I^+(p)$. This presents a realistic example illustrating that there do exist null geodesics emanating from p that can get into $I^+(p)$ before meeting a point conjugate to p. All results are generalized to a class of spacetimes.

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I. INTRODUCTION

It is well known in Riemannian geometry that the existence of conjugate points characterizes the stage at which a geodesic fails, locally, to be the minimum length curve connecting two points. In Lorentzian geometry (a manifold with a Lorentzian metric), two sorts of physically interesting geodesics, namely time-like geodesics and null geodesics, might admit conjugate points, both having important physical relevance. The existence of conjugate points on a time-like geodesic characterizes the stage at which it fails to be a local maximum of proper time between two points, while the physical significance of conjugate points on null geodesics are twofold. The first involves the Fermat's principle in general relativity. Suppose $\gamma(t)$ is the world line of an observer. Consider all null curves starting at $p \notin \gamma(t)$ and intersecting $\gamma(t)$ at some point $\gamma(\hat{t})$ ($\hat{t}$ is referred to as the "arrival time" of the specific null curve η), then it is shown in Ref. [1] that, (i) the arrival time is an extremum if and only if η is a null geodesic; (ii) it is a strict local minimum if η is free of conjugate points, and is a saddle point (i.e., neither a minimum nor a maximum) if η passes through a conjugate point before arriving at γ. The second physical significance of conjugate points on null geodesics is that a null geodesic η emanating from p will fail to remain on the boundary of $I^+(p)$, the chronological future of p, and enter into $I^+(p)$. This phenomenon has a lot of interesting physical relevance. For instance, if p is the event of a lightning, η and ξ (a time-like curve) are, respectively, the world lines of its light and sound, $q \in I^+(p)$ is the intersecting point

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of η and ξ, then, contrary to what is well known by ordinary physics in flat spacetime, an observer with a world line containing q will receive the light and the sound at the same time! As a second example, consider a movie projector (together with its screen) with ξ as its world line and one turns its lens back to the screen, the light signal emitted from the projector will arrive at the screen and form the movie picture on it some time later![2] Conjugate points on null geodesics also have effects in astrophysics, an important example being the phenomenon of gravitational lens. An interesting question then arise: Is passing a point conjugate to its starting point p a necessary condition for a null geodesic to enter $I^+(p)$? The answer is negative, as is shown by a simple example[3] in which a null geodesic in a two-dimensional Einstein static universe emanating from p enters $I^+(p)$ without passing any conjugate point (there is no conjugate point at all since the metric is flat). Nonetheless, this example is too artificial and, in particular, the entrance of the null geodesic into $I^+(p)$ is achieved only because one pastes the R^1 to obtain an S^1. It is then natural to ask whether there is a more realistic spacetime (say, a solution to vacuum Einstein equations) with a background manifold as simple as R^4 in which a null geodesic from p does contain a point $\widetilde{p}$ conjugate to p but it enters $I^+(p)$ before meeting $\widetilde{p}$. We will show in Sec.II that Taub's plane symmetric vacuum spacetime $(\mathrm{R}^4, g_{\mathrm{Taub}})$ is such a spacetime, and the result will then be generalized to a class of spacetimes. We, therefore, conclude that, although a null geodesic starting at p will enter $I^+(p)$ after meeting a point $\widetilde{p}$ conjugate to p, in many cases the entrance into $I^+(p)$ cannot be attributed to the existence of the conjugate point $\widetilde{p}$. As a by-product, a detailed investigation on the location (distribution) of conjugate points along null geodesics in Taub's vacuum spacetime is given and put on record.

II. LOCATION OF CONJUGATE POINTS ALONG NULL GEODESICS IN TAUB SPACETIME

The line element of Taub's plane symmetric vacuum metric reads

$$\mathrm{d}S^2_{\mathrm{Taub}} = z^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}z^2) + z(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \quad z > 0\,. \tag{1}$$

Since null geodesics and conjugate points along null geodesics are conformally invariant[4], we consider the spacetime (R^4, g) with the following conformal metric instead

$$\mathrm{d}S^2 = z^{-3/2}(-\mathrm{d}t^2 + \mathrm{d}z^2) + (\mathrm{d}x^2 + \mathrm{d}y^2)\,, \quad z > 0\,. \tag{2}$$

The existence of the Killing vector fields $\partial/\partial t$, $\partial/\partial x$, and $\partial/\partial y$ gives rise to the following constants E, P_1, and P_2 along a geodesic:

$$E := z^{-3/2}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\,, \quad P_1 := \frac{\mathrm{d}x}{\mathrm{d}\lambda}\,, \quad P_2 := \frac{\mathrm{d}y}{\mathrm{d}\lambda}\,. \tag{3}$$

For a null geodesic one has

$$z^{-3/2}[-(\frac{\mathrm{d}t}{\mathrm{d}\lambda})^2 + (\frac{\mathrm{d}z}{\mathrm{d}\lambda})^2] + (\frac{\mathrm{d}x}{\mathrm{d}\lambda})^2 + (\frac{\mathrm{d}y}{\mathrm{d}\lambda})^2 = z^{-3/2}[-E^2z^3 + (\frac{\mathrm{d}z}{\mathrm{d}\lambda})^2] + P^2 = 0\,,$$

where

$$P^2 := P_1^2 + P_2^2 ,$$

and hence

$$(\frac{\mathrm{d}z}{\mathrm{d}\lambda})^2 = E^2 z^{3/2}(z^{3/2} - Q) , \tag{4}$$

where

$$Q := P^2/E^2 . \tag{5}$$

It follows from Eqs. (4) and (5) that each null geodesic with $Q \neq 0$ has a turning point p with

$$z_T = Q^{2/3} ,$$

while each null geodesic with $Q = 0$ lies on a t-z plane and reaches the singularity $z = 0$. It is manifest that null geodesics with $Q = 0$ contain no conjugate points, therefore the essential issue is to find out the condition under which two points on a null geodesic with $Q \neq 0$ are conjugate to each other along that geodesic.

Let $\eta(\lambda)$ be a future directed null geodesic with $Q \neq 0$ and λ an affine parameter. let $p \in \eta(\lambda)$. Denote by $\gamma(\lambda)$ and $\beta(\lambda)$, respectively, the projections of $\eta(\lambda)$ on a t-z plane Σ_{tz} and an x-y plane Σ_{xy}. Since the induced metric on $\sum_{tz}$ is independent of x and y and that on Σ_{xy} independent of t and z, $\gamma(\lambda)$ and $\beta(\lambda)$ are also geodesics in Σ_{tz} and Σ_{xy}, respectively, with λ an affine parameter. The fact that the induced metric on Σ_{xy} is flat ensures that $\beta(\lambda)$ contains no conjugate points, thus it follows from the definition of conjugate points[5,6] that $\widetilde{p} \in \eta$ is conjugate to p along η if and only if the projection of $\widetilde{p}$ on Σ_{tz} is conjugate to that of p along $\gamma(\lambda)$. Therefore the problem is reduced to searching for the condition under which two points on a time-like geodesic $\gamma(\lambda)$ in a two-dimensional manifold Σ_{tz} with metric

$$\mathrm{d}S_{tz}^2 = z^{-3/2}(-\mathrm{d}t^2 + \mathrm{d}z^2)$$

are conjugate to each other along $\gamma(\lambda)$. Such a time-like geodesic is characterized by two constants Q and E, which are still defined by Eqs. (3) and (5), respectively, where $-P^2$ can also be understood as the norm of its tangent vector, and Eq. (4) still holds. Note that E depends on the affine parameter λ, while Q is λ-independent and depends only on the image of the map $\gamma : I \subset \mathrm{R} \to \mathrm{R}^4$ where I is an interval of R. Note also that γ is symmetric about the turning point.

Let $p_0 = (t_0, z_0)$ and $p_1 = (t_1, z_1)$ with $t_0 < t_1$ be any two points on a future directed time-like geodesic γ. Denote by (a, b) the coordinate components of a vector v at p_0 and consider $\exp_{p_0} : (a, b) \to (t, z)$. Let $v_1 = (a_1, b_1) = \exp_{p_0}^{-1}(p_1)$ with v_1 tangent to γ at p_0. According to the definition[7] of conjugate points equivalent to that of Refs. [5] and [6], p_1 is conjugate to p_0 along γ if and only if

$$J(a_1, b_1) = \left| \begin{array}{cc} \frac{\partial t}{\partial a} & \frac{\partial t}{\partial b} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} \end{array} \right|_{(a_1, b_1)} = 0 .$$

Let λ be the affine parameter of γ satisfying $v_1 = \partial/\partial\lambda|_{p_0}$ with $\lambda = 0$ at p_0, (Q_γ, E_1) be the characterizing constants of $\gamma(\lambda)$, (Q, E) be those constants of future directed time-like geodesics $\rho(\lambda)$ with $\rho(0) = p_0$, and $t = t(\lambda), z = z(\lambda)$ be the equations of $\rho(\lambda)$, then

$$a = \left.\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right|_0 = Ez_0^{3/2} ,$$
$$b = \left.\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right|_0 = \pm Ez_0^{3/4}(z_0^{3/2} - Q)^{1/2} , \tag{6}$$

with

$$a_1 = E_1 z_0^{3/2} ,$$
$$b_1 = \pm E_1 z_0^{3/4}(z_0^{3/2} - Q_\gamma)^{1/2} .$$

Equation (6) establishes a one-one correspondence between (a, b) and (Q, E) provided that the sign of $b = \mathrm{d}z/\mathrm{d}\lambda|_0$ is fixed, thus

$$J(a,b) = \begin{vmatrix} \frac{\partial t}{\partial Q} & \frac{\partial t}{\partial E} \\ \frac{\partial z}{\partial Q} & \frac{\partial z}{\partial E} \end{vmatrix} \begin{vmatrix} \frac{\partial Q}{\partial a} & \frac{\partial Q}{\partial b} \\ \frac{\partial E}{\partial a} & \frac{\partial E}{\partial b} \end{vmatrix} = K(Q,E) \cdot 2b/a ,$$

where

$$K(Q,E) := \begin{vmatrix} \frac{\partial t}{\partial Q} & \frac{\partial t}{\partial E} \\ \frac{\partial z}{\partial Q} & \frac{\partial z}{\partial E} \end{vmatrix} .$$

Since in what follows $\exp_{p_T}$ is never involved and hence b_1 never vanishes, $J(a_1, b_1) = 0$ if and only if $K(a_1, b_1) = 0$. To proceed further one has to distinguish the following three cases.

CASE 1. Both p_0 and p_1 are located on the same side of the turning point p_T (either $t_0 < t_1 < t_T$ or $t_T < t_0 < t_1$).

In the situation $t_0 < t_1 < t_T$, one has from Eq. (4) that

$$(\frac{\mathrm{d}z}{\mathrm{d}\lambda})_\gamma = -E_1 z^{3/4}(z^{3/2} - Q_\gamma)^{1/2} .$$

It then follows from $p_1 = \exp_{p_0}(a_1, b_1)$ that

$$\frac{1}{E_1}\int_{z_1}^{z_0} z^{-3/4}(z^{3/2} - Q_\gamma)^{-1/2}\mathrm{d}z = 1 .$$

If $p = (t, z) = \exp_{p_0}(a, b)$ with (a, b) sufficiently close to (a_1, b_1), then $t_0 < t < t_T$ and hence

$$E^{-1}\int_z^{z_0} u^{-3/4}(u^{3/2} - Q)^{-1/2}\mathrm{d}u = 1 , \tag{7}$$

and

$$t - t_0 = \int_z^{z_0} u^{3/4}(u^{3/2} - Q)^{-1/2}\mathrm{d}u , \tag{8}$$

where (Q, E) is the correspondence to (a, b). Differentiation of Eq. (8) with respect to Q and E, respectively, yields

$$\partial t/\partial Q = \int_z^{z_0} \frac{\partial}{\partial Q}[u^{3/4}(u^{3/2}-Q)^{-1/2}]\mathrm{d}u - z^{3/4}(z^{3/2}-Q)^{-1/2}\partial z/\partial Q \ ,$$

$$\partial t/\partial E = -z^{3/4}(z^{3/2}-Q)^{-1/2}\partial z/\partial E \ ,$$

and differentiation of Eq. (7) with respect to E gives

$$\partial z/\partial E = -z^{3/4}(z^{3/2}-Q)^{1/2} \ .$$

Hence

$$K(Q_\gamma, E_1) = -\frac{1}{2}z_1^{3/4}(z_1^{3/2}-Q_\gamma)^{1/2}\int_{z_1}^{z_0} u^{3/4}(u^{3/2}-Q_\gamma)^{-3/2}\mathrm{d}u \neq 0 \ .$$

Therefore, it is impossible for p_1 to be conjugate to p_0 along $\gamma(\lambda)$. It is easily seen that the same result holds for the situation $t_T < t_0 < t_1$.

CASE 2. One of p_0 and p_1 coincides with the turning point p_T.

Suppose $p_1 = p_T$. Let $p_1' = (t_1', z_1') \in \gamma$ with $t_0 < t_1' < t_T$ and $p_1' = \exp_{p_0}(a_1', b_1')$ with $E_1' = a_1' z_0^{-3/2}$, then

$$K(Q_\gamma, E_T) = \lim_{E_1' \to E_T} K(Q_\gamma, E_1')$$

$$= \lim_{\substack{z_1' \to z_T \\ E_1' \to E_T}} \left[-\frac{1}{2}z_1'^{3/4}(z_1'^{3/2}-Q_\gamma)^{1/2}\int_{z_1'}^{z_0} u^{3/4}(u^{3/2}-Q_\gamma)^{-3/2}\mathrm{d}u\right] = -\frac{2}{3}z_T \neq 0 \ .$$

Therefore, it is impossible for p_1 to be conjugate to p_0 along $\gamma(\lambda)$. One readily sees that this result holds also for the situation $p_0 = p_T$ by setting $p_0' = (t_0', z_0') \in \gamma$ with $t_T < t_0' < t_1$ and viewing $p_0' = \exp_{p_1}(a_0', b_0')$ for some vector (a_0', b_0') at p_1 and then taking the limit.

CASE 3. p_0 and p_1 are located on different sides of $p_T (t_0 < t_T < t_1)$.

If $p = (t, z) = \exp_{p_0}(a, b)$ with (a, b) sufficiently close to (a_1, b_1), then $t_T < t$ and hence

$$1 = E^{-1}\int_{z_T}^{z_0} u^{-3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u + E^{-1}\int_{z_T}^{z} u^{-3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u$$

$$= E^{-1}\int_{Q^{2/3}}^{z_0} u^{-3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u + E^{-1}\int_{Q^{2/3}}^{z} u^{-3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u \ ,$$

and

$$t - t_0 = \int_{Q^{2/3}}^{z_0} u^{3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u + \int_{Q^{2/3}}^{z} u^{3/4}(u^{3/2}-Q)^{-1/2}\mathrm{d}u \ ,$$

which can be simplified by setting $v = Q^{-2/3}u$:

$$E^{-1}Q^{-1/3}\left[\int_1^{z_0 Q^{-2/3}} v^{-3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v + \int_1^{zQ^{-2/3}} v^{-3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v\right] = 1 \ , \quad (9)$$

$$t - t_0 = Q^{2/3}\left[\int_1^{z_0 Q^{-2/3}} v^{3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v + \int_1^{zQ^{-2/3}} v^{3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v\right] . \quad (10)$$

Differentiation of Eq. (10) with respect to Q yields

$$\partial t/\partial Q = \frac{2}{3}Q^{-1/3}[\alpha(z_0 Q^{-2/3}) + \alpha(zQ^{-2/3})] + (zQ^{-2/3})^{3/4}[(zQ^{-2/3})^{2/3} - 1]^{-1/2}\partial z/\partial Q ,$$

where

$$\alpha(w) := \int_1^w v^{3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v - w^{7/4}(w^{3/2}-1)^{-1/2} , \quad 1 < w < \infty . \quad (11)$$

Differentiation of Eqs. (10) and (9) with respect to E, respectively, gives

$$\partial t/\partial E = (zQ^{-2/3})^{3/4}[(zQ^{-2/3})^{3/2} - 1]^{-1/2}\partial z/\partial E ,$$

and

$$\partial z/\partial E = z^{3/4}(z^{3/2} - Q)^{1/2} \neq 0 .$$

Hence

$$K(Q,E) = \frac{2}{3}Q^{-1/3}[\alpha(z_0 Q^{-2/3}) + \alpha(zQ^{-2/3})]\partial z/\partial E ,$$

and $K(Q_\gamma, E_1) = 0$ if and only if

$$\alpha(z_0 Q_\gamma^{-2/3}) + \alpha(z_1 Q_\gamma^{-2/3}) = 0 . \quad (12)$$

Equation (12) is then the necessary and sufficient condition for $p_0, p_1 \in \gamma$ to be conjugate along γ.

The function $\alpha(w)$ can be recast into a more tractable form:

$$\alpha(w) = \frac{1}{4}\int_1^w v^{-3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v - w^{1/4}(w^{3/2}-1)^{-1/2} , \quad 1 < w < \infty . \quad (13)$$

Equation (13) implies that the function $\alpha(w)$ has the following properties:

(a) $\alpha(1) = -\infty$;

(b) $\mathrm{d}\alpha/\mathrm{d}w = 3w^{3/4}/4(w^{3/2}-1)^{3/2} > 0 , \quad \forall w \in (1,\infty)$;

(c) $0 < \alpha(\infty) = \frac{1}{4}\int_1^\infty v^{-3/4}(v^{3/2}-1)^{-1/2}\mathrm{d}v < \infty$.

It follows from the combination of properties (a), (b) and (c) that (i) there exist a unique $w_s \in (1,\infty)$ such that $\alpha(w_s) = 0$, and a unique $w_c \in (1, w_s)$ such that $\alpha(w_c) + \alpha(\infty) = 0$; (ii) for any $w_1 \in (w_c, w_s)$ there is a unique $w_2 \in (w_s, \infty)$ such that $\alpha(w_1) + \alpha(w_2) = 0$, while for any $w_1 \in (1, w_c)$ no such w_2 exists. Therefore every $\gamma(\lambda)$ contains four special points

$p_s = (t_s, z_s), \widetilde{p}_s = (\widetilde{t}_s, z_s), p_c = (t_c, z_c)$, and $\widehat{p}_c = (\widehat{t}_c, z_c)$, where

$$z_s = w_s Q_\gamma^{2/3} ,$$
$$z_c = w_c Q_\gamma^{2/3} ,$$
$$t_s = t_T - Q^{2/3} \int_1^{w_s} v^{3/4}(v^{3/2} - 1)^{-1/2} \mathrm{d}v ,$$
$$\widetilde{t}_s = t_T + Q^{2/3} \int_1^{w_s} v^{3/4}(v^{3/2} - 1)^{-1/2} \mathrm{d}v ,$$
$$t_c = t_T - Q^{2/3} \int_1^{w_s} v^{3/4}(v^{3/2} - 1)^{-1/2} \mathrm{d}v ,$$

and

$$\widetilde{t}_c = t_T + Q^{2/3} \int_1^{w_s} v^{3/4}(v^{3/2} - 1)^{-1/2} \mathrm{d}v .$$

In terms of these special points Eq. (12) then implies the following.

PROPOSITION 1. The following holds for every time-like geodesic $\gamma(\lambda)$ with $Q \neq 0$ in Σ_{tz}:

(i) p_s and $\widetilde{p}_s$ comprise a pair of symmetric conjugate points along $\gamma(\lambda)$, i.e., $\widetilde{t}_s - t_T = t_T - t_s$;

(ii) Since $w_s > w_c$, one has $t_c > t_s$ and $\widetilde{t}_s > \widehat{t}_c$ and hence there exist segments

$$\mathcal{M} = \{(t,z) \in \gamma | t_s < t < t_c\} , \qquad \widetilde{\mathcal{M}} = \{(t,z) \in \gamma | \widetilde{t}_s < t < \infty\} ,$$
$$\mathcal{N} = \{(t,z) \in \gamma | -\infty < t < t_s\} , \quad \text{and} \quad \widetilde{\mathcal{N}} = \{(t,z) \in \gamma | \widehat{t}_c < t < \widetilde{t}_s\} ,$$

such that each $p \in \mathcal{M}$ (resp. each $p \in \mathcal{N}$) has a unique conjugate point along $\gamma(\lambda)$ that is located in $\widetilde{\mathcal{M}}$ (resp. $\widetilde{\mathcal{N}}$) and vice versa;

(iii) $p = (t,z) \in \gamma(\lambda)$ has no conjugate point along γ if and only if $t_c \leq t \leq \widehat{t}_c$.

This proposition is readily generalized to the four-dimensional Taub spacetime, the only change being the replacement of "time-like geodesic $\gamma(\lambda)$" by "null geodesic $\eta(\lambda)$", and this generalization is also referred to as proposition 1 hereafter.

It follows from proposition 1 that if $\eta(\lambda)$ is a future directed null geodesic with $Q \neq 0$ in Taub spacetime, then any $p \in \mathcal{M} \cup \mathcal{N} \subset \eta$ has a conjugate point $\widetilde{p} \in J^+(p)$ along $\eta(\lambda)$. Therefore, $\eta(\lambda)$ with $\lambda > \lambda(\widetilde{p})$ will certainly be within $I^+(p)$. However, as is proved in the following proposition, the conjugate point $\widetilde{p}$ itself is also in $I^+(p)$, and by virtue of proposition 1 there is no point other than $\widetilde{p}$ conjugate to p along η, so that the entrance of the null geodesic η emanating from p into $I^+(p)$ cannot be attributed to the existence of the conjugate point to p.

PROPOSITION 2. If $\eta(\lambda)$ is null geodesic in Taub spacetime and $p_0, p_1 \in \eta$ with $t_1 > t_0$ are a pair of conjugate points along $\eta(\lambda)$, then we have $p_1 \in I^+(p_0)$.

PROOF: According to Lemma 1 of Ref. [8] by Kuang *et al.*, any two points p_1 and p_2 in Taub spacetime satisfy $p_2 \in I(p_1)$ if $t_1 - z_1 > t_2 + z_2$ (note that there is an error in Ref. [7], in which the condition is given as $t_1 - z_1 \geq t_2 + z_2$, the equal sign being incorrect). Therefore

to show our proposition it suffices to show $t_1 - z_1 > t_0 + z_0$. Since

$$t_1 - z_1 = t_T + \int_{z_T}^{z_1} u^{3/4}(u^{3/2} - Q_\eta)^{-1/2}\mathrm{d}u - z_1$$
$$= t_T + Q_\eta^{2/3}\left[\int_1^{z_1 Q_\eta^{-2/3}} v^{3/4}(v^{3/2} - 1)^{-1/2}\mathrm{d}v - z_1 Q_\eta^{-2/3}\right],$$

$$t_0 + z_0 = t_T - \int_{z_T}^{z_0} u^{3/4}(u^{3/2} - Q_\eta)^{-1/2}\mathrm{d}u + z_0$$
$$= t_T - Q_\eta^{2/3}\left[\int_1^{z_0 Q_\eta^{-2/3}} v^{3/4}(v^{3/2} - 1)^{-1/2}\mathrm{d}v - z_0 Q_\eta^{-2/3}\right],$$

$t_1 - z_1 > t_0 + z_0$ if and only if

$$\mu(z_1 Q_\eta^{-2/3}) + \mu(z_0 Q_\eta^{-2/3}) > 0 ,$$

where

$$\mu(w) := \int_1^w v^{3/4}(v^{3/2} - 1)^{-1/2}\mathrm{d}v - w .$$

Since $w < w^{7/4}(w^{3/2} - 1)^{-1/2}$ for any $w \in (1, \infty)$, one has $\mu(w) > \alpha(w)$ and hence

$$\mu(z_1 Q_\eta^{-2/3}) + \mu(z_0 Q_\eta^{-2/3}) > \alpha(z_1 Q_\eta^{-2/3}) + \alpha(z_0 Q_\eta^{-2/3}) .$$

The right hand side vanishes due to the fact that p_0 and p_1 are conjugate along $\eta(\lambda)$, implying that the left hand side is greater than zero.

III. GENERALIZATION OF THE RESULTS

The results in Sec.II can be generalized to a class of spacetimes. Consider first the spacetime (R^4, g_σ) with metric

$$\mathrm{d}S = z^{-2\sigma}(-\mathrm{d}t^2 + \mathrm{d}z^2) + (\mathrm{d}x^2 + \mathrm{d}y^2) , \quad \sigma > 0 , \quad z > 0 . \tag{14}$$

The generalization of cases 1 and 2 is straightforward and the conclusion is the same: no conjugate points exist in these two cases. As to case 3, the definition of the function $\alpha(w)$ is generalized to be

$$\alpha(w) := \int_1^w v^\sigma(v^{2\sigma} - 1)^{-1/2}\mathrm{d}v - w^{1+\sigma}(w^{2\sigma} - 1)^{-1/2}$$
$$= (1 - \sigma)\int_1^w v^{-\sigma}(v^{2\sigma} - 1)^{-1/2}\mathrm{d}v - w^{1-\sigma}(w^{2\sigma} - 1)^{-1/2} , \quad 1 < w < \infty , \tag{13}'$$

and $K(Q_\eta, E_1) = 0$ if and only if

$$\alpha(z_1 Q_\eta^{-1/2\sigma}) + \alpha(z_0 Q_\eta^{-1/2\sigma}) > 0 .$$

It is easily checked that properties (a) and (b) of $\alpha(w)$ in Sec.II still hold for the $\alpha(w)$ in this section, but the generalized property (c) should be formulated as follows (the proof is straightforward and thus omitted):

(c_1) $\alpha(\infty) \leq 0$ if $\sigma \geq 1$;

(c_2) $0 < \alpha(\infty) < \infty$ if $1/2 < \sigma < 1$;

(c_3) $\alpha(\infty) = \infty$ if $\sigma \leq 1/2$.

Properties (a), (b), and (c_1)-(c_3) of $\alpha(w)$ imply the following.

PROPOSITION 3. In a spacetime (R^4, g_σ) with metric (14), the existence and location of conjugate points along a null geodesic $\eta(\lambda)$ with $Q \neq 0$ can be classified as follows:

(A) if $\sigma \geq 1$, then there exist no conjugate points along $\eta(\lambda)$;

(B) if $1/2 < \sigma < 1$, then each null geodisic $\eta(\lambda)$ with $Q \neq 0$ contains four special points $p_s, \widetilde{p}_s, p_c$, and $\widehat{p}_c$ and four segments $\mathcal{M}, \widetilde{\mathcal{M}}, \mathcal{N}$, and $\widetilde{\mathcal{N}}$ defined similarly as those in Sec.II, and the existence and location of conjugate points along $\eta(\lambda)$ obey the same rule as that in proposition 1;

(C) if $\sigma \leq 1/2$, then (i) each null geodisic $\eta(\lambda)$ with $Q \neq 0$ contains a pair of symmetric conjugate points p_s and $\widetilde{p}_s$, defined similarly as those in Sec.II; (ii) there exist segments $\mathcal{M} = \{(t,z,x,y) \in \eta | t_s < t < t_T\}$, $\widetilde{\mathcal{M}} = \{(t,z,x,y) \in \eta | \widetilde{t}_s < t < \infty\}$, $\mathcal{N} = \{(t,z,x,y) \in \eta | -\infty < t < t_s\}$, and $\widetilde{\mathcal{N}}\{(t,z,x,y) \in \eta | t_T < t < \widetilde{t}_s\}$ such that each $p \in \mathcal{M}$ (resp. $\mathcal{N}$) has a unique conjugate point along $\eta(\lambda)$ that is located in $\widetilde{\mathcal{M}}$ (resp. $\widetilde{\mathcal{N}}$) and vice versa; (iii) $p \in \eta(\lambda)$ has no conjugate points if and only if $p = p_T$.

It is easily checked that the lemma 1 of Ref. [8] applies also to spacetime (R^4, g_σ), hence proposition 2 can be generalized as follows.

PROPOSITION 4. If η is a null geodesic in spacetime (R^4, g_σ) and $p_1 \in \eta$ is a conjugate point to $p_0 \in \eta$ along η with $t_1 > t_0$, then $p_1 \in I^+(p_0)$.

Since all the conclusions obtained previously are conformally invariant, propositions 3 and 4 can be further generalized to all spacetimes conformal to (R^4, g_σ), implying that those conclusions can be applied to a class of spacetimes which are allowed to be non-stationary and non-plane-symmetric.

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